1 The theorems of Paley and Wiener

Consider the identity

$$\frac{\sin \pi x}{\pi x} = \int_{-1/2}^{1/2} e^{2\pi i x t} dt,$$

where $x \in \mathbb{R}$. From the previous investigations we recognize this as the Fourier transform pair

$$f(x) = \frac{\sin \pi x}{\pi x}$$

and

$$\widehat{f}(t) = \chi_{[-1/2,1/2]}(t).$$

(Indeed, it is easy to verify that the identity is true by calculating the integral on the right.) Evidently, both functions are elements of $L^2(\mathbb{R})$. However, there is more to be said: Both sides of the identity make sense for complex $x \in \mathbb{C}$. In fact, both sides are entire functions! This can be seen directly for the left side from a power series expansion, and for the right side by an application of Morera's theorem.

In fact, we have for $z \in \mathbb{C}$ that

$$\frac{\sin \pi z}{\pi z} = \int_{-1/2}^{1/2} e^{2\pi i z t} dt$$

This is not entirely surprising, since we know that for every $t \in \mathbb{R}$ the function

$$z \mapsto e^{2\pi i z t}$$

is an entire function of $z \in \mathbb{C}$. Hence,

$$z \mapsto \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i z t} dt$$

extends by Morera's theorem as an analytic function to every region $\Omega \subseteq \mathbb{C}$ for which

$$\int_T \int_{\mathbb{R}} \widehat{f}(t) e^{2\pi i z t} dt dz = \int_{\mathbb{R}} \int_T \widehat{f}(t) e^{2\pi i z t} dz dt$$

for every triangular path $T \subseteq \Omega$.

This leads to the following two questions that have far reaching applications in analysis:

1. What are conditions on a Fourier transform which guarantee that the Fourier integral extends as an analytic function to a given region Ω?
2. If f is in some $L^p(\mathbb{R})$ space and extends to an analytic function on some subset of Ω, what are the conclusions about the Fourier transform that we can draw?

To get a feeling for the type of results that are availabe, we consider some examples.

1. If $\widehat{f}$ is bounded and has compact support contained in $[a, b]$, then f is entire, and is in fact given by

$$f(z) = \int_a^b e^{2\pi izt} \widehat{f}(t)dt.$$

2. If $\widehat{f}$ satisfies

$$|\widehat{f}(t)| \leq Ce^{2\pi a|t|},$$

for some positive constant C, then f is analytic in $|\Im z| < a$, and satisfies

$$f(z) = \int_{\mathbb{R}} e^{2\pi izt} \widehat{f}(t)dt$$

in that strip.To see this, note that

$$|e^{2\pi i(x+iy)t} \widehat{f}(t)| \leq |e^{2\pi y|t|} \widehat{f}(t)| \leq e^{2\pi(y-a)|t|},$$

and this is in $L^1 \cap L^\infty(\mathbb{R})$ if $|y| < |a|$. Hence the change of integration described above in the application of Morera's theorem is justified.

We set

$$\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}.$$

Lemma 1. *Let $F \in L^2(0, \infty)$. Then f defined by*

$$f(z) = \int_0^\infty F(t)e^{2\pi izt}dt$$

is analytic in $\mathbb{C}^+$.

Proof. We note first that for $z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$,

$$|e^{2\pi izt}| = e^{-2\pi yt},$$

hence $t \mapsto F(t)e^{2\pi itz} \in L^1(\mathbb{R})$. We show next that f is continuous on $\mathbb{C}^+$. Let $z \in \mathbb{C}^+$ and let $z_n \to z$. There exists $\delta > 0$ so that $\Im z > \delta > 0$, and we may assume that $\Im z_n > \delta$ as well. (Delete the first n_0 elements of z_n if this is not the case.)

Cauchy-Schwarz inequality implies

$$\begin{aligned} |f(z) - f(z_n)|^2 &= \left| \int_0^\infty F(t)(e^{2\pi itz} - e^{2\pi itz_n})dt \right|^2 \\ &\leq ||F||_2^2 \int_0^\infty |e^{2\pi itz} - e^{2\pi itz_n}|^2 dt, \end{aligned}$$

and we have

$$|e^{2\pi itz} - e^{2\pi itz_n}|^2 \leq 4e^{-2\pi\delta t}$$

which is integrable on $[0, \infty)$ and independent of z_n. Lebesgue dominated convergence is applicable and gives

$$\lim_{n\to\infty}(f(z) - f(z_n)) = \int_0^\infty F(t) \lim_{n\to\infty}(e^{2\pi itz} - e^{2\pi itz_n})dt = 0$$

which shows that f is continuous. Finally, let T be a triangular path in $\mathbb{C}^+$. Fubini's theorem gives

$$\int_T \int_0^\infty F(t)e^{2\pi itz}dtdz = \int_0^\infty F(t) \int_T e^{2\pi izt}dzdt = 0,$$

since $z \mapsto e^{2\pi izt}$ is entire. □

Consider now $f(z)$ as a function of x for fixed y, i.e., consider

$$h_y(x) = f(x + iy) = \int_0^\infty F(t)e^{-2\pi ty}e^{2\pi itx}dt.$$

Then

$$\int_{-\infty}^\infty |h_y(x)|^2 dx = \int_0^\infty |F(t)|^2 e^{-2ty}dt \leq ||F||_2^2.$$

We have shown

Proposition 1. *Under the assumptions of the previous lemma, the set of restrictions $f_y(x) = f(x + iy)$ to horizontal lines is a bounded set in $L^2(\mathbb{R})$.*

The first theorem of Paley and Wiener has as its content that the converse is true as well.

Theorem 1. *Suppose $f : \mathbb{C}^+ \to \mathbb{C}$ is analytic, and*

$$\sup_{0<y<\infty} \int_{-\infty}^{\infty} |f(x+iy)|^2 dx = C < \infty.$$

Then there exists $F \in L^2(0,\infty)$ such that for all $z \in \mathbb{C}^+$

$$f(z) = \int_0^\infty F(t) e^{2\pi i t z} dt$$

and

$$||F||_2^2 = C.$$

Some intuition. Given f, we want F so that $f(x+iy)$ is the Fourier transform of $t \mapsto F(t)e^{-2\pi t y}$. We can find a candidate for F purely formal by Fourier inversion: if

$$f(x+iy) = \int_0^\infty F(t) e^{-2\pi y t} e^{2\pi i x t} dt,$$

then $F(t)e^{-2\pi y t}$ should be the Fourier transform of $f_y(x) = f(x+iy)$. This means, for $z = x+iy$ we should have

$$F(t) = \int_{-\infty}^{\infty} f(x+iy) e^{2\pi t y} e^{-2\pi i t x} dx = \int_{\Im z = y} f(z) e^{-2\pi i t z} dz,$$

where the last integral is a path integral. So far, this is only a heuristic argument; the left-hand side should independent of y, but the right-hand side looks as if it depends on y. However, we will prove that changing the path of integration from $\Im z = y_1$ to $\Im z = y_2$ does not change the value of the integral.

Proof of Theorem 1. To distinguish different horizontal lines, set $f_y(x) = f(x+iy)$. By assumption, $f_y \in L^2(\mathbb{R})$ for every $y > 0$. We use $y = 1$ to define $F : \mathbb{R} \to \mathbb{C}$ by

$$F(t) = e^{2\pi t} \widehat{f_1}(t).$$

We need to show that

$$f_y(x) = f(x+iy) = \int_{-\infty}^{\infty} F(t) e^{-2\pi t y} e^{2\pi i t x} dt,$$

and that $F(t) = 0$ for $t < 0$.

We prove this by showing that $F(t) = e^{2\pi t y} \widehat{f_y}(t)$ holds almost everywhere.

In order to apply Cauchy's theorem, we need to show this on the transform side, i.e., need to prove first that $\widehat{f_1} = \widehat{f_y}$, and then apply Fourier inversion.

For ease of notation we only consider the case $y > 1$. The case $y < 1$ is analogous by reversing a couple of integral bounds. Let $k > 0$. Define Γ_k to be the rectangle with corners $\pm k + i$ and $\pm k + iy$ traced counterclockwise. Fix real t. Cauchy's theorem implies that

$$\int_{\Gamma_\alpha} f(z)e^{2\pi itz}dz = 0.$$

In order to show that

$$F(t) = \int_{-\infty}^{\infty} f(x+iy)e^{2\pi it(x+iy)}dx$$

for every $y > 1$, we have to prove that the integrals over the vertical line segments in the above contour integral go to zero. We cannot (quite) do this for every k, but we are able to show that there exists a sequence of positive values k_j for which the integrals over the corresponding line segments go to zero.

Define

$$V(k) := \int_{k+i}^{k+iy} f(z)e^{-itz}dz.$$

We would like that $V(\pm k)$ goes to zero as $k \to \pm\infty$. We cannot quite prove that, though.

Lemma 2. *There exists a sequence $k_j \to \infty$ with*

$$V(\pm k_j) \to 0.$$

Proof. Using Cauchy-Schwarz inequality,

$$|V(k)|^2 \leq \int_1^y |f(k+iu)|^2du \int_1^y e^{2tu}du.$$

The second integral is independent of k. We know that

$$\int_1^y \int_{-\infty}^{\infty} |f(x+iu)|^2dxdu \leq C(y-1)$$

by assumption on the value of the supremum. Change order of integration:

$$\int_{-\infty}^{\infty} \int_1^y |f(x+iu)|^2dudx \leq C(y-1)$$

From this we get at least a sequence of $k_j \to \infty$ so that

$$\int_1^y |f(\pm k_j + iu)|^2 du \to 0,$$

(if such a sequence did not exist, the value of the double integral would be infinite), and hence

$$V(\pm k_j) \to 0$$

as $j \to \infty$. □

Define

$$\varphi_{j,y}(t) = \int_{-k_j}^{k_j} f(x+iy) e^{-2\pi i t x} dx.$$

Cauchy's theorem and $V(\pm k_j) \to 0$ imply that

$$\lim_{j \to \infty} (e^{2\pi t y} \varphi_{j,y}(t) - e^{2\pi t} \varphi_{j,1}(t)) = 0.$$

Recall that $f_y(x) = f(x+iy)$. From L^2-theory (Plancherel's theorem) it follows that

$$\varphi_{j,y}(t) \to \widehat{f}_y(t) \text{ in } L^2(\mathbb{R}),$$

and from Real Analysis we obtain the existence of a subsequence of the j's so that $\varphi_{j_n,y}(t) \to \widehat{f}_y(t)$ pointwise almost everywhere. We had defined F by

$$F(t) = e^{2\pi t} \widehat{f}_1(t),$$

and we obtain now almost everywhere

$$\begin{aligned} F(t) = e^{2\pi t} \widehat{f}_1(t) &= \lim_{j \to \infty} e^{2\pi t} \varphi_{j,1}(t) \\ &= \lim_{n \to \infty} e^{2\pi y t} \varphi_{j_n,y}(t) = e^{2\pi y t} \widehat{f}_y(t) \end{aligned}$$

for every $y > 1$. We now need to prove the required properties of F. Apply $||f_y||_2 = ||\widehat{f}_y||_2$ to get

$$\int_{-\infty}^{\infty} e^{-4\pi t y} |F(t)|^2 dt = \int_{-\infty}^{\infty} |\widehat{f}_y(t)|^2 dt = \int_{-\infty}^{\infty} |f(x+iy)|^2 dx \leq C.$$

This holds for every $y > 1$. In particular, if $y \to \infty$ the integral on the left remains bounded, but the exponential converges to infinity uniformly on every interval $(-\infty, -\delta]$ with $\delta > 0$. Hence, if there exists a set $A \subseteq (-\infty, 0]$

of positive measure, so that $F(t) \neq 0$ for all $t \in A$, then the integral must diverge to zero.

Since it does not do this, we must have that $F(t) = 0$ for almost every negative t. Letting $y \to 0$ shows that $|F|^2$ is integrable (monotone convergence theorem). Cauchy-Schwarz implies for $y > 0$ that $\widehat{f_y}$ is in L^1, hence

$$f(x+iy) = \int_{-\infty}^{\infty} \widehat{f_y}(t) e^{2\pi itx} dt = \int_0^{\infty} F(t) e^{-2\pi ty} e^{2\pi itx} dt,$$

and the exponent is $2\pi izt$.

2 The Paley Wiener space

The second class of functions that we consider is given by the collection $PW_{2\pi A}$ of $f \in L^2(\mathbb{R})$ such that

$$f(z) = \int_{-A}^{A} F(t) e^{2\pi itz} dt$$

where $0 < A < \infty$ and $F \in L^2(-A, A)$. These functions are entire and satisfy the growth condition

$$|f(z)| \leq e^{2\pi A|y|} \int_{-A}^{A} |F(t)| dt =: Ce^{2\pi A|y|}.$$

Before we get to the Paley-Wiener theorem for this class, we investigate the structure of this space. We recall that $L^2([-A, A])$ is a Hilbert space with basis

$$\left\{ \frac{1}{2A} e^{\pi int/A} : n \in \mathbb{Z} \right\}.$$

Let $\widehat{f} \in L^2([-A, A])$. Expand $\widehat{f}$ into its Fourier series: We have

$$\widehat{f}(t) = \sum_{n \in \mathbb{Z}} a_n e^{\pi int/A}$$

where

$$a_n = \frac{1}{2A} \int_{-A}^{A} \widehat{f}(t) e^{\pi int/A} dt = \frac{1}{2A} f\left(\frac{n}{2A}\right).$$

Now take the Fourier inverse transform of $\widehat{f}$ and plug in the Fourier series of $\widehat{f}$. We obtain that

$$f(z) = \sum_{n \in \mathbb{Z}} a_n \int_{-A}^{A} e^{2\pi it(z - n/(2A))} dt = \sum_{n \in \mathbb{Z}} f\left(\frac{n}{2A}\right) \frac{\sin \pi(n - 2Az)}{\pi(n - 2Az)}$$

where convergence takes place in $L^2(\mathbb{R})$. This means in particular that $f \in PW_{2\pi A}$ is completely determined by its values $f(n/(2A)$, where $n \in \mathbb{Z}$. Historically, this is one of the reasons why the Paley-Wiener class is important in applications; it allows reconstruction of the function from its values at a discrete set of points. The reonstruction formula that we just developed is often called the 'Shannon-Whittaker interpolation formula', and has important applications in signal processing.

The second Paley-Wiener theorem gives a characterization of membership in $PW_{2\pi A}$. The crucial insight of Paley and Wiener was the recognition that membership of an entire function in this class can be checked just by looking at the increase of $|f(z)|$ as $|z| \to \infty$.

Theorem 2. *Suppose A and C are positive constants and f is entire with $|f(z)| \leq Ce^{2\pi A|z|}$ for all z and $||f||_{L^2(\mathbb{R})} < \infty$. Then there exists $F \in L^2(-A, A)$ so that*

$$f(z) = \int_{-A}^{A} F(t)e^{2\pi itz}dt.$$

for all z.

The idea of the proof is as follows. Consider (only formally) the integral

$$\int_{-k}^{k} f(x)e^{-2\pi ixt}dx.$$

Split this integral at the origin, and consider it as the difference of two path integrals starting at the origin traced outwards. Complete both paths to a closed contour that includes the positive imaginary axis. Apply the residue theorem.

For each real α we define Γ_α to be a ray starting at the origin so that the angle of the ray with the x-axis has angle $2\pi\alpha$ traced outwards. Parametrize:

$$\Gamma_\alpha(s) = se^{i\alpha},$$

where $0 \leq s < \infty$. (We are mainly interested in the real axis and the imaginary positive axis, i.e, Γ_0, Γ_π, and $\Gamma_{\pi/2}$.)

Define

$$\Pi_\alpha = \{w : \Re(we^{i\alpha}) > A\}.$$

Note that if we write $e^{i\alpha}w = z$, then $\Pi_\alpha = \{ze^{-i\alpha} : \Re z > A\}$, i.e, Π_α is the image of a right half-plane under the rotation by $e^{-i\alpha}$.

We define next the path integrals

$$\Phi_\alpha(w) = \int_{\Gamma_\alpha} f(z)e^{-2\pi wz}dz = e^{i\alpha}\int_0^\infty f(se^{i\alpha})e^{-2\pi wse^{i\alpha}}ds.$$

We shall show more than needed, namely, that there is an analytic function Φ in the upper half-plane of which the Φ_α are all analytic continuations, i.e., $\Phi = \Phi_\alpha$ on the domain of Φ_α.

We note that

$$\Phi_0(it) - \Phi_\pi(it) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixt}dx,$$

hence we aim to show that $\Phi_0(it) = \Phi_\pi(it)$ for $|t| \geq A$. If we can show that all the Φ_α are analytic continuations of the same analytic function, then this statement is proved.

Lemma 3. *Φ_α is analytic on Π_α. More is true for $\alpha \in \{0, \pi\}$: Φ_0 is analytic in $\Re w > 0$ and Φ_π is analytic in $\Re w < 0$.*

Proof. We use Morera's theorem to show the analyticity of Φ_α in Π_α. Note $s > 0$, hence

$$|f(se^{i\alpha})e^{-2\pi iwe^{i\alpha}}| \leq Ce^{-2\pi As}e^{-\Re(we^{i\alpha}s)} = Ce^{-[\Re(we^{i\alpha})-2\pi A]s}.$$

The exponential is of the form $s \mapsto e^{-\tau s}$ with $\tau > 0$ (and $s > 0$) provided $w \in \Pi_\alpha$, hence if T is a triangular path in Π_α, we may interchange the integrals in $\int_T \Phi_\alpha(w)dw$ to obtain that this integral equals zero, i.e., Φ_α is analytic in Π_α.

In particular, Φ_0 is analytic in $\Re w > 2\pi A$ and Φ_π is analytic in $\Re w < -2\pi A$. More is true in this case since $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$: Cauchy-Schwarz and another application of Morera's theorem give that Φ_0 is analytic in $\Re w > 0$ and Φ_π is analytic in $\Re w < 0$. □

Lemma 4. *Let $0 < \beta - \alpha < \pi$. Then $\Phi_\alpha = \Phi_\beta$ on $\Pi_\alpha \cap \Phi_\beta$.*

Proof. We only need to prove the identity on a dense set in the intersection, and we choose the ray $\Gamma_{(\alpha+\beta)/2)}$. Hence, we assume that $|w| > 2\pi A/\cos((\beta-\alpha)/2)$ and

$$w = |w|e^{-i\frac{\alpha+\beta}{2}}.$$

Cut Γ_α and Γ_β at $s = r$. Close the finite segments by adding the arc Γ defined by $\Gamma(u) = re^{iu}$, $\alpha \leq u \leq \beta$.

Hence, take $w = |w|e^{-i(\alpha+\beta)/2}$ and $z = re^{it} \in \Gamma$. Then we have

$$\Re(-wz) = -|w|r\cos(t - 2^{-1}(\alpha+\beta)) \leq -|w|r\cos((\beta-\alpha)/2),$$

and hence

$$|f(z)e^{-wz}| \leq Ce^{(2\pi A-|w|\cos((\beta-\alpha)/2))r}$$

For sufficiently large $|w|$ the right hand side decays exponentially. Since the arc has length bounded by $2\pi r$, the contribution from the arc goes to zero as $r \to \infty$, which implies that the integrals along the two rays are equal. □

This is almost what we need, since (only formally!)

$$\Phi_0(it) - \Phi_\pi(it) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx.$$

Proof. Consider $f_\varepsilon(x) := f(x)e^{-\varepsilon|x|}$ where $\varepsilon > 0$. We note first:

$$\int_{-\infty}^{\infty} |f(x) - f_\varepsilon(x)|^2 dx = \int_{-\infty}^{\infty} (1 - e^{\varepsilon|x|})|f(x)|^2 dx,$$

and as $\varepsilon \to 0$, Lebesgue dominated convergence with $|f|^2 \in L^1$ shows that $f_\varepsilon \to f$ in $L^2(\mathbb{R})$. Hence, it suffices to show that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f_\varepsilon(x) e^{-2\pi i x t} dx = 0$$

for all real t with $|t| > A$. Consider $t > A$. We have

$$\begin{aligned}\int_{-\infty}^{\infty} f_\varepsilon(x) e^{-2\pi i x t} dx &= \Phi_0(\varepsilon + 2\pi i t) - \Phi_\pi(-\varepsilon + 2\pi i t) \\ &= \Phi_{-\pi/2}(\varepsilon + 2\pi i t) - \Phi_{-\pi/2}(-\varepsilon + 2\pi i t),\end{aligned}$$

and let $\varepsilon \to 0$.

□

The class of analytic functions that we investigated in the first of the two Paley-Wiener theorems is the Hardy space H_2 of the upper half-plane. It is an analogue of the Hardy space $H_2(\mathbb{T})$ of the unit disk, and many of the statements of the disk space have analogues for the upper half-plane space.