1 Basic Theory

Let $1 \leq p < \infty$. We define $L^p(X)$ (usually $X = \mathbb{R}$ or $X$ an interval) to be the vector space of (equivalence classes of functions such that

$$
\|f\|_p = \left(\int_X |f(x)|^p dx\right)^{1/p}
$$

is finite. Lebesgue measure may be replaced by any Borel measure $d\mu(x)$, but we won’t need it. As with $L^1$, we identify two functions if they are equal almost everywhere. For $p = \infty$ we replace the integral norm with the essential supremum.

Obviously,

1. $\|f\|_p = 0$ if and only if $f = 0$ a.e.,

2. $\|cf\|_p = |c| \|f\|_p$.

In order to show that $\|f\|_p$ defines a norm, we require the triangle inequality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. This is called Minkowski’s inequality. We prove first Hölder’s inequality.

**Theorem 1.** Suppose $1 < p < \infty$ and $1/p + 1/q = 1$. Assume $f \in L^p$ and $g \in L^q$. Then

$$
\|fg\|_1 \leq \|f\|_p \|g\|_q.
$$

**Proof.** We may assume that neither function is zero a.e and that $\|f\|_p = \|g\|_q = 1$. We note that for $a \geq 0$, $b \geq 0$ and $0 < \lambda < 1$

$$
a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b.
$$

(To see this, set $t = a/b$ and use calculus to show that the maximum $1 - \lambda$ of $t^\lambda - \lambda t$ is at $t = 1$.) Apply this with $a = |f(x)|^p$, $b = |g(x)|^q$ and $\lambda = 1/b$. We get

$$
\|fg\|_1 \leq |f(x)|^{p-1}|f(x)|^p + |g(x)|^{q-1}|g(x)|^q.
$$

Now integrate both sides over $X$. \qed
**Theorem 2.** (Minkowski) If $1 \leq p < \infty$ and $f, g \in L^p$, then
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p.
\]

**Proof.** We note that
\[
|f + g|^p \leq (|f| + |g|)|f + g|^{p-1}.
\]
Now apply Hölder:
\[
\int |f + g|^p \leq \|f\|_p \|f + g|^{p-1}\|_q + \|g\|_p \|f + g|^{p-1}\|_q.
\]
Since \(\|f + g|^{p-1}\|_q = \left(\int |f + g|^p\right)^{1/q}\), we obtain the statement after division by the integral on the right and using that \((p - 1)q = p\). \(\Box\)

In particular, \(\|f\|_p\) is a norm on \(L^p\).

**Theorem 3.** \(L^p\) is complete, i.e., every Cauchy sequence converges.

**Proof.** We show first that if \(\sum_k \|f_k\|_p < \infty\), then \(\sum_k f_k\) converges in \(L^p\)-norm to an element in \(L^p\). Define
\[
G_n(x) = \sum_{j=1}^n |f_j(x)|,
\]
\[
G(x) = \sum_{j=1}^\infty |f_j(x)|
\]
and observe that \(\|G_n\|_p \leq \sum \|f_j\|_p < \infty\) by assumption. Monotone convergence: \(G \in L^p\). In particular: \(G(x) < \infty\) a.e., i.e., \(\sum f_k\) converges at least pointwise a.e. Denote the sum by \(F\). We have \(|F| \leq G\), hence \(F \in L^p\). Moreover,
\[
|F(x) - \sum_{j=1}^n f_j(x)|^p \leq (2G(x))^p \in L^1.
\]
Use this to show that \(\|F - \sum^n f_j\|_p \to 0\), i.e., \(\sum f_j\) converges to \(F\) in \(L^p\).

Now, if \(F_n\) is a Cauchy sequence in \(L^p\), choose a sequence \(n_k\) so that \(\|F_n - F_m\| < 2^{-j}\) for \(m, n \geq n_j\). Set \(f_1 = F_{n_1}\) and \(f_j = F_{n_j} - F_{n_{j-1}}\) for \(j > 1\). Check that
\[
F_{n_k} = \sum_{n=1}^k f_n.
\]
Prove that the series converges in \(L^p\), and use the fact that \(F_n\) is Cauchy to show that \(F_n\) and \(F_{n_k}\) have the same limit. \(\Box\)
Consequence: All $L^p$ spaces are normed complete vector spaces. These are also called Banach spaces.

2 Hilbert spaces

Let $H$ be a complex vector space. An inner product is a map $(x, y) \mapsto \langle x, y \rangle \in \mathbb{C}$ such that

1. $\langle ax + by, z \rangle = a \langle x, z \rangle + b \langle y, z \rangle$ for all $x, y, z \in H$ and $a, b \in \mathbb{C}$,

2. $\langle y, x \rangle = \overline{\langle x, y \rangle}$ for $x, y \in H$,

3. $\langle x, x \rangle \in (0, \infty)$ for all nonzero $x \in H$.

We define $\|x\| = \sqrt{\langle x, x \rangle}$.

For our purpose, the important examples of Hilbert spaces are $\mathbb{C}^n$ with the Euclidean norm, and $L^2(X)$ with scalar product $\langle f, g \rangle = \int_X f(u)\overline{g(u)}du$.

Proposition 1 (Parallelogram Law). For all $x, y \in H$,

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Proof. Uncollected exercise. $\square$

Theorem 4. Let $x, y \in H$.

1. (Schwarz inequality) $|\langle x, y \rangle| \leq \|x\| \|y\|$.

2. $x \mapsto \|x\|$ is a norm.

Proof. Assume the scalar product is not zero. Let $|\alpha| = 1$ so that $z = \alpha y$ satisfies $\langle x, z \rangle = \langle z, x \rangle = |\langle x, y \rangle|$. Observe

$$0 \leq \langle x - tz, x - tz \rangle = \|x\|^2 - 2t|\langle x, y \rangle| + t^2\|y\|^2.$$

The absolute minimum occurs at $t = \frac{|\langle y\rangle|^2}{\|y\|^2}$ (use Calculus). Plug this value of $t$ into the inequality to get

$$0 \leq \|x\|^2 - |\langle y\rangle|^2|\langle x, y \rangle|^2,$$

and solve.

To prove the triangle inequality, note that $\|x + y\|^2 = \|x\|^2 + 2\Re\langle x, y \rangle + \|y\|^2$, and apply the Schwarz inequality to the middle term. This gives $\|x + y\|^2 \leq (\|x\| + \|y\|^2)$. $\square$
If \( H = L^2(X) \), the Schwarz inequality is
\[
\left| \int fg \right|^2 \leq \int |f|^2 \int |g|^2,
\]
i.e., Hölder’s inequality. Without proof we note that \( L^p \) for \( p \neq 2 \) is never a Hilbert space.

Since \( H \) is a metric (even normed) space, \( T : H \to \mathbb{C} \) is continuous, if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) so that \( \|x_1 - x_2\| < \delta \) implies \( |T(x_1) - T(x_2)| < \varepsilon \).

**Theorem 5.** Let \( y \in H \). The mappings \( x \mapsto \langle x, y \rangle \) and \( x \mapsto \|x\| \) are continuous on \( H \).

**Proof.** We prove the first statement. Let \( x_1, x_2 \in H \). Then
\[
|\langle x_1, y \rangle - \langle x_2, y \rangle| = |\langle x_1 - x_2, y \rangle| \leq \|x_1 - x_2\| \|y\|
\]
by Schwarz. Hence the first map is uniformly continuous.

For the second statement use triangle inequality on \( x_1 - x_2 + x_2 \) to prove first \( \|x_1\| - \|x_2\| \leq \|x_1 - x_2\| \), and then interchange the roles of \( x_1 \) and \( x_2 \) to conclude that this is true even with absolute values on the left. \( \square \)

A subset \( M \) of \( H \) is called a subspace, if \( M \) is itself a vector space. It is called closed, if it is closed with respect to \( \|\| \). If \( \langle x, y \rangle = 0 \) we say that \( x \) and \( y \) are orthogonal, \( x \perp y \). We denote by \( M^\perp \) the set of all elements in \( H \) that are orthogonal to all elements in \( M \).

A set \( E \subseteq H \) is called convex, if for all \( x, y \in E \) and \( 0 < t < 1 \) we have \((1-t)x + ty \in E \). Evidently every subspace \( M \) and every translate \( x + M \) for fixed \( x \in H \) is convex.

**Theorem 6.** Every nonempty, closed, convex set \( E \subseteq H \) contains a unique element of smallest norm.

**Proof.** Let
\[
\delta = \inf \{\|x\| : x \in E\}.
\]
Parallelogram Law:
\[
\frac{1}{4} \|x - y\|^2 = \frac{1}{2} \|x\|^2 + \frac{1}{2} \|y\|^2 - \left\| \frac{x + y}{2} \right\|^2
\]
Note that \( (x + y)/2 \in E \). Hence,
\[
\|x - y\|^2 \leq 2 \|x\|^2 + 2 \|y\|^2 - 4\delta^2.
\]
(This already gives uniqueness; if $\delta = \|x\| = \|y\|$, then the left side is zero.) We still need existence: Let $y_n$ a sequence with $\|y_n\| \to \delta$. Apply the previous inequality to $y_n$ and $y_m$ to see that $y_n$ is a Cauchy sequence, and use completeness to show that a limit $x \in E$ exists. Then use continuity of the norm to show that $\|x\| = \delta$.

**Theorem 7.** Let $M$ be a closed subspace of $H$.

1. Every $x \in H$ has a unique decomposition $x = Px + Qx$ with $Px \in M$, $Qx \in M^\perp$.
2. $Px$ and $Qx$ are the nearest points to $x$ in $M$ and $M^\perp$, respectively.
3. The maps $P : H \to M$ and $Q : H \to M^\perp$ are linear
4. $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.

$P$ and $Q$ are called the orthogonal projections from $H$ onto $M$ and $M^\perp$.

**Proof.** Existence: $x + M$ is convex and closed, hence has a unique element of smallest norm. Call it $Qx$ and define $Px = x - Qx$. Since $Qx = x + M$, we get $Px \in M$. (We will show $Qx \in M^\perp$ below.)

Uniqueness: Two such decompositions with $Px + Qx$ and $P'x + Q'x$, say, lead to $Px - P'x \in M$, $Qx - Q'x \in M^\perp$ and

$$0 = Px - P'x + Qx - Q'x.$$

Since $M \cap M^\perp = \{0\}$, we get $Px = P'x$ and $Qx = Q'x$.

We have seen that $P : H \to M$. We prove now $\langle Qx, y \rangle = 0$ for all $x \in H$ and $y \in M$. Set $z = Qx$. Since $z \in x + M$ we have $z - \alpha y \in x + M$, all $\alpha \in \mathbb{C}$. Since $Qx$ is the element of smallest norm in $x + M$, we get

$$\|z\|^2 \leq \langle z - \alpha y, z - \alpha y \rangle$$

for all scalars $\alpha$. Multiply out the right side and simplify to get

$$0 \leq -\alpha \langle y, z \rangle - \overline{\alpha} \langle z, y \rangle + |\alpha|^2.$$

Set $\alpha = \langle z, y \rangle$. Then $0 \leq -|\langle z, y \rangle|^2$, i.e., the scalar product must be zero. Thus $Qx \in M^\perp$.

We show next that $Px$ is the closest element to $x$ in $M$. Let $y \in M$. Since $Qx \in M^\perp$, we have

$$\|x - y\|^2 = \|Qx\|^2 + \|Px - y\|^2.$$
This is minimized if \( y = Px \). (Similarly to show that \( Qx \) is the closest element to \( x \) from \( M^\perp \).) Apply the decomposition to \( x, y, \) and \( \alpha x + \beta y \) to get

\[
P(\alpha x + \beta y) - \alpha Px - \beta Py = \alpha Qx + \beta Qy - Q(\alpha x + \beta y)
\]

and note that these can only be equal if both sides equal zero.

For the last property plug the decomposition into the left side, multiply it out, and note that the cross terms vanish by orthogonality. 

**Theorem 8.** If \( L \) is a continuous linear map \( L : H \to \mathbb{C} \) (i.e., a continuous linear functional), then there exists unique \( y \in H \) with

\[
Lx = \langle x, y \rangle
\]

for all \( x \in H \).

**Proof.** May assume that there exists \( z \in H \) with \( Lz \neq 0 \) (if not, take \( y = 0 \)). Define \( M = \{ x : Lx = 0 \} \) and note that this is a proper subspace of \( H \). Since \( L \) is continuous, \( M \) is closed. There exists \( z \in M^\perp \) with \( \|z\| = 1 \). Define

\[
u = (Lx)z - (Lz)x.
\]

Check that \( Lu = 0 \), hence \( u \in M \). Thus, \( \langle u, z \rangle = 0 \). Plug the definition of \( u \) into this equation. We get

\[
Lx \langle z, z \rangle = (Lz) \langle x, z \rangle.
\]

The left side is \( Lx \), hence take \( y = (Lz)z \).