1 The Fourier transform

Let $f \in L^1(\mathbb{R})$. We define

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t} dx,$$

and we note that $\widehat{f}(t)$ is always in $\mathbb{C}$, since the integral converges absolutely. Moreover, $\widehat{f}$ is evidently an element of $L^\infty(\mathbb{R})$ with

$$\|\widehat{f}\|_\infty \leq \|f\|_1.$$

In functional analysis terms: $\mathcal{F}$ defined by $\mathcal{F}(f) = \widehat{f}$ is a bounded linear operator from $L^1(\mathbb{R})$ into $L^\infty(\mathbb{R})$ with norm ≤ 1.

Goals:

1. The Fourier transform maps L^1 into, but not onto L^∞. In fact, we will prove that
$$\lim_{t \to \infty} \widehat{f}(t) = 0$$
if $f \in L^1(\mathbb{R})$ (compare homework 1).

2. What is the Fourier transform of a function in $L^2(\mathbb{R})$? At this point it is not even clear how to define the Fourier transform of an L^2-function!

   We know that the Fourier transform can be defined on $L^1 \cap L^2(\mathbb{R})$, and we will use density statements to extend it from this set to all of L^2. Along the way we will see that the limit
$$\widehat{f}(t) = \lim_{N \to \infty} \int_{-N}^{N} f(x) e^{-2\pi i x t} dx$$
exists for all $f \in L^2(\mathbb{R})$, and we will prove that this limit can be taken as the definition of the Fourier transform of f.

3. The exponentials $x \mapsto e^{-2\pi i t x}$ for $t \in \mathbb{R}$ are not elements of L^p for $p < \infty$. Hence the Hilbert space approach via maximal orthonormal systems cannot be used. Nonetheless,
$$\langle f, g \rangle = \int_{\mathbb{R}} f(u) \overline{g(u)} du$$

defines a scalar product on $L^2(\mathbb{R})$, and this space is closed with respect to the induced norm. We shall prove that even in this situation the Fourier transform is an isometry on L^2, and that

$$\mathcal{F}(\widehat{f})(x) = f(-x).$$

2 Basic Properties

Since we will go from $L^1 \cap L^2$ to L^2, we start with properties of the Fourier transform on L^1. For $f, g \in L^1(\mathbb{R})$ and $\alpha, \lambda \in \mathbb{R}$ we note the following properties.

1. If $g(x)f(x)e^{2\pi i\alpha x}$, then $\widehat{g}(t) = \widehat{f}(t-\alpha)$. (Follows by direct substitution.)
2. If $g(x) = f(x-\alpha)$, then $\widehat{g}(t) = \widehat{f}(t)e^{-2\pi i\alpha t}$.
3. If $h = f * g$, then $h \in L^1(\mathbb{R})$ and $\widehat{h}(t) = \widehat{f}(t)\widehat{g}(t)$. (Fubini and substitution)
4. If $g(x) = \overline{f(-x)}$, then $\widehat{g}(t) = \overline{\widehat{f}(t)}$.
5. If $g(x) = f(\lambda x)$ for $\lambda > 0$, then $\widehat{g}(t) = \lambda^{-1}\widehat{f}(\lambda^{-1}t)$.
6. If $g(x) = -2\pi i x f(x) \in L^1(\mathbb{R})$, then $\widehat{f}$ is differentiable, and $\widehat{g}(t) = \widehat{f}'(t)$.

   For this part we note first that

   $$\frac{\widehat{f}(s) - \widehat{f}(t)}{s-t} = \int_{\mathbb{R}} f(x)e^{-2\pi ixt}\varphi(x, s-t)dx$$

   where

   $$\varphi(x,u) = \frac{e^{-2\pi ixu} - 1}{u}.$$

   After multiplication and division by $|x|$ we see that $|\varphi(x,u)| \leq C\max(1,|x|)$ for some positive $C > 0$. (In fact, it is $\leq |x|$.) By assumption $|xf(x)| \in L^1$, hence we may apply dominated convergence when letting $s \to t$ and move the limit inside the integral.

   $$\lim_{u\to 0}\varphi(x,u) = -2\pi ix.$$

   we obtain the stated identity.

3 Inversion

Consider pointwise inversion of Fourier series. We emphasize that we did not do this when talking about Fourier series since we were interested in inversion in $L^2([0,1])$. What conditions are necessary to obtain pointwise that

$$f(x) = \sum_n c_n e^{2\pi inx}$$

when

$$c_n = \int_{-1/2}^{1/2} f(x) e^{-2\pi inx} dx$$

is the nth Fourier coefficient? A pointwise proof requires using Fubini or uniform convergence in the double 'integral'

$$\int_{-1/2}^{1/2} \sum_n c_n e^{2\pi inx} e^{-2\pi inx} dx,$$

which is to say, we would need $f \in L^1([0,1])$ and $c_n \in \ell^1(\mathbb{Z})$.

The corresponding condition for $f \in L^1(\mathbb{R})$ would be $\widehat{f} \in L^1(\mathbb{R})$, but note that the analogue of the above approach will fail since

$$\int_{\mathbb{R}} e^{-2\pi i(x-t)u} du$$

is not a convergent integral. We need to proceed differently. What should happen is that the integral is 'equal' to the Dirac measure δ. This suggest to employ a tool that we used now twice already, namely replacing δ by convolution with an approximate identity, i.e., a family φ_t of nonnegative function with integral value 1 such that

$$\lim_{t\to\infty} f * \varphi_t = f$$

in L^1. We start with the Riemann-Lebesgue lemma for Fourier transforms. (This gives in particular that $\mathcal{F} : L^1 \to L^\infty$ is not onto.)

Theorem 1. *If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is continuous, $\|\widehat{f}\|_\infty \le \|f\|_1$ and*

$$\lim_{t\to\infty} \widehat{f}(t) = 0$$

Proof. We had shown the first inequality already. To prove continuity, let $t \in \mathbb{R}$ and consider a sequence $t_n \to t$. We have

$$|\widehat{f}(t) - \widehat{f}(t_n)| = \left| \int_{\mathbb{R}} f(x)(e^{-2\pi ixt} - e^{-2\pi ixt_n})dx \right|.$$

Check assumptions of dominated convergence and pull the limit $t_n \to t$ inside the integral.

We could prove the limit statement in the same way as the analogous statement for Fourier coefficients on the first assignment. We will give a different proof using uniform convergence of the translation operator. We note first that

$$\widehat{f}(t) = -\int_{\mathbb{R}} f(x)e^{-2\pi it(x+\frac{1}{2t})}dx = -\int_{\mathbb{R}} f\left(x - \frac{1}{2t}\right) e^{-2\pi itx}dx.$$

Hence

$$\widehat{f}(t) = \frac{1}{2}\int_{\mathbb{R}} \left(f(x) - f\left(x - \frac{1}{2t}\right)\right) e^{-2\pi ixt}dx$$

and it follows that

$$|\widehat{f}(t)| \leq \frac{1}{2}\|f - f_{1/(2t)}\|_1,$$

where $f_y(x) = f(x-y)$. The proof is completed by the following lemma. □

Lemma 1. *If $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$, then $y \mapsto f_y$ is a uniformly continuous mapping from $\mathbb{R}$ into $L^p(\mathbb{R})$.*

It is worthwhile to write out what this statement really means: For every $\varepsilon > 0$ there exists $\delta > 0$ so that for all $s, t \in \mathbb{R}$

$$|s - t| < \delta \Longrightarrow \|f_s - f_t\|_p < \varepsilon.$$

In particular, for $f \in L^1(\mathbb{R})$,

$$\lim_{t\to\infty} \|f - f_{1/(2t)}\|_1 = 0.$$

Proof. The crucial feature is that if $|s-t| < \delta$, then $|(x-s)-(x-t)| < \delta$ for all x. To get this fact into the game, we approximate arbitrary L^1 functions by continuous functions with bounded support. Let $\varepsilon > 0$. There exists $A > 0$ and continous g supported in $[-A, A]$ such that

$$\|f - g\|_1 < \varepsilon.$$

Uniform continuity of g implies that there exists $\delta \in (0, A)$ such that $|s - t| < \delta$ implies

$$|g(s) - g(t)| < (3A)^{-1/p}\varepsilon.$$

If $|s - t| < \delta$, then $|(x - s) - (x - t)| < \delta$ for all x, so that

$$\int_{\mathbb{R}} |g(x - s) - g(x - t)|^p dx < (3A)^{-1}\varepsilon^p(2A + \delta) < \varepsilon^p,$$

hence,

$$\|g_s - g_t\|_p < \varepsilon.$$

We note that $\|h_t\|_p = \|h\|_p$ for all $h \in L^p$. We get

$$\|f_s - f_t\|_p \leq \|(f - g)_t\|_p + \|g_s - g_t\| + \|(f - g)_s\|_p < 3\varepsilon.$$

This finishes the proof. □

We need a family of functions that are nonnegative, have all integral value 1, and converge to zero uniformly away from the origin. There are many choices. Note that

$$\int_{\mathbb{R}} e^{-2\pi ixt} d\delta(x) = 1$$

for all t, hence a good choice for such a family has also the property that its Fourier transforms converge to 1 from below. Recall that a function $H(\lambda t)$ has transform $\lambda^{-1}\widehat{h}(x/\lambda)$. So we start with a function H such that $H(\lambda t)$ goes to 1 from below as $\lambda \to \infty$. We will use the choice H defined by

$$H(t) = e^{-2\pi|t|}.$$

(Another popular choice is the Gaussian $e^{-\pi x^2}$.) We define

$$h_\lambda(x) = \int_{-\infty}^{\infty} H(\lambda t)e^{2\pi itx} dt$$

and note that

$$h_\lambda(x) = \frac{\lambda}{\pi(x^2 + \lambda^2)}.$$

Moreover,

$$\int_{-\infty}^{\infty} h_\lambda(x) dx = 1$$

and

$$\widehat{h}_\lambda(t) = H(\lambda t).$$

We will use the family $\{h_\lambda\}_{\lambda>0}$ as our approximate identity. (We will let $\lambda \to 0+$.) Since the transform of a convolution of L^1 functions is the product of the transforms, we obtain for $f \in L^1(\mathbb{R})$ that

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t)\widehat{f}(t)e^{2\pi ixt}dt.$$

We also have for every $g \in L^\infty(\mathbb{R})$ that is continuous at x_0 that

$$\lim_{\lambda\to 0+}(g * h_\lambda)(x_0) = g(x_0).$$

(The proof is the usual combination of integrals. The estimation step is simpler, since the assumptions imply that after a substitution the integrand is bounded by $2\|g\|_\infty h_1(s)$, which means that we may use dominated convergence to pass to the limit under the integral sign.)

For the next theorem we need an inequality of the following form:

$$\left|\int_{\mathbb{R}} g(u)d\mu(u)\right|^p \leq \int |g(u)|^p d\mu(u)$$

where μ is a Borel measure with $\mu(\mathbb{R}) = 1$. This is obtained from a useful inequality for convex functions, namely Jensen's inequality:

Lemma 2. *Let $\Omega \subseteq \mathbb{R}^n$ with $\mu(\Omega) = 1$. If $f \in L^1(\mu)$ is real valued, $\varphi : (a,b) \to \mathbb{R}$ is convex, and $a < f < b$ on Ω, then*

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega \varphi(f(x))d\mu(x).$$

Proof. Some intuition first: We want to show that

$$\int_\Omega \varphi(f(x))d\mu(x) - \varphi\left(\int_\Omega f d\mu\right) \geq 0.$$

Set $s = f(x)$ and $t = \int_\Omega f d\mu$. If it were true that

$$\varphi(s) - \varphi(t) \geq 0,$$

for all s and t, then plugging in the values of s and t, integrating over x, and using that $\mu(\Omega) = 1$ would give the claim.Evidently, this can never be true, unless φ is constant. However, we don't need this inequality for all s and t. We have with the above choices that

$$\int_\Omega (s-t)d\mu(x) = \int_\Omega f(x)d\mu(x) - t = 0,$$

i.e., any occurence of $s-t$ will not affect the inequalities that we need! This is promising since we do have the following: convexity of φ implies (and is in fact equivalent to) that for $a<z<t<u<b$ we have

$$\frac{\varphi(t)-\varphi(z)}{t-z} \leq \frac{\varphi(u)-\varphi(t)}{u-t}.$$

Note that for $s=f(x)$ we have to consider the possibilities $s<t$ and $t<s$. To do this, let

$$\beta = \sup_{a<z<t} \frac{\varphi(t)-\varphi(z)}{t-z},$$

and note the inequality above gives

$$\beta \leq \frac{\varphi(u)-\varphi(t)}{u-t}$$

for $t<u<b$. Hence

$$\varphi(s) \geq \varphi(t)+\beta(t-s)$$

for *any* $s \in (a,b)$. Apply this with $s=f(x)$. We get

$$\varphi(f(x))-\varphi(t)-\beta(f(x)-t) \geq 0.$$

Integrate both sides with respect to μ and note that the term with factor β becomes zero for our choice of t and s. □

Theorem 2. *If* $1 \leq p < \infty$ *and* $f \in L^p(\mathbb{R})$, *then*

$$\lim_{\lambda \to 0+} \|f * h_\lambda - f\|_p = 0.$$

Proof. Let $1 \leq p < \infty$, and note that $h_\lambda \in L^q(\mathbb{R})$ where $p^{-1}+q^{-1}=1$. Hence by Hölder's inequality, $f * h_\lambda$ is defined for every x. Since $\int h_\lambda = 1$, we get

$$f * h_\lambda(x) - f(x) = \int_{\mathbb{R}} (f(x-u)-f(x)) h_\lambda(u) du.$$

We obtain from Jensen's inequality with $\varphi(t)=|t|^p$ and $d\mu(x)=h_\lambda(x)dx$ that

$$|f * h_\lambda(x) - f(x)|^p \leq \int_{-\infty}^{\infty} |f(x-u)-f(x)|^p h_\lambda(u) du.$$

Integration in x and application of Fubini's theorem gives

$$\|f * h_\lambda - f\|_p^p \leq \int_{\mathbb{R}} \|f_u - f\|_p^p h_\lambda(u) du.$$

Define

$$g(y) = \|f_{-y} - f\|_p^p$$

and note that

$$g * h_\lambda(0) = \int_{\mathbb{R}} g(-u) h_\lambda(u) du = \int_{\mathbb{R}} \|f_u - f\|_p^p h_\lambda(u) du.$$

We had shown that g is continuous and that the convolution of a continuous function g with h_λ converges to g pointwise. Since $g(0) = 0$, we obtain that

$$\lim_{\lambda \to 0+} \|f * h_\lambda - f\|_p = 0.$$

□

We are now in a position to prove the inversion theorem.

Theorem 3. *If $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$, then*

$$\int_{-\infty}^{\infty} \widehat{f}(t) e^{2\pi i x t} dt = f(x)$$

almost everywhere. Moreover, f is continuous and satisfies $\lim_{|x| \to \infty} f(x) = 0$.

Proof. Most of the work has been done. We start with

$$f * h_\lambda(x) = \int_{-\infty}^{\infty} H(\lambda t) \widehat{f}(t) e^{2\pi i x t} dt.$$

We had seen before that the left side converges in $L^1(\mathbb{R})$ to f. This means that there exists a subsequence λ_n for which we have almost everywhere convergence to f. It follows that for almost every x

$$f(x) = \lim_{n \to \infty} f * h_{\lambda_n}(x) = \lim_{n \to \infty} \int_{-\infty}^{\infty} H(\lambda_n t) \widehat{f}(t) e^{2\pi i x t} dt,$$

and since $|H| \leq 1$ and $\widehat{f} \in L^1(\mathbb{R})$, dominated convergence shows that the right hand side converges to

$$\int_{-\infty}^{\infty} \widehat{f}(t) e^{2\pi i x t} dt,$$

which gives the claimed identity. The remaining statements follow from the assumption that $\widehat{f} \in L^1(\mathbb{R})$. □

4 Parseval's identity

Theorem 4. *Let $f \in L^2(\mathbb{R})$. There exists $\widehat{f} \in L^2(\mathbb{R})$ such that the following properties hold.*

1. *If $f \in L^1 \cap L^2(\mathbb{R})$, then $\widehat{f}$ is the previously defined Fourier transform of f.*
2. *For every $f \in L^2(\mathbb{R})$ the identity $\|f\|_2 = \|\widehat{f}\|_2$ holds.*
3. *The mapping $f \mapsto \widehat{f}$ is a Hilbert space isomorphism of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.*
4. *If*

$$\varphi_A(t) = \int_{-A}^{A} f(x)e^{-2\pi ixt}dx,$$
$$\psi_A(x) = \int_{-A}^{A} \widehat{f}(t)e^{2\pi ixt}dt,$$

then $\|\varphi_A - \widehat{f}\|_2 \to 0$ and $\|\psi_A - f\|_2 \to 0$ as $A \to \infty$.

Proof. Let $f \in L^1 \cap L^2(\mathbb{R})$. (So far we know that $\widehat{f}$ is continuous, bounded, and converges to zero as $|t| \to \infty$. We do not yet know the inversion formula, since the transform might not be integrable.)

Our first goal is $\|f\|_2 = \|\widehat{f}\|_2$. We recall that for $h_1, h_2 \in L^1(\mathbb{R})$ we have

$$(h_1 * h_2)^\wedge(t) = \widehat{h}_1(t)\widehat{h}_2(t).$$

Here's a question, whose answer contains the proof for the proposed identity: Which convolution has Fourier transform $|\widehat{f}|^2 = \widehat{f}\overline{\widehat{f}}$? Heuristically, we are going to solve the identity

$$h_1 * h_2(x) = \int_{-\infty}^{\infty} |\widehat{f}(t)|^2 e^{2\pi itx}dt, \tag{1}$$

but of course we do not yet know that $\widehat{f} \in L^2(\mathbb{R})$.

This immediately leads to the definition $g = h_1 * h_2$, where $h_1(x) = f(x)$ and $h_2(x) = \overline{f(-x)}$. From the properties of the Fourier transform we then obtain $\widehat{h}_2(t) = \overline{\widehat{f}(t)}$, and the value of the convolution is

$$g(x) = \int_{\mathbb{R}} f(x-u)\overline{f(-u)}du.$$

Let us collect properties of g. Fubini implies that $g \in L^1(\mathbb{R})$, hence this is well defined. Moreover, for fixed x we have $g(x) = \langle f, f_{-x} \rangle$, hence Cauchy-Schwarz and using the assumption $f \in L^2(\mathbb{R})$ implies that g is bounded. We recall that $x \mapsto f_{-x}$ is a continuous mapping from $\mathbb{R}$ to $L^2(\mathbb{R})$, and that the scalar product is continuous. Hence g is continuous. Evidently $g(0) = \|f\|_2^2$. The question to be solved can now be reformulated as follows: Is the identity

$$g(0) = \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt$$

true?

As before, consider $g * h_\lambda$. Continuity and boundedness of g imply that

$$\lim_{\lambda \to 0} g * h_\lambda(0) = g(0) = \|f\|_2^2.$$

Since $g \in L^1(\mathbb{R})$, we have

$$(g * h_\lambda)(t) = \int_{\mathbb{R}} \widehat{g}(t) H(\lambda t) e^{2\pi i t x} dt.$$

We emphasize that for this identity we only needed that $\widehat{g}$ is bounded; we did not have to require that $\widehat{g} \in L^1(\mathbb{R})$. We have $\widehat{g}(t) = |\widehat{f}(t)|^2 \geq 0$. Crucial fact: if $x = 0$, then in order to let $\lambda \to 0$ in the above identity we do not need to use dominated convergence, we may use monotone convergence instead. Hence

$$\lim_{\lambda \to 0+} g * h_\lambda(0) = \int_{\mathbb{R}} |\widehat{f}(t)|^2 dt,$$

regardless of whether or not the right hand side is finite or infinite! With the previous identity for $g(0)$ it follows now that

$$\|f\|_2 = \|\widehat{f}\|_2$$

and hence both sides must be finite, and in particular $\widehat{f} \in L^2(\mathbb{R})$.

This was the major part of the proof. For the remaining pieces, let

$$Y = \{\widehat{f} : f \in L^1 \cap L^2(\mathbb{R})\}$$

We have shown that $Y \subseteq L^2(\mathbb{R})$. We prove next that if $w \perp g$ for all $g \in Y$, then $w = 0$. Note that this implies that Y is dense in $L^2(\mathbb{R})$ by the decomposition for closed subspaces.

We note that $t \mapsto h_\lambda(t - \alpha) \in Y$ since this is the Fourier transform of $e^{2\pi i \alpha x} H(\lambda x)$, which is in $L^1 \cap L^2(\mathbb{R})$. Hence

$$\int_{\mathbb{R}} h_\lambda(t - \alpha) \overline{w(t)} dt = 0$$

for all $\alpha \in \mathbb{R}$. But the left side is $h_\lambda * \overline{w}(\alpha)$, and we had shown before that

$$\lim_{\lambda \to 0} \|h_\lambda * \overline{w} - \overline{w}\|_2 = 0.$$

Hence there exists a subsequence of λ's such that the difference converges pointwise almost everywhere, hence $w = 0$ a.e.

Almost done: The Fourier transform provides an isometry from the dense subspace $L^1 \cap L^2(\mathbb{R})$ onto the dense subspace Y of $L^2(\mathbb{R})$. By the extension theorem for isometries it follows that it extends to some isometry from $L^2(\mathbb{R})$ onto itself.

For the explicit representation it is enough to note that for $f \in L^2(\mathbb{R})$ we have $\chi_{[-A,A]} f \in L^1 \cap L^2(\mathbb{R})$ for all $A > 0$ (and by definition $\varphi_A = (\chi_{[-A,A]} f)^\wedge$), and then apply the previous limit relations. $\square$