INTERPOLATION FORMULAS WITH DERIVATIVES IN DE BRANGES SPACES II

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Abstract. We consider the problem of reconstruction of entire functions of exponential type $\tau$ that are elements of certain weighted $L^p(\mu)$-spaces from their values and the values of their derivatives up to order $\nu$. In this paper we extend the interpolation results of [24] in which the case $\nu = 1$ was solved. Using the theory of de Branges spaces we find a discrete set $\mathcal{T}_{\tau,\nu}$ of points on the real line and a frame $\mathcal{G}_{\tau,\nu}$ from an associated de Branges space that allow reconstruction of the function from information at the points in $\mathcal{T}_{\tau,\nu}$ via an interpolation series. If $p = 2$ we show that the series converges in $L^2(\mu)$-norm while for $p \neq 2$ we prove convergence on compact subsets of $\mathbb{C}$. Finally, we give an application to sampling/interpolation theory in Paley-Wiener spaces.

1. Introduction

One of the classical problems in complex analysis is to reconstruct an entire function from a countable set of data. For example, the Weierstrass factorization reconstructs a given entire function $F(z)$ using its set of zeros.

This article considers reconstruction of an entire function $F(z)$ from its values and the values of its derivatives up to a specified order at a discrete set of points on the real line. To accomplish this we use an interpolation formula. Some assumptions about the growth of $F(z)$ at infinity will be required in order to achieve complete characterizations.

Given two real numbers $p \in [1, \infty)$ and $\tau > 0$ the Paley-Wiener space $PW^p(\tau)$ is defined as the space of entire functions of exponential type at most $\tau$ such that their restriction to the real axis belong to $L^p$. The space $PW^p(\tau)$ is a Banach space, and it is a Hilbert space for $p = 2$. These are special spaces with a reproducing kernel structure. The reproducing kernel of $PW^p(\tau)$ is given by

$$K(w, z) = \frac{\sin \tau(z - w)}{\pi(z - w)}.$$
and

\[ F(w) = \int_{\mathbb{R}} F(x) K(w, x) dx \]

for every \( F \in PW^p(\tau) \). We aim to generalize classical interpolation formulas for \( PW^p(\tau) \) to the setting of so called de Branges spaces.

A basic result of the theory of Paley-Wiener spaces is that for all \( F \in PW^p(\tau) \) we have

\[
F(z) = \frac{\sin(\tau z)}{\tau} \sum_{n \in \mathbb{Z}} (-1)^n F(\pi n/\tau) \frac{1}{z - \pi n/\tau} \tag{1.1}
\]

where the sum converges uniformly in compact sets of \( \mathbb{C} \). (This is sometimes called the Shannon-Whittaker interpolation formula.) The reproducing kernel structure is intrinsically related with the above formula by the fact that when \( p = 2 \) the functions \( \{K(\pi n/\tau, z)\}_{n \in \mathbb{Z}} \) form an orthogonal basis of the space and formula (1.1) is a simple representation of the function in terms of this basis. The existence of interpolation formulas using derivatives is also known in Paley-Wiener spaces. In [33, Theorem 9] J. Vaaler proved that

\[
F(z) = \left( \frac{\sin \tau x}{\tau} \right)^2 \sum_{n \in \mathbb{Z}} \left( \frac{F(\pi n/\tau)}{(z - \pi n/\tau)^2} + \frac{F'(\pi n/\tau)}{z - \pi n/\tau} \right) \tag{1.2}
\]

for every \( F \in PW^p(2\tau) \). Vaaler’s proof of (1.2) is based on Fourier Analysis, and his method does not generalize to the function spaces that we consider below.

The Paley-Wiener spaces are particular cases of a wider class of spaces of entire functions \( H^p(E) \) called de Branges spaces (see [3] for the case \( p = 2 \) and [21] for an exposition of the case \( p \neq 2 \)). These spaces are uniquely determined by a structure function \( E(z) \) of Hermite-Biehler class (see Section 1.3). They contain the Paley-Wiener space \( PW^p(\tau) \) as a special case that can be recovered by using the Hermite-Biehler function \( E(z) = e^{-i\tau x} \). Generalizations of (1.1) and (1.2) are valid in de Branges spaces; an analogue of formula (1.1) is proved in [3, Theorem 22], while the analogue of formula (1.2) was proved recently by the first named author in [21, Theorem 1].

Formulas like (1.1) and (1.2) have been used to construct special functions called extremal functions. These are functions \( F \in PW^1(2\tau) \), or more generally that belong to \( H^1(E^2) \), that minimize the \( L^1(\mathbb{R}, |E(x)|^{-2} dx) \)-distance from a given function \( g(x) \) and such that \( F(x) \) lies below or above \( g(x) \). These functions possess special properties that are very useful in applications to analytical number theory, being the key to provide sharp (or improved) estimates to known objects. For instance in connection to: large sieve inequalities [25, 33], Erdős-Turán inequalities [14, 33], Hilbert-type inequalities [11, 13, 14, 23, 27, 33], Tauberian theorems [23] and bounds in the theory of the Riemann zeta-function and general
1.1. Problem formulation. This paper deals with reproducing kernel Hilbert spaces $H$ of entire functions $F(z)$ with reproducing kernel $K(w,z)$. We require that the space $H$ is closed under differentiation, that is, $F' \in H$ whenever $F \in H$. This last assumption will imply that the function $\partial^j w K(w,z)$ is the reproducing kernel for the differential operator $\partial^j z$, that is,

$$F^{(j)}(w) = \langle F(\cdot), \partial^j w K(w,\cdot) \rangle_H$$

for all $F \in H$.

We recall that a system $\{\varphi_n\}_{n \in \mathbb{Z}}$ in $H$ is called a weighted frame if there exists $C > 0$ and $\lambda_n > 0$ such that

$$C^{-1} \|F\|^2_H \leq \sum_{n \in \mathbb{Z}} \lambda_n |\langle F, \varphi_n \rangle_H|^2 \leq C \|F\|^2_H$$

(1.3)

for all $F \in H$, and the frame is called exact if (1.3) fails to hold for some $F \in H$ and all $C > 0$ if one of the terms in the series is removed.

In this terminology, for given $\nu \in \mathbb{N}$ we seek to find a discrete set of points $T_\nu \subseteq \mathbb{R}$ such that the collection $D_\nu$ of functions $D_{\nu,j}(z,t)$ defined by

$$D_{\nu,j}(z,t) = \partial^j w K(w,z) \bigg|_{w=t} \quad t \in T_\nu, \quad j = 0, \ldots, \nu - 1$$

(1.4)

forms an exact, weighted frame of $H$. In order to obtain an interpolation series we also seek to find a dual frame $G_\nu$ consisting of functions $z \mapsto G_{\nu,j}(z,t) \in H$ such that an inequality of the form (1.3) holds for $G_\nu$, and

$$F(z) = \sum_{t \in T} \sum_{j=0}^{\nu-1} F^{(j)}(t) G_{\nu,j}(z,t)$$

(1.5)

for all $F \in H$, where the convergence of the series to $F(z)$ is in the norm of $H$.

1.2. Overview of the article. In Section 2 we prove an interpolation formula with derivatives for functions of a reproducing kernel Hilbert space $H$ of entire functions (this will be a de Branges space), showing that the interpolating functions form a frame in the space. In Section 3 we aim to extend these formulas to Banach spaces of entire functions $H^p$ (these spaces will also be a de Branges space), but now proving the validity of formula (1.5) only in the point-wise sense. Finally, in Section 4 we give applications related to sampling/interpolation in Paley-Wiener spaces.
1.3. De Branges spaces. We briefly review the basics of de Branges’ theory of Hilbert spaces of entire functions [3]. A function \( F(z) \) analytic in the open upper half plane

\[ \mathbb{C}^+ = \{ z \in \mathbb{C} : \Im(z) > 0 \} \]

has bounded type if it can be written as the quotient of two functions that are analytic and bounded in \( \mathbb{C}^+ \). If \( F(z) \) has bounded type in \( \mathbb{C}^+ \) then, according to [3, Theorems 9 and 10], we have

\[ \limsup_{y \to \infty} y^{-1} \log |F(iy)| = v(F) < \infty. \]

The number \( v(F) \) is called the mean type of \( F(z) \). We say that an entire function \( F : \mathbb{C} \to \mathbb{C} \), not identically zero, has exponential type if

\[ \limsup_{|z| \to \infty} |z|^{-1} \log |F(z)| = \tau(F) < \infty. \]

In this case, the non-negative number \( \tau(F) \) is called the exponential type of \( F \). If \( F : \mathbb{C} \to \mathbb{C} \) is entire we define \( F^* : \mathbb{C} \to \mathbb{C} \) by \( F^*(z) = \overline{F(z)} \) and if \( F(z) = F^*(z) \) we say that it is real entire.

A Hermite-Biehler function \( E : \mathbb{C} \to \mathbb{C} \) is an entire function that satisfies the inequality

\[ |E^*(z)| < |E(z)| \]

for all \( z \in \mathbb{C}^+ \). We define the de Branges space \( \mathcal{H}(E) \) to be the space of entire functions \( F : \mathbb{C} \to \mathbb{C} \) such that

\[ \|F\|_2^2 := \int_{-\infty}^{\infty} |F(x)|^2 |E(x)|^{-2} \, dx < \infty, \]

and such that \( F/E \) and \( F^*/E \) have bounded type and nonpositive mean type in \( \mathbb{C}^+ \). This is a Hilbert space with respect to the inner product

\[ \langle F, G \rangle_E := \int_{-\infty}^{\infty} F(x) \overline{G(x)} |E(x)|^{-2} \, dx. \]

The Hilbert space \( \mathcal{H}(E) \) has the special property that, for each \( w \in \mathbb{C} \), the map \( F \mapsto F(w) \) is a continuous linear functional on \( \mathcal{H}(E) \). Therefore, there exists a function \( z \mapsto K(w, z) \) in \( \mathcal{H}(E) \) such that

\[ F(w) = \langle F, K(w, \cdot) \rangle_E. \quad (1.6) \]

The function \( K(w, z) \) is called the reproducing kernel of \( \mathcal{H}(E) \). If we write

\[ A(z) := \frac{1}{2} \{ E(z) + E^*(z) \} \quad \text{and} \quad B(z) := \frac{i}{2} \{ E(z) - E^*(z) \}, \]
then \(A(z)\) and \(B(z)\) are real entire functions with only real zeros and \(E(z) = A(z) - iB(z)\). The reproducing kernel is then given by \[3\, \text{Theorem 19}\]

\[\pi(z - \bar{w})K(w, z) = B(z)A(\bar{w}) - A(z)B(\bar{w}),\]

or alternatively by

\[2\pi i(\bar{w} - z)K(w, z) = E(z)E^*(\bar{w}) - E^*(z)E(\bar{w}). \tag{1.7}\]

When \(z = \bar{w}\) we have

\[\pi K(z, z) = B'(z)A(z) - A'(z)B(z). \tag{1.8}\]

We may apply the Cauchy-Schwartz inequality in (1.6) to obtain that

\[|F(w)|^2 \leq \|F\|_2^2 K(w, w), \tag{1.9}\]

for every \(F \in \mathcal{H}(E)\). Also, it is not hard to show that \(K(w, w) = 0\) if and only if \(w \in \mathbb{R}\) and \(E(w) = 0\) (see \[23\, \text{Lemma 11}\]).

We denote by \(\varphi(z)\) a phase function associated to \(E(z)\). This is an analytic function in a neighborhood of \(\mathbb{R}\) defined by the condition \(e^{i\varphi(x)}E(x) \in \mathbb{R}\) for all real \(x\). Using (1.7) we obtain that

\[\varphi'(x) = \frac{\pi K(x, x)}{|E(x)|^2} > 0 \tag{1.10}\]

for all real \(x\) and thus \(\varphi(x)\) is an increasing function of real \(x\). We also have that

\[e^{2i\varphi(x)} = \frac{A(x)^2}{|E(x)|^2} - \frac{B(x)^2}{|E(x)|^2} + 2i\frac{A(x)B(x)}{|E(x)|^2}\]

for all real \(x\). As a consequence, the points \(t \in \mathbb{R}\) such that \(\varphi(t) \equiv 0 \mod \pi\) coincide with the real zeros of \(B(z)/E(z)\) and the points \(s \in \mathbb{R}\) such that \(\varphi(s) \equiv \pi/2 \mod \pi\) coincide with the real zeros of \(A(z)/E(z)\) and by (1.10), these zeros are simple. In other words, the function \(B(z)/A(z)\) has only simple real zeros and simple real poles that interlace.

Denote by \(T_B\) the set of real zeros of the function \(B(z)\). This set plays a special role in the theory of de Branges associated with a function \(E(z)\) with no real zeros. In this case, by \[1.8\] and the reproducing kernel property we easily see that the functions \(\{B(z)/(z - t)\}_{t \in T_B}\) form an orthogonal set in \(\mathcal{H}(E)\) and, by \[23\, \text{Theorem 22}\] forms a basis if and only if \(B \notin \mathcal{H}(E)\). In that case the identities

\[\|F\|^2_{\mathcal{H}(E)} = \sum_{t \in T_B} \frac{|F(t)|^2}{K(t, t)} \tag{1.11}\]

and

\[F(z) = B(z) \sum_{t \in T_B} \frac{F(t)}{B'(t)(z - t)}\]
hold for all $F \in \mathcal{H}(E)$. Evidently this is the case $\nu = 1$ in (1.3), and we show that $T_B$ in fact may be used for any $\nu \in \mathbb{N}$.

Finally, we say that $\mathcal{H}(E)$ is closed under differentiation if $F' \in \mathcal{H}(E)$ whenever $F \in \mathcal{H}(E)$. Inequality (1.9) together with the fact that $w \in \mathbb{C} \mapsto K(w, w)$ is a continuous function implies that convergence in the norm of $\mathcal{H}(E)$ implies uniform convergence on compact sets of $\mathbb{C}$. As a consequence, differentiation defines a closed linear operator on $\mathcal{H}(E)$ and therefore by the Closed Graph Theorem defines a bounded linear operator on $\mathcal{H}(E)$.

1.4. Main Result. Let $\nu \in \mathbb{N}$ and $E(z)$ be a Hermite-Biehler function with no real zeros (hence the zeros of $B(z)$ are simple). In order to obtain the desired interpolation series we need to work in $\mathcal{H}(E^\nu)$. Denote by $A_\nu(z)$ and $B_\nu(z)$ the real entire functions that satisfy $E(z)^\nu = A_\nu(z) - iB_\nu(z)$ and by $K_\nu(w, z)$ the reproducing kernel associated with $\mathcal{H}(E^\nu)$.

We define the collection $B_\nu$ of functions $z \mapsto B_{\nu,j}(z, t)$ given by

$$B_{\nu,j}(z, t) = B(z)^\nu \frac{(z - t)^j}{(z - t)^j} j!$$

where $t \in T_B$ and $1 \leq j \leq \nu$. For $\ell \in \mathbb{N}$ we denote by $P_{\nu,\ell}(z, t)$ the Taylor polynomial of degree $\ell$ of $B_{\nu,\nu}(z, t)^{-1}$ expanded into a power series at $z = t$ as a function of $z$. Finally, we denote by $G_\nu$ the collection of functions $z \mapsto G_{\nu,j}(z, t)$ defined by

$$G_{\nu,j}(z, t) = B_{\nu,\nu-j}(z, t) \frac{P_{\nu,\nu-1}(z, t)}{j!}.$$  \hspace{1cm} (1.13)

We note that

$$G_{\nu,j}(z, t) = \frac{(z - t)^j}{j!} - B(z)^\nu \sum_{n \geq \nu-j} b_{\nu,n}(t)(z - t)^{n+j-\nu}$$

where the quantity $b_{\nu,n}(t)$ is the coefficient of $(z - t)^n$ in the Taylor expansion of $1/B_{\nu,\nu}(z, t)$ about the point $z = t$. We easily see that these functions satisfy the following property

$$G_{\nu,j}^{(\ell)}(s, t) = \delta_0(s-t) \delta_{0}(\ell-j)$$  \hspace{1cm} (1.14)

for $\ell, j = 0, \ldots, \nu - 1$ and $s, t \in T_B$.

The next result essentially says that $D_\nu$ defined in (1.4) is an exact, weighted frame for $\mathcal{H}(E^\nu)$ with dual frame $G_\nu$. As part of the proof we will also show that $B_\nu$ is a frame (not weighted) for $\mathcal{H}(E^\nu)$. We emphasize that $K(w, z)$ is the reproducing kernel of $\mathcal{H}(E)$ while $K_\nu(w, z)$ is the reproducing kernel of $\mathcal{H}(E^\nu)$.

**Theorem 1.** Let $E(z)$ be a Hermite-Biehler function with phase function $\varphi(z)$. Let $\nu \geq 2$ be an integer with the property that the space $\mathcal{H}(E^\nu)$ is closed under
differentiation. Assume also that neither $A(z)$ nor $B(z)$ belong to $\mathcal{H}(E)$ and that there exists $\delta > 0$ such that $\varphi'(t) \geq \delta$ for all $t \in \mathcal{T}_B$. Then the following statements hold.

(1) For every $F \in \mathcal{H}(E^\nu)$

$$F(z) = \sum_{t \in \mathcal{T}_B} \sum_{j=0}^{\nu-1} F^{(j)}(t) G_{\nu,j}(z,t)$$

(1.15)

where the series converges to $F(z)$ in the norm of $\mathcal{H}(E^\nu)$.

(2) There exist a positive constant $C > 0$ such that

$$C^{-1} \|F\|_{\mathcal{H}(E^\nu)} \leq \sum_{t \in \mathcal{T}_B} \sum_{j=0}^{\nu-1} \frac{|F^{(j)}(t)|^2}{K_{\nu,t}(t,t)} \leq C \|F\|_{\mathcal{H}(E^\nu)}$$

(1.16)

and

$$C^{-1} \|F\|_{\mathcal{H}(E^\nu)} \leq \sum_{t \in \mathcal{T}_B} \sum_{j=0}^{\nu-1} K_{\nu,t}(t,t)|\langle F, G_{\nu,j}(.,t) \rangle_{\mathcal{H}(E^\nu)}|^2 \leq C \|F\|_{\mathcal{H}(E^\nu)}$$

(1.17)

for all $F \in \mathcal{H}(E^\nu)$.

(3) If any of the terms of the series in (1.16) are removed, the modified formula fails to hold for some $F \in \mathcal{H}(E^\nu)$.

Remarks.

(i) A simple application of [3, Theorem 22] shows that the statement $A, B \notin \mathcal{H}(E)$ is equivalent to $AB \notin \mathcal{H}(E^2)$. This is required since (1.16) is established by an induction on $\nu$ for which the induction start $\nu = 2$ does not hold without this condition (see [21]).

(ii) For $\nu = 1$ the two frames agree, and (1.16) holds with $C = 1$ without any assumption on the phase and the differentiation operator (see [3, Theorem 22]).

(iii) Conditions for the boundedness of the differentiation operator were given by A. Baranov in [1, 2]. It was also shown there that $E(z)$ has exponential type and no real zeros if $\mathcal{H}(E)$ is closed under differentiation.

(iv) The fact that $\mathcal{T}_B \subset \mathcal{T}_{B^\nu}$ for every $\nu \geq 1$ is a crucial ingredient in the proof of the proposed theorem.

1.5. Notation used. Given two positive quantities $Q$ and $Q'$ and $N$ elements $r_1, ..., r_N$ of a set $\Omega$ we write $Q \ll_{r_1, ..., r_N} Q'$ when $Q \leq C(r_1, ..., r_N)Q'$ where $C : \Omega \to (0, \infty)$ is some function. We also write $Q \approx_{r_1, ..., r_N} Q'$ when both $Q \ll_{r_1, ..., r_N} Q'$ and $Q' \ll_{r_1, ..., r_N} Q$ hold. Often, the quantities $Q$ and $Q'$ will depend on a function $F$ and other quantities. We write $Q(F) \ll Q'(F)$ when there exists a constant $C > 0$, which does not depend on $F$, such that $Q(F) \leq CQ'(F)$. 
2. The $L^2$ case

2.1. Preliminaries. The recipe for the proof of Theorem 1 is the following.

1. We show that the span of the collection $\mathcal{G}_\nu$ is dense in $\mathcal{H}(E^\nu)$, which in turn by (1.14) implies that there exists a dense set of functions in $\mathcal{H}(E^\nu)$ for which (1.15) holds.

2. We derive estimates involving the inner products of the collection $\mathcal{G}_\nu$ in order to prove Theorem 1 item (2) for a dense set of functions (and hence for the whole space).

This section contains some auxiliary statements. The proof of Theorem 1 is presented in Subsection 2.2. First we prove density statements for the classes $\mathcal{B}_\nu$ and $\mathcal{G}_\nu$.

**Lemma 2.** Let $E(z)$ be a Hermite-Biehler function with no real zeros and assume that $\mathcal{B} \notin \mathcal{H}(E)$. Then the span of $\mathcal{B}_\nu$ and the span of $\mathcal{G}_\nu$ defined in (1.12) and (1.13) are both dense in $\mathcal{H}(E^\nu)$ for every integer $\nu \geq 1$.

*Proof.* First we show via induction on $\nu$ that the span of the collection $\mathcal{B}_\nu$ is dense in $\mathcal{H}(E^\nu)$. It follows from [3, Theorem 22] that the span of $\mathcal{B}_1$ is dense in $\mathcal{H}(E)$. Let $\nu \in \mathbb{N}$ and assume that the span of $\mathcal{B}_\nu$ is dense in $\mathcal{H}(E^\nu)$. It follows from [1, Lemma 4.1] that if $E_a(z)$ and $E_b(z)$ are two Hermite-Biehler functions then

$$
\mathcal{H}(E_aE_b) = E_a^*\mathcal{H}(E_b) \oplus E_b\mathcal{H}(E_a)
$$

where the sum is orthogonal. This implies that

$$
\mathcal{H}(E^{\nu+1}) = A\mathcal{H}(E^\nu) \oplus B\mathcal{H}(E^\nu).
$$

Therefore, the span of the collection $\mathcal{C} = AB_\nu \cup BB_\nu$ is dense in $\mathcal{H}(E^{\nu+1})$. Evidently $BB_\nu$ is a subset of $\mathcal{B}_{\nu+1}$, and it remains to show that the collection $AB_\nu$ can be arbitrarily approximated in the norm of $\mathcal{H}(E^{\nu+1})$ by elements of the span of $\mathcal{B}_{\nu+1}$.

Now note that for $t \in \mathcal{T}_B$

$$
A(z)B_{\nu,v}(z,t) = \frac{A(t)}{B'(t)}[B_{\nu+1,v+1}(z,t) - B(z)G(z,t)]
$$

where

$$
G(z,t) = B_{\nu-1,v-1}(z,t) \frac{A(t)B(z) - B'(t)(z-t)A(z)}{A(t)(z-t)^2}.
$$

Evidently $G(z,t) \in \mathcal{H}(E^\nu)$ for $t \in \mathcal{T}_B$. Hence, for any given $\varepsilon > 0$ there exists $H(z)$ belonging to the span of $\mathcal{B}_\nu$ such that

$$
\|G(z,t) - H(z)\|_{E^\nu} < \varepsilon B'(t)/A(t).
$$
It follows that $[B_{\nu+1}\nu+1(z, t) - B(z)H(z)]$ belongs to the span of $B_{\nu+1}$ and
\[
\left\| A(z)B_{\nu,\nu}(z, t) - \frac{A(t)}{B'(t)}[B_{\nu+1}\nu+1(z, t) - B(z)H(z)] \right\|_{E^\nu+1} = \frac{A(t)}{B'(t)} \left\| B(z)[G(z, t) - H(z)] \right\|_{E^\nu+1} \leq \frac{A(t)}{B'(t)} \left\| G(z, t) - H(z) \right\|_{E^\nu} < \varepsilon.
\]

We conclude that $A(z)B_{\nu,\nu}(z)$ is an element of the closure of the span of $B_{\nu+1}$. If $1 \leq j < \nu$ then evidently $A(z)B_{\nu,\nu}(z, t) = B(z)H(z)$ for some $H \in \mathcal{H}(E^\nu)$ and the same argument above could be replicated. This proves the first part of the lemma.

Now, denote by $a_{\nu,j}(t)$ the coefficient of $(z-t)^j$ in the Taylor series representation of $1/B_{\nu,\nu}(z, t)$ as a function of $z$ about $z = t$. Then the function $G_{\nu,k}(z, t)$ may be represented as
\[
G_{\nu,k}(z, t) = \frac{1}{k!} \sum_{j=0}^{\nu-k-1} a_{\nu,j}(t)B_{\nu,\nu-j-k}(z, t)
= \frac{1}{k!} \sum_{m=1}^{\nu-k} a_{\nu,\nu-m-k}(t)B_{\nu,m}(z, t).
\]

Suppressing the arguments $t$ and $z$, this is in matrix notation
\[
(G_{\nu,k})_{0 \leq k \leq \nu-1} = \begin{pmatrix}
\frac{a_{\nu,\nu-1}}{0!} & \cdots & \frac{a_{\nu,1}}{0!} & \frac{a_{\nu,0}}{0!} \\
0 & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & \frac{a_{\nu,0}}{\nu-1!} & 0 \\
0 & \cdots & 0 & 0
\end{pmatrix} (B_{\nu,m})_{1 \leq m \leq \nu}.
\]

Since $a_{\nu,0}(t) = 1/B'(t)^\nu \neq 0$, it follows that the above matrix is invertible and, in particular, any element of $B_{\nu}$ is a linear combination of elements from $G_{\nu}$ and vice versa. This concludes the lemma.

For the proof of item (2) of Theorem 1 we will need to estimate the norm of a linear combination of elements from $B_{\nu}$. The following four lemmas collect the necessary upper bounds for each term that will appear.

**Lemma 3.** Let $\nu \geq 2$ be an integer and let $E(z)$ be a Hermite-Biehler function such that $\mathcal{H}(E^\nu)$ is closed under differentiation and denote by $D$ the norm of the differentiation operator on $\mathcal{H}(E^\nu)$. Then

1. The derivative of the phase function is bounded. In fact we have
   \[
   \varphi'(x) \leq D\sqrt{\nu}
   \] (2.1)
   for all real $x$. 

(2) If \( t > s \) are two consecutive elements of \( T_B \), then
\[
t - s \geq \frac{\sqrt{2/\nu}}{D}.
\]

(3) For all real \( x \) and \( t \in T_B \),
\[
\left| \frac{B(x)}{E(x)(x-t)} \right| \leq D \sqrt{\nu}.
\]

Proof. Item (1). Using the reproducing kernel definition (1.7) we deduce that
\[
K_{\nu}(x,x) = \nu \vert E(x) \vert^{2(\nu-1)} K(x,x)
\]
for all real \( x \), where \( K_{\nu}(w,z) \) and \( K(w,z) \) are respectively the reproducing kernels associated with \( \mathcal{H}(E^\nu) \) and \( \mathcal{H}(E) \). This implies that
\[
K_{\nu}(t,t) = \frac{\nu}{\pi} A(t)^{2\nu-1} B'(t)
\]
for all \( t \in T_B \). We prove first (2) for \( x = t \in T_B \). Consider the entire function \( F(z) \) defined by
\[
F(z) = \pi E(z)^{\nu-2} B(z) K(t, z).
\]
Evidently, \( F \in \mathcal{H}(E^\nu) \) and by (1.9) we obtain
\[
\vert F'(t) \vert^2 \leq \Vert F' \Vert_{E^\nu}^2 K(t,t) \leq D^2 \Vert F \Vert_{E^\nu}^2 K_{\nu}(t,t).
\]
Now observe that
\[
\Vert F \Vert_{E^\nu}^2 \leq \pi^2 \Vert K(t, z) \Vert_{E}^2 = \pi B'(t) A(t)
\]
and that \( F'(t) = A(t)^{\nu-1} B'(t)^2 \). Using identity (1.10) we easily obtain (2) for \( x = t \in T_B \).

Now, let \( \alpha \in \mathbb{R} \) be arbitrary and denote by \( \varphi_\alpha(z) \) the phase of \( E_\alpha(z) = e^{i\alpha} E(z) = A_\alpha(z) - iB_\alpha(z) \). Observe that \( \varphi_\alpha(x) = \varphi(x) + \alpha \) for all real \( x \) and \( E^\nu(z) \) and \( E^\nu_\alpha(z) \) generate the same space. Thus \( E_\alpha(z) \) does not have real zeros and the real zeros of \( B_\alpha(z) \) coincide with the points \( \varphi(x) \equiv \alpha \mod \pi \). Hence, the above argument for the space \( \mathcal{H}(E^\nu_{\alpha}) \) gives the claim for arbitrary \( x \in \mathbb{R} \).

Item (2). Let \( u \) be the real zero of \( A(z) \) between \( s \) and \( t \) (recall that the zeros of \( A(z) \) and \( B(z) \) interlace, see the discussion in Section 1.3). The function \( F(z) = A(z)K(u, z) \) is an element of \( \mathcal{H}(E^2) \) and by (1.9) we obtain
\[
\left| A(t)^2 B(u) \right|^{2} \pi(t-u) \leq \Vert F(z) \Vert_{E^{2\nu}}^2 K_2(t,t) \leq \Vert K(u, z) \Vert_{E^{2\nu}}^2 K_2(t,t)
\]
\[
= K(u, u) K_2(t,t) = \left| \frac{2A(t)^3 B'(t)A'(u)B(u)}{\pi^2} \right|
\]
Using identity (1.10), the last inequality is equivalent to
\[
(t - u)^{-2} \leq 2\varphi'(u) \varphi'(t).
\]
By item (1) of Lemma 3, we conclude that
\[ t - u \geq \frac{1}{D\sqrt{2\nu}}. \]
The same argument shows that \( u - s \geq \frac{1}{D\sqrt{2\nu}} \), hence \( t - s \geq \frac{\sqrt{2}}{\nu}/D \).

**Item (3).** By inequality (1.9) we obtain \( |K(w, z)|^2 \leq K(w, w)K(z, z) \) for all \( w, z \in \mathbb{C} \). We obtain
\[ \left| \frac{B(x)}{E(x)(x-t)} \right|^2 = \frac{\pi^2 K(t, x)^2}{A(t)^2|E(x)|^2} \leq \varphi(t)\varphi'(x) \leq D^2\nu. \]
This concludes the lemma.

**Lemma 4.** Let \( E(z) \) be a Hermite-Biehler function, and let \( \nu \geq 1 \) be an integer. Then for all \( s, t \in T_B \) with \( s \neq t \) we have
\[ \langle B_{\nu,1}(\cdot, s), B_{\nu,1}(\cdot, t) \rangle_{E^\nu} = 0. \]

**Proof.** Define an entire function \( I(z) \) by
\[ I(z) = \frac{B(z)^{2\nu}}{(E(z)(E^*(z))^\nu(z-s)(z-t)} \]
where \( s \) and \( t \) are two zeros of \( B(z) \). We aim to show that
\[ \int_{-\infty}^{\infty} I(x)dx = 0. \]

Define a contour \( \Gamma_K \) in \( \mathbb{C} \) by replacing in \([-K, K]\) the segments \([t - \delta, t + \delta]\) and \([s - \delta, s + \delta]\) with semicircles in the lower half-plane of radius \( \delta \) and centers \( s \) and \( t \), respectively, traced counterclockwise. (Here \( \delta \) is chosen so small that the disks of radius \( \delta \) about \( s \) and \( t \) contain no zero of \( E(z) \) or \( E^*(z) \).) Since \( I(z) \) is entire, the integrals of \( I(z) \) over \([-K, K]\) and over \( \Gamma_K \) are equal by the residue theorem.

We note that
\[ \frac{B(z)^{2\nu}}{(E(z)(E^*(z))^\nu(z-s)(z-t)} = \frac{i\nu}{2\nu} \sum_{j=0}^{2\nu} \binom{2\nu}{j} \frac{E(z)^j}{(E(z)(E^*(z))^\nu} \]
\[ = \frac{i\nu}{2\nu} \sum_{j=0}^{2\nu} \binom{2\nu}{j} \left( \frac{E(z)}{E^*(z)} \right)^{j-\nu}. \]

Define \( I_j(z) \) for \( j \in \{0, ..., 2\nu\} \) by
\[ I_j(z) = \frac{1}{(z-s)(z-t)} \left( \frac{E(z)}{E^*(z)} \right)^{j-\nu}. \]
Each \( I_j(z) \) is a meromorphic function in \( \mathbb{C} \) with poles at \( z = s \) and \( z = t \), and with additional poles in the lower or upper half-plane depending on whether \( j < \nu \) or \( j > \nu \). For \( j > \nu \) the function \((z-t)(z-s)I_j(z) \) is bounded and analytic in the lower half-plane. We close the contour \( \Gamma_K \) by a semicircle with center at the origin and radius \( K \) in the lower half-plane, traced counterclockwise. Since none of the
Lemma 5. Assume all hypotheses of Theorem 1. Then the following statements hold.

1. For all \( s, t \in \mathcal{T}_B \) with \( s \neq t \) and \( j = 2, \ldots, \nu \)
   \[ |\langle B_{\nu,j}(\cdot, s), B_{\nu,j}(\cdot, t)\rangle_{E^\nu}| \ll_{\mathcal{D},\nu} \frac{1}{(s-t)^2}. \]

2. For all \( t \in \mathcal{T}_B \) we have
   \[ \|B_{\nu,j}(\cdot, t)\|_{E^\nu} \ll_{\mathcal{D},\nu} 1. \]
   for all \( j = 1, \ldots, \nu \) and all \( t \in \mathcal{T}_B \).

3. Denote by \( a_{\nu,j}(t) \) the coefficient of \( (z-t)^j \) in the Taylor series expansion of \( B_{\nu,v}(z, t) \) about the point \( z = t \). Then
   \[ |a_{\nu,j}(t)|^2 \ll_{\mathcal{D},\nu,\delta} \frac{1}{K_{\nu}(t, t)}. \]
   for all \( t \in \mathcal{T}_B \) and \( j = 0, \ldots, \nu - 1 \).

Proof. Item (1). For \( j = \nu = 2 \) this is present in the Step 1 of the proof of [21 Theorem 1] together with [21 Lemma 5]. Assume that for some integer \( \nu \geq 2 \) the statement holds for all \( 2 \leq j \leq \nu \). For \( j = 2 \) we use that \( |B(x)| \leq |E(x)| \) for all real \( x \) to get
   \[ \langle B_{\nu+1,2}(\cdot, s), B_{\nu+1,2}(\cdot, t)\rangle_{E^{\nu+1}} \leq \langle B_{\nu,2}(\cdot, s), B_{\nu,2}(\cdot, t)\rangle_{E^{\nu}}. \]
   For \( 2 < j \leq \nu + 1 \) with \( j \) odd we can apply item (3) of Lemma 3 to obtain
   \[ |\langle B_{\nu+1,j}(\cdot, s), B_{\nu+1,j}(\cdot, t)\rangle_{E^{\nu+1}}| \ll_{\mathcal{D},\nu} \langle B_{\nu,j-1}(\cdot, s), B_{\nu,j-1}(\cdot, t)\rangle_{E^{\nu}}. \]
   For \( 2 < j \leq \nu + 1 \) with \( j \) even then \( \nu \geq 3 \) and in the same way we obtain
   \[ |\langle B_{\nu+1,j}(\cdot, s), B_{\nu+1,j}(\cdot, t)\rangle_{E^{\nu+1}}| \ll_{\mathcal{D},\nu} \langle B_{\nu,j-2}(\cdot, s), B_{\nu,j-2}(\cdot, t)\rangle_{E^{\nu-1}}. \]
The item follows by induction.

Item (2). We note that for \( j = 1, \ldots, \nu \)

\[
\frac{B(x)'}{(x-t)E(x)'} \leq \frac{B(x)}{(x-t)E(x)} \ll D, \nu
\]

for all real \( x \), by item (2) of Lemma 3. Since \( B(z)/(z-t) = \pi K(t, z)/A(t) \), it follows that

\[
\|B_{\nu,j}(z,t)\|_{E^\nu}^2 \ll D, \nu \|\pi K(t, z)/A(t)\|_{E^\nu}^2 = \frac{\pi B'(t)/A(t) = \varphi'(t)}{D\sqrt{\nu}}.
\]

Item (3). Denote by \( b_{\nu,j}(t) \) the coefficients of the power series expansion of \( B_{\nu,\nu}(z) \) as a function of \( z = t \). We have \( b_{\nu,0}(t) = B'(t)\nu \) and for \( j = 1, \ldots, \nu \)

\[
|b_{\nu,j}(t)|^2 = \frac{1}{(j!)^2} |B^{(j)}(t)|^2 \leq \frac{1}{(j!)^2} \|B^{(j)}(t)\|_{E^\nu}^2 K_{\nu}(t, t) \ll D, \nu K_{\nu}(t, t). \tag{2.2}
\]

We obtain

\[
\frac{|b_{\nu,j}(t)|^2}{|b_{\nu,0}(t)|^2} \ll D, \nu \frac{K_{\nu}(t, t)}{|B'(t)|^{2\nu}} \leq \frac{\pi A(t)^{2\nu-1} B'(t)}{\nu B'(t)^{2\nu}} = \frac{\pi}{\nu} \varphi'(t)^{1-2\nu} \ll_{\delta, \nu} 1 \tag{2.3}
\]

since \( A(t)/B'(t) \) equals \( 1/\varphi'(t) \) which we assumed to be bounded by \( \delta \). Now note that for \( \ell = 1, \ldots, \nu - 1 \)

\[
0 = \partial^\ell_z \left[ \frac{B_{\nu,\nu}(z,t)}{B_{\nu,\nu}(z,t)} \right]_{z=t} = \ell! \sum_{j=0}^{\ell} a_{\nu,j}(t) b_{\nu,\ell-j}(t).
\]

Hence the relation between \( a_{\nu,j}(t) \) and \( b_{\nu,j}(t) \) is given by a triangular matrix with diagonal terms equal to \( b_{\nu,0}(t) \). Using (2.2) and (2.3) we conclude that

\[
|a_{\nu,j}(t)|^2 \ll_{D, \nu, \delta} 1/K_{\nu}(t, t).
\]

This concludes the lemma. \( \square \)

The next lemma estimates the norm of the linear combination of elements from \( G_{\nu} \). This is one of the two inequalities needed to show that this collection is a (weighted) frame.

**Lemma 6.** Assume all hypothesis of Theorem 1. Let \( c_j(t) \in \mathbb{C} \) for \( t \in T_B \) and \( j \in \{0, \ldots, \nu - 1\} \) be such that

\[
\sum_{t \in T_B} \sum_{j=0}^{\nu-1} |c_j(t)|^2 < \infty.
\]

Then

\[
\left\| \sum_{t \in T_B} \sum_{j=0}^{\nu-1} c_j(t) G_{\nu,j}(\cdot, t) \right\|_{H(E^\nu)} \ll_{D, \nu, \delta} \sum_{t \in T_B} \sum_{j=0}^{\nu-1} |c_j(t)|^2 K_{\nu}(t, t)
\]
Proof. Define
\[ d_m(t) = \sum_{j=0}^{\nu-m} a_{\nu,\nu-m-j}(t) \frac{c_j(t)}{j!}. \]

By item (3) of Lemma \[5\] we trivially obtain
\[ \sum_{t \in T_B} \nu \sum_{m=1}^{\nu} |d_m(t)|^2 \ll_{D,\nu,\delta} \sum_{t \in T_B} \nu \sum_{m=1}^{\nu-1} \frac{|c_j(t)|^2}{K_{\nu}(t,t)} < \infty. \tag{2.4} \]

For a given \( T > 0 \) we obtain
\[
\left\| \sum_{t \in T_B} \sum_{j=0}^{\nu-1} c_j(t)g_{\nu,j}(z,t) \right\|_{E^{\nu}}^2 = \left\| \sum_{t \in T_B} \sum_{j=0}^{\nu-1} c_j(t) \sum_{m=1}^{\nu-j} a_{\nu,\nu-m-j}(t)B_{\nu,m}(z,t) \right\|_{E^{\nu}}^2 \\
\ll_{D,\nu,\delta} \sum_{m=1}^{\nu} \sum_{t \in T_B} |d_m(t)|^2 + \sum_{m=2}^{\nu} \sum_{s,t \in T_B, s \neq t} \frac{|d_m(t)||d_m(s)|}{(t-s)^2} \\
\ll_{D,\nu,\delta} \sum_{m=1}^{\nu} \sum_{|t| \leq T} |d_m(t)|^2 \\
\ll_{D,\nu,\delta} \sum_{t \in T_B} \sum_{|t| \leq T} \frac{|c_j(t)|^2}{K_{\nu}(t,t)},
\]

where the first inequality is due to Lemmas \[3\] and \[5\] and the third inequality follows from \( (2.4) \). The second term on the right hand side of the third line in the above calculation is in the form of a Hilbert-Type inequality. By item (2) of Lemma \[3\] the points \( T_B \) are uniformly separated, hence the second inequality above follows by \[13\] Corollary 22. The statement follows by letting \( T \) to infinity. \( \square \)

2.2. Proof of Theorem \[1\]

Step 1. Let \( F \in \mathcal{H}(E^{\nu}) \). Since \( \mathcal{H}(E^{\nu}) \) is closed under differentiation, it follows that \( F^{(j)} \in \mathcal{H}(E^{\nu}) \) for every \( j \geq 0 \) and using the fact that \( T_B \subseteq T_{B'} \) we may apply \( (1.11) \) to obtain
\[
\sum_{j=0}^{\nu-1} \sum_{t \in T_B} \frac{|F^{(j)}(t)|^2}{K_{\nu}(t,t)} \ll_{D} \|F\|_{E^{\nu}}^2. \tag{2.5}
\]

Thus, we may apply Lemma \[6\] to deduce that the interpolation formula \( (1.15) \) converges in \( \mathcal{H}(E^{\nu}) \) to a function \( F_0(z) \), hence also uniformly on compact sets of \( \mathbb{C} \). We also conclude by Lemma \[6\] and property \( (1.14) \) that for every function
\( H \in \mathcal{H}(E^\nu) \) such that the interpolation formula (1.15) holds we must have

\[
\|H\|_{E^\nu} \ll_{D,\nu,\delta} \sum_{j=0}^{\nu-1} \sum_{t \in T_B} |H^{(j)}(t)|^2 / K_{\nu}(t,t).
\]  

(2.6)

We claim that \( F_0(z) = F(z) \). By Lemma 2 the span of \( \mathcal{G}_\nu \) is dense in \( \mathcal{H}(E^\nu) \), hence for any given \( \varepsilon > 0 \) there exists a function \( H \in \mathcal{H}(E^\nu) \) such that the interpolation formula holds and \( \|F - H\|_{E^\nu} < \varepsilon \). We obtain

\[
\|F - F_0\|^2_{E^\nu} \leq 2\varepsilon^2 + 2\|F - F_0\|^2_{E^\nu}
\]

where the second inequality is due to (2.6) and the third one due to (2.5). Since \( \varepsilon > 0 \) is arbitrarily, the claim follows. This also proves inequality (1.16) of item (2).

**Step 2.** We prove next inequality (1.17) of Item (2). Define

\[ \Delta_{\nu,j,t}(z) = K_{\nu}(t,t)^{-1/2} \mathcal{D}_{\nu,j}(z,t). \]

Equation (1.16) implies that \( \{\Delta_{\nu,j,t} : j = 0, ..., \nu - 1; t \in T_B\} \) is an (unweighted) frame for \( \mathcal{H}(E^\nu) \). Consider the frame operator \( U : \mathcal{H}(E^\nu) \to \mathcal{H}(E^\nu) \) defined by

\[
UF(z) = \sum_{t \in T_B} \sum_{j=0}^{\nu-1} \langle F, \Delta_{\nu,j,t} \rangle \mathcal{H}(E^\nu) \Delta_{\nu,j,t}(z).
\]

It is a basic result of frame theory (see [24, Corollary 5.1.3]) that \( U \) is invertible and positive, and that the collection \( \{U^{-1}\Delta_{\nu,j,t} : j = 0, ..., \nu - 1; t \in T_B\} \) is also a frame, sometimes called the canonical dual frame. By applying the operator \( U \) on both sides of (1.5) it follows immediately from (1.16) that

\[ UG_{\nu,j} = K_{\nu}(t,t)^{-1/2} \Delta_{\nu,j,t}(z), \]

which implies that \( \{K_{\nu}(t,t)^{1/2} G_{\nu,j}(.,t) : j = 0, ..., \nu - 1; t \in T_B\} \) is a dual frame of \( \mathcal{D}_{\nu} \). This implies (1.17). We remark that since for every fixed \( t \in T_B \) the functions \( G_{\nu,j}(z,t) \) and \( B_{\nu,j}(z,t) \) are connected via an invertible matrix transformation, the inequalities can also be shown from the bounds for \( B_{\nu} \) established in Lemma 5. Finally, Item (3) is a direct consequence of property (1.14). The proof of Theorem 1 is complete.

3. The \( L^p \) case

3.1. Preliminaries. De Branges spaces are closely related to Hardy spaces in the upper-half plane \( \mathbb{C}^+ = \{ \Im z > 0 \} \). For a given \( p \in [1, \infty] \) the Hardy space \( H^p(\mathbb{C}^+) \)
is defined as the space of holomorphic functions $F : \mathbb{C}^+ \to \mathbb{C}$ such that
\[
\sup_{y > 0} \|F(x + iy)\|_{L^p} < \infty.
\] (3.1)
where $\| \cdot \|_{L^p}$ denotes the standard $L^p$–norm in the variable $x$. This space endowed with the norm (3.1) defines a Banach space of holomorphic functions on the upper-half plane (see [30]).

Given a Hermite-Biehler function $E(z)$ one can prove that an entire function $F(z)$ belongs to $H(E)$ if and only if $F/E$ and $F/E^*$ belong to the space $L^2(\mathbb{C}^+)$ (hence $H^2(\mathbb{C}^+) = H(E)$). Using the fact that $H^p(\mathbb{C}^+)$ is a Banach space one can easily prove that $H^p(E)$ is indeed a Banach space of entire functions with norm given by
\[
\|F\|_{E,p} = \left( \int_{\mathbb{R}} \left| \frac{F(x)}{E(x)} \right|^p \frac{dx}{E(x)} \right)^{1/p}
\]
for finite $p$, or
\[
\|F\|_{E,\infty} = \sup_{x \in \mathbb{R}} \left| \frac{F(x)}{E(x)} \right|
\]
for $p = \infty$. For all these facts see [21, Section 3] and [1, 2].

Evidently $\|K(w, \cdot)\|_{E,q} < \infty$ for every $1 < q \leq \infty$ and $w \in \mathbb{C}$. It follows that these spaces still have a reproducing kernel structure, in fact one can prove, using the Cauchy formula for functions of Hardy spaces (see [30] Theorems 13.2 and 13.5), that if $p \in [1, \infty)$ and $F \in H^p(E)$ then
\[
F(w) = \int_{\mathbb{R}} \frac{F(x)K(w,x)}{E(x)} \frac{dx}{E(x)^2}
\] (3.2)
for every $w \in \mathbb{C}$, where $K(w, z)$ is defined in [17]. Using Hölder’s inequality we obtain an important estimate
\[
|F(w)| \leq \|F\|_{E,p}\|K(w, \cdot)\|_{E,p'},
\] (3.3)
where $p'$ is the conjugate exponent of $p$. Using the know fact that the space $H^p'(\mathbb{C}^+)$ can be identified with the dual space of $H^p(\mathbb{C}^+)$ for $p \in (1, \infty)$ one can deduce that $H^p(E)' = H^p(E)$ if $p \in (1, \infty)$, that is, if $\Lambda$ is a functional over $H^p(E)$ then there exists a function $\Lambda \in H^p(E)'$ such that
\[
\langle \Lambda, F \rangle = \int_{\mathbb{R}} \frac{F(x)\Lambda(x)}{|E(x)|^2} dx
\]
for all $F \in \mathcal{H}^p(E)$. The proof of this duality result deals with model spaces which diverges from the purposes of this article. For the interested reader we refer to [2, Proposition 1.1] and [16, Lemma 4.2].

3.2. Main Result. In what follows we will need $E(z)$ to satisfy some special conditions, named

(C1) The mean type of $E^*(z)/E(z)$ is negative.
(C2) There exist some $h > 0$ such that all the zeros of $E(z)$ lie in the half-plane $\Re z \leq -h.$

**Theorem 7.** Let $\nu \in \mathbb{N}$ with $\nu \geq 2$. Let $E(z)$ be a Hermite-Biehler function with the property that $\mathcal{H}^2(E^\nu)$ is closed under differentiation. Assume that $A, B \notin \mathcal{H}(E)$, and that there exists $\delta > 0$ so that the phase function $\varphi(x)$ satisfies $\varphi'(t) \geq \delta$ whenever $B(t) = 0$. Then if $p \in [1, 2)$ and $F \in \mathcal{H}^p(E^\nu)$ we have

$$F(z) = \sum_{t \in \mathbb{T}_n} \sum_{j=0}^{\nu-1} F^{(j)}(t) G_{\nu,j}(z,t). \quad (3.4)$$

where the formula converges uniformly in compact sets of $\mathbb{C}$. Furthermore, formula (3.4) is also valid in the case $p \in (2, \infty)$ if we additionally assume that $E(z)$ satisfies condition (C1) or (C2).

Remarks.

(i) We note that closure under differentiation of the space $\mathcal{H}^p(E)$ does not imply, in general, closure under differentiation of the space $\mathcal{H}^q(E)$ with $q \neq p$. However, this implication will be true if $E(z)$ satisfies conditions (C1) and (C2) (see Theorem 11).

(ii) The functions $E(z)$ that satisfy all the properties of the previous theorem in general don’t have closed analytic expressions. A better way to construct such functions is to give them by their Weierstrass factorization formula. A special subfamily of Hermite-Biehler functions with a manageable Weierstrass factorization formula is the Pólya class. This class is defined as those entire functions that can be arbitrarily approximated in any compact set of $\mathbb{C}$ by polynomials with no zeros in the upper-half plane (see [3, Section 7]).

In fact, a function of Pólya class can be characterized by its Hadamard factorization formula. A function $E(z)$ with non-zero zeros $w_n = x_n - iy_n$, belongs to the Pólya class if and only if it has the following factorization

$$E(z) = E^{(r)}(0)(z^r/r!) e^{-az^2 - ibz} \prod_n \left(1 - \frac{z}{w_n}\right) e^{zh_n},$$
where \( a \geq 0, \Re b \geq 0, \) \( h_n = \frac{x_n}{|w_n|^2}, \) \( y_n \geq 0 \) and
\[
\sum_n \frac{1 + y_n}{|w_n|^2} < \infty.
\]
In this situation, condition (C1) is equivalent to \( \Re b > 0. \)

3.3. Technicalities. Before proving Theorem 7 we need some technical lemmas.

The next result indicate that when the derivative of the associated phase function is bounded, the \( L^p \) de Branges space in question behave similarly to a Paley-Wiener space with respect to inclusion and summability issues.

**Lemma 8.** Let \( E(z) \) be a Hermite-Biehler function such that the associated phase function \( \varphi(x) \) has bounded derivative on \( \mathbb{R} \). Then for \( 1 \leq p < q < \infty \) \( \mathcal{H}^p(E) \subset \mathcal{H}^q(E) \) continuously. Also, if \( p > 1 \) then \( \mathcal{H}^p(E) \) is dense in \( \mathcal{H}^q(E) \).

**Proof.** The inclusion part is [21, Lemma 9]. The second part is an application of the Hahn-Banach Theorem. Let \( \Lambda \) be a functional on \( \mathcal{H}^q(E) \) that is zero in \( \mathcal{H}^p(E) \).

By duality there exists an entire function \( \Lambda(z) \) belonging to \( \mathcal{H}^q(E) \) such that
\[
\langle F, \Lambda \rangle = \int_{\mathbb{R}} \frac{F(x)\Lambda(x)}{|E(x)|^2} \, dx
\]
for every \( F \in \mathcal{H}^q(E) \). Since \( p > 1 \), the function \( K(w, z) \) belongs to \( \mathcal{H}^p(E) \) for every \( w \in \mathbb{C} \), which by the reproducing kernel property (3.2) implies that \( \Lambda \equiv 0 \). We conclude, by the Hahn-Banach theorem that \( \mathcal{H}^p(E) \) is dense in \( \mathcal{H}^q(E) \). \( \square \)

**Lemma 9.** Let \( E(z) \) be a Hermite-Biehler function satisfying condition (C1) and such that \( \tau = \sup_x \varphi'(x) < \infty \). Then for \( p \in (2, \infty) \) and \( F \in \mathcal{H}^p(E) \) we have
\[
\sum_{t \in T(\alpha)} \frac{|F(t)|}{(1 + |t|)K(t, t)^{1/2}} \ll_{\tau, p} \|F\|_{E, p}.
\]

**Proof.** This is [21, Lemma 11] \( \square \)

**Remark.** The implied constant also depends on \( v(E^* / E) \).

We say that a sequence of real numbers \( \{\lambda_n\} \) is \( \varepsilon \)-separated, for some \( \varepsilon > 0 \), if \( |\lambda_m - \lambda_n| \geq \varepsilon \) for every \( m \neq n \). We now prove a generalized (weighted) version of the Pólya-Plancherel theorem (see [32]).

**Proposition 10.** Let \( E(z) \) be a Hermite-Biehler function satisfying condition (C2) for some \( h > 0 \) and such that \( \tau = \sup_x \varphi'(x) < \infty \). Then if \( \{\lambda_n\} \) is a \( \varepsilon \)-separated sequence of real numbers, \( p \in [1, \infty) \) and \( F \in \mathcal{H}^p(E) \) we have
\[
\sum_n \left| \frac{F(\lambda_n)}{E(\lambda_n)} \right|^p \leq \frac{1 + \varepsilon^{8 \tau p}}{\pi \alpha} \int_{\mathbb{R}} \left| \frac{F(x)}{E(x)} \right|^p \, dx
\]
where \( \alpha = \min\{\varepsilon/2, h/2\} \).
Proof. Step 1. Since $E^*/E$ is bounded on the upper-half plane and has modulo one in the real line, by Nevanlinna’s factorization (see [3, Theorem 8]) we obtain

$$\Theta(z) := \frac{E^*(z)}{E(z)} = e^{2\pi i z} \prod_n \frac{1 - z/w_n}{1 - z/\bar{w}_n}$$

where $\bar{w}_n = x_n - iy_n$ with $y_n \geq h$, are the zeros of $E(z)$ and $2a = -\nu(E^*/E) \geq 0$. If $z = x + iy$ with $y \geq 0$ we have the following identity

$$\frac{1}{2} \partial_y \log |\Theta^*(z)| = a + \sum_n \frac{y_n[(x - x_n)^2 + y_n^2]}{|z - w_n|^2|z - \bar{w}_n|^2}.$$

If $0 \leq y \leq h/2$ then $y_n^2 - y^2 \leq 4(y_n - y)^2$ and we deduce that

$$\frac{1}{2} \partial_y \log |\Theta^*(z)| \leq a + 4 \sum_n \frac{y_n}{(x - x_n^2) + y_n^2} = -3a + \frac{1}{2} \partial_y \log |\Theta^*(x)| = -3a + 4\varphi'(x) \leq 4r.$$

Integrating in $y$ we obtain $|\Theta^*(z)| \leq e^{8r_y}$ if $0 \leq y \leq h/2$.

Step 2. Let $\alpha = \min\{h/2, \varepsilon/2\}$. Since the function $|F(z)/E(z)|^p$ is sub-harmonic in the half-plane $\Im z > -h$, its value at the center of a disk is not greater than its mean value over the disk. We obtain

$$|F(\lambda_n)/E(\lambda_n)|^p \leq \frac{1}{\pi \alpha^2} \int_{-\alpha}^{\alpha} \int_{0}^{2\pi} |F(\lambda_n + \rho e^{it})/E(\lambda_n + \rho e^{it})|^p dt d\rho$$

$$\leq \frac{1}{\pi \alpha^2} \int_{0}^{\lambda_n + \alpha} \int_{-\alpha}^{\lambda_n - \alpha} |F(x + iy)/E(x + iy)|^p dx dy.$$

Using the separability of $\{\lambda_n\}$ we can sum the above inequality for all values of $n$ to obtain

$$\sum_n |F(\lambda_n)/E(\lambda_n)|^p \leq \frac{1}{\pi \alpha^2} \int_{-\alpha}^{\alpha} \int_{R} |F(x + iy)/E(x + iy)|^p dx dy$$

$$= \frac{1}{\pi \alpha^2} \int_{0}^{\alpha} \int_{R} |F(x + iy)/E(x + iy)|^p dx dy$$

$$+ \frac{1}{\pi \alpha^2} \int_{0}^{\alpha} \int_{R} |F^*(x + iy)/E^*(x + iy)|^p dx dy.$$

Since $|\Theta^*(z)| \leq e^{8r_y}$ for $0 < y < h/2$ we conclude that

$$\sum_n |F(\lambda_n)/E(\lambda_n)|^p \leq \frac{1}{\pi \alpha^2} \int_{0}^{\alpha} \int_{R} |F(x + iy)|^p + e^{8r_y}p |F^*(x + iy)|^p dx dy.$$

By definition $F/E$ and $F^*/E$ belong to the Hardy space $H^p(C^+)$. A basic property of Hardy spaces is that

$$\sup_{y > 0} \|G(x + iy)\|_{L^p} = \lim_{y \to 0} \|G(x + iy)\|_{L^p}.$$
for every $G \in H^p(\mathbb{C}^+)$ (see [39] Theorems 13.2 and 13.5]). Using this last fact we obtain

$$\int_{\mathbb{R}} \frac{|F(x+iy)|^p + e^{8\tau_0}|F^*(x+iy)|^p}{|E(x+iy)|^p} \, dx \leq (1 + e^{8\tau_0}) \int_{\mathbb{R}} \frac{|F(x)|^p}{|E(x)|} \, dx$$

for every $y > 0$. This concludes the lemma. $\square$

For the sake of completeness we state a result about boundeness of the differentiation operator due to A. Baranov (see [2, Theorem A]). For analogous results in model spaces we refer to K. M. Dyakonov papers [17, 18].

**Theorem 11.** Let $E(z)$ be a Hermite-Biehler function satisfying conditions (C1) or (C2). For $p \in (1, \infty)$ the following statements are equivalent

1. $H^p(E)$ is closed under differentiation.
2. $E(z)$ is of exponential type and $E'(z)/E(z) \in H^\infty(\mathbb{C}^+)$.

**Remark.** The fact that condition (2) above is independent of $p \in (1, \infty)$ implies that if $E(z)$ satisfy condition (C1) or (C2) then $H^p(E)$ is closed under differentiation if and only if $H^2(E)$ is closed under differentiation.

### 3.4. Proof of Theorem 7

The following is a very technical proof and for this reason we split the proof into several steps. Steps 0, 1 and 2 are the crucial ones.

**Step 0.** A simple, but crucial consequence of formula (1.5) is that the singular part of the function $F(z)/B(z)^\nu$ at a given zero $t \in T_B$ is

$$\sum_{j=0}^{\nu-1} F^{(j)}(t) \frac{G_{\nu,j}(z,t)}{B^\nu(z)} = \sum_{j=0}^{\nu-1} F^{(j)}(t) \frac{P_{\nu,j-1}(z,t)}{j!(z-t)^{\nu-j}}.$$  

To see this, define for any complex number $w$ a linear operator $S_w$ on the space of meromorphic functions by

$$S_w(G)(z) = \sum_{n \leq -1} \frac{g_n}{(z-w)^n}.$$  

if $G(z)$ has the following series representation

$$G(z) = \sum_{n \in \mathbb{Z}} \frac{g_n}{(z-w)^n}.$$  

about $z = w$. That is, $S_w(G)$ is defined as the singular part of the function $G(z)$ at the point $z = w$. Since $G(z)$ is meromorphic, $S_w(G)(z)$ is always a rational function. It is a simple, but useful characterization that $S_w(G)(z)$ is the unique rational function $R(z)$ having exactly one pole which is located at $z = w$, such that $G(z) - R(z)$ has a removable singularity at the point $z = w$ and

$$\lim_{|z| \to \infty} \frac{R(z)}{(z-w)^j} = 0$$

(3.5)
for every integer \( j \geq 0 \). Recalling that
\[
\frac{(z-t)^\nu}{B(z)\nu} = \sum_{n=0}^{\infty} a_{\nu,n}(t)(z-t)^n \quad \text{and} \quad P_{\nu,j}(z,t) = \sum_{n=0}^{j} a_{\nu,n}(t)(z-t)^n
\]
we obtain
\[
S_t\left(\frac{F}{B^\nu}\right)(z) = \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{P_{\nu,j}(z,t)}{j!(z-t)^{\nu-j}} = \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{G_{\nu,j}(z,t)}{B(z)\nu}. \tag{3.6}
\]

Now, given a complex number \( w \in \mathbb{C} \) we define another linear operator \( \mathcal{M}_w \) on the space of entire functions \( F(z) \) by
\[
\mathcal{M}_w(F)(z) = \frac{F(z)B(w)\nu - B(z)\nu F(w)}{z-w}.
\]

We observe that for every \( t \in T_B \) and every \( w \in \mathbb{C} \setminus T_B \) we have
\[
S_t\left(\frac{\mathcal{M}_w(F)(\cdot)}{B(w)\nu B(z)\nu}\right)(z) = \frac{S_t(F/B^\nu)(z) - S_t(F/B^\nu)(w)}{z-w}. \tag{3.7}
\]
One can deduce this last identity by observing that
\[
\frac{\mathcal{M}_w(F)(z)}{B(w)\nu B(z)\nu} - \frac{S_t(F/B^\nu)(z) - S_t(F/B^\nu)(w)}{z-w}
\]
has a removable singularity at the point \( z = w \) and also that the right hand side of (3.7) is a rational function in the variable \( z \) with exactly one pole, which is located at \( z = t \) and satisfies condition (3.5).

**Step 1.** We begin with the case \( p \in [1,2] \). By Lemma 3 the assumption that \( \mathcal{H}^2(E^\nu) \) is closed under differentiation implies that \( \tau = \sup_x \varphi'(x) < \infty \). We can apply Lemma 8 to conclude that \( \mathcal{H}^p(E^\nu) \subset \mathcal{H}^2(E^\nu) \), hence formula (3.4) is a direct consequence of Theorem 1.

**Step 2.** Now, we deal with the case \( p \in (2,\infty) \). A crucial observation is that if \( F \in \mathcal{H}^p(E^\nu) \) then \( \mathcal{M}_w(F) \in \mathcal{H}^2(E^\nu) \). Thus, we can apply Theorem 1 together with (3.6) to obtain
\[
\mathcal{M}_w(F)(z) = \sum_{t \in T_B} B(z)\nu S_t\left(\frac{\mathcal{M}_w(F)(\cdot)}{B(\cdot)\nu}\right)(z)
\]
where the last sum converges uniformly in the variable \( z \) in every compact subset of \( \mathbb{C} \) for every \( w \in \mathbb{C} \). By (3.6) and (3.7) we conclude that
\[
\frac{F(z)}{B(z)\nu} - \frac{F(w)}{B(w)\nu} = \sum_{t \in T_B} \left\{ \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{G_{\nu,j}(z,t)}{B(z)\nu} - \sum_{j=0}^{\nu-1} F^{(j)}(t)\frac{G_{\nu,j}(w,t)}{B(w)\nu} \right\} \tag{3.8}
\]
for every \( w,z \in \mathbb{C} \setminus T_B \). We claim that (3.8) implies that
\[
F(z) = \Lambda(F)B(z)\nu + \sum_{t \in T_B} \sum_{j=0}^{\nu-1} F^{(j)}(t)G_{\nu,j}(z,t) \tag{3.9}
\]
for some constant $\Lambda(F)$, where the sum converges uniformly in compact sets of $\mathbb{C}$.

Assuming that (3.9) is valid, clearly the map $F \mapsto \Lambda(F)$ defines a linear functional in the space $\mathcal{H}^p(E^\nu)$. In Steps 4 and 5 we’ll show that formula (3.9) holds and that $F \mapsto \Lambda(F)$ is a continuous functional that vanishes in a dense set of functions in $\mathcal{H}^p(E^\nu)$, hence identically zero and concluding the theorem.

Step 3. Recall that $a_{\nu,j}(t)$ is defined as the coefficients of the Taylor expansion of $B_{\nu,\nu}(z,t)^{-1}$ at the point $z = t$ with $t \in T_B$. By Lemma 3 these coefficients satisfy the estimate

$$|a_{\nu,j}(t)|^2 \ll_{D,\nu} \frac{1}{K_{\nu}(t,t)}$$

for $j = 0, ..., \nu - 1$, where $D$ is the norm of the differentiation operator in $\mathcal{H}^2(E^\nu)$ and $K_{\nu}(w,z)$ is the reproducing associated with $E(z)^\nu$. Since

$$G_{\nu,j}(z,t) = \frac{B(z)^\nu}{j!} \sum_{\ell=0}^{\nu-1-j} \frac{a_{\nu,\ell}(t)}{(z-t)^{\nu-\ell-j}},$$

we obtain

$$K_{\nu}(t,t)|G_{\nu,j}(i,t)|^2 \ll_{D,\nu} \frac{1}{1 + t^2}$$

(3.10)

for every $t \in T_B$ and $j = 0, ..., \nu - 1$.

Step 4. Assume that $E(z)$ satisfy condition (C2) for some $h > 0$. By the remark after Theorem 11 we have that $\mathcal{H}^p(E^\nu)$ is closed under differentiation. Also, the assumption that $\varphi'(t) \geq \delta$ for all $t \in T_B$ together with formula (1.10) implies that

$$|E(t)|^{2\nu} \ll_{\delta} K_{\nu}(t,t).$$

(3.11)

for all $t \in T_B$. By Lemma 3 items (1) and (2) we have $\varphi'(x) \ll 1$ and the zeros of $B(z)$ are separated with width of separation at least $\sqrt{\nu D}$. Thus, we can apply Proposition 10 together with (3.11) to obtain

$$\sum_{t \in T_B} \left| \frac{F(t)}{K_{\nu}(t,t)^{1/2}} \right|^p \ll_{D,\delta,h,p} \int_{\mathbb{R}} \left| \frac{F(x)}{E(x)^\nu} \right|^p dx.$$  

(3.12)

for every $F \in \mathcal{H}^p(E^\nu)$. Finally, we obtain the following estimate

$$\sum_{j=0}^{\nu-1} \left| F^{(j)}(i,t)G_{\nu,j}(i,t) \right| \leq \sum_{j=0}^{\nu-1} \left[ \sum_{t \in T_B} \left| \frac{F^{(j)}(t)}{K_{\nu}(t,t)^{1/2}} \right|^p \right]^{1/p} \left[ \sum_{t \in T_B} \left| \frac{G_{\nu,j}(i,t)}{K_{\nu}(i,t)^{-1/2}} \right| \right]^{p'/1} \ll \sum_{j=0}^{\nu-1} \left[ \sum_{t \in T_B} \left| \frac{F^{(j)}(t)}{K_{\nu}(t,t)^{1/2}} \right|^p \right]^{1/p} \ll \|F(x)/E(x)^\nu\|_{L^p},$$

(3.13)
where the first inequality is Holder’s inequality, the second one due to \(3.10\) and the separation of \(T_B\), the third one due to \(3.12\) and the closure under differentiation of \(H^p(E^\nu)\).

Estimate \(3.13\) together with formula \(3.8\) clearly implies that \(3.9\) is valid and, by using estimate \(3.3\) we can easily deduce that \(F \mapsto \Lambda(F)\) is a continuous functional over \(H^p(E^\nu)\). By Lemmas 2 and 8 the functions \(\{G_{\nu,j}(z,t)\}\) for \(j = 0, \ldots, \nu - 1\) and \(t \in T_B\) form a dense set in \(H^p(E^\nu)\) and trivially \(\Lambda(G_{\nu,j}(z,t)) = 0\). Hence \(\Lambda\) vanishes identically.

**Step 5.** Assume that \(E(z)\) satisfies (C1) with \(-a = \nu(E^*/E)\). By Theorem 11 we again conclude that \(H^p(E^\nu)\) is closed under differentiation. We can apply Lemma 9 together with estimation \(3.10\) to deduce that 

\[
\sum_{j=0}^{\nu-1} \sum_{t \in T_B} |F^{(j)}(t)G_{\nu,j}(i,t)| \ll \|F(x)/E(x)^\nu\|_{L^p}
\]

for every \(F \in H^p(E^\nu)\). In the same way as the previous step, we would conclude that formula \(3.9\) is valid and \(F \mapsto \Lambda(F)\) is a continuous functional over \(H^p(E^\nu)\) that is identically zero in a dense set of functions. Hence \(\Lambda\) vanishes identically and this concludes the theorem.

4. **Applications**

We give here one application of the results obtained for the \(L^2\) case related to sampling/interpolation theory in Paley-Wiener spaces. We say that a sequence of real numbers \(\{\lambda_m\}\) is **sampling of order \(\nu\)** for \(PW^2(\tau)\) if the norm 

\[
\eta_\nu(F) = \left[ \sum_{n=0}^{\nu-1} \sum_{m} |F^{(n)}(\lambda_m)|^2 \right]^{1/2}
\]

defines an equivalent norm in \(PW^2(\tau)\) (the \(L^2(\mathbb{R})\)-norm). Recall that \(PW^2(\tau)\) is defined as the space of entire functions \(F(z)\) of exponential type at most \(\tau > 0\) that belong to \(L^2(\mathbb{R})\) when restricted to the real line. We say that \(\{\lambda_m\}\) is a non-redundant sampling sequence if by extracting one element of \(\{\lambda_m\}\) the norm \(\eta_\nu\) is no-longer equivalent to the \(L^2(\mathbb{R})\)-norm.

We say that \(\{\lambda_m\}\) is **interpolating of order \(\nu\)** in \(PW^2(\tau)\) if the system 

\[
F^{(n)}(\lambda_m) = f_{n,m}
\]

has a solution for all \((\{f_{0,m}\}, \{f_{1,m}\}, \ldots, \{f_{\nu-1,m}\}) \in \ell^2(\mathbb{Z}) \times \cdots \times \ell^2(\mathbb{Z}).\) We say that \(\{\lambda_m\}\) is a complete interpolating sequence if the solution is unique.

In [31], J. Ortega-Cerdà and K. Seip gave necessary and sufficient conditions for a sequence be sampling/interpolating with no derivatives in \(PW^2(\tau)\). In [24]
Corollary 2] the first named author of this article gave the first known sufficient condition for sampling/interpolating with first order derivatives in $PW^2(\tau)$. Both these characterizations rely heavily upon the representation $PW^2(\tau) = H^2(E)$ for some Hermite-Biehler function $E(z)$. The following theorem is a generalization of the result in [24, Corollary 2].

**Theorem 12.** Let $E(z) = A(z) - iB(z)$ be a Hermite-Biehler function, $\nu \geq 2$ be an integer and denote by $\{t_n\}_{n \in \mathbb{Z}}$ the real zeros of $B(z)$. Assume that $PW^2(\tau) = H^2(E^\nu)$ as sets and there exists $C > 0$ such that $|A(t_n)| \leq C$ for all $n$. Then the map

$$F(z) \mapsto (F(t_n), F'(t_n), \cdots, F^{(\nu-1)}(t_n))$$

defined $\nu$ times defines a continuous linear isomorphism from $PW^2(\tau)$ to $\ell^2(\mathbb{Z}) \times \cdots \times \ell^2(\mathbb{Z})$. In particular, the sequence $\{t_n\}$ is complete interpolating and a non-redundant sampling sequence.

**Proof.** Since convergence in the space implies uniform convergence in compacts of $\mathbb{C}$, we can apply the Closed Graph Theorem to obtain that

$$\int_{\mathbb{R}} \frac{|F(x)|^2}{|E(x)^\nu|} \, dx \simeq \int_{\mathbb{R}} |F(x)|^2 \, dx$$

for all $F \in PW^2(\tau) = H^2(E^\nu)$. Also, it’s a known fact that the Paley-Wiener spaces are closed under differentiation (one can use Fourier inversion to see that). The statement follows from Theorem 1 items (1) and (2) once we verify the estimates

$$1 \ll \psi'(t_n)$$

and

$$K_\nu(t_n, t_n) \simeq 1.$$ 

Using the reproducing kernel structure of $PW^2(\tau)$ one can easily prove that

$$K_\nu(x, x) = \sup_{\|F/E^\nu\|_{L^2} \leq 1} |F(x)|^2 \simeq \sup_{\|F\|_{L^2} \leq 1} |F(x)|^2 = \tau/\pi,$$

for all real $x$. This last fact together with formula

$$\nu \psi'(x)|E(x)|^{2\nu} = \pi K_\nu(x, x)$$

and the hypothesis that $|A(t_n)| \ll 1$ proves the desired estimates and concludes the proof. \hfill $\square$

**Remark.** In [31, Theorem 4], Y. Lyubarskii and K. Seip give necessary and sufficient conditions for the representation $PW^2(\tau) = H^2(E)$. 

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