L^1-APPROXIMATION TO LAPLACE TRANSFORMS OF SIGNED MEASURES

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Abstract. For functions that are piecewise equal to one-sided Laplace transforms of signed measures we construct interpolations that are entire functions with interpolation points that are zeros of Laguerre-Pólya entire functions.

If the interpolated function f is sufficiently regular, these interpolations are best approximations in L^1-norm to f by functions of fixed exponential type. This is demonstrated for the example $f_{a,b}(x) = e^{ax}(1+e^{bx})^{-1}$ with $0 < a < b$.

1. Introduction

An entire function f is of exponential type $\eta > 0$ iff for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ with $|f(z)| \le C_\varepsilon e^{|z|(\eta+\varepsilon)}$ for every $z \in \mathbb{C}$. We define $\mathcal{A}(\eta)$ to be the class of all entire functions of exponential type at most η.

We study an instance of the problem of finding the best $L^1(\mathbb{R})$-approximation in $\mathcal{A}(\eta)$ to given $f \in L^1(\mathbb{R})$. E. Carneiro and J. D. Vaaler [3], [4] investigated this problem for functions of the form

$$f(x) = \int_0^\infty e^{-\lambda|x|} d\nu(\lambda), \tag{1}$$

where ν is a positive measure. To obtain a solution they constructed the extremal functions for $x \mapsto e^{-\lambda|x|}$ and then integrated over the parameter λ.

In this article we consider certain signed measures ν. For a signed measure ν of finite total variation we define

$$f_\nu(z) = \begin{cases} \displaystyle\int_{t>0} e^{-zt} d\nu(t) & \text{for } \Re z > 0, \\ \displaystyle -\int_{t<0} e^{-zt} d\nu(t) & \text{for } \Re z < 0. \end{cases} \tag{2}$$

The function f is normalized by setting $f(0) = 2^{-1}\{f(0+) + f(0-)\}$.

The construction of explicit best approximations of exponential type to "special" functions goes back to B. Sz.-Nagy [15] and M. G. Krein [10], an account of their methods can be found in A. F. Timan [16].

The interpolation method employed in this paper has been introduced in connection with the problem of one sided approximation. We refer to S. W. Graham and J. D. Vaaler [8], Vaaler [17], Littmann [11, 12], E. Carneiro *et. al.* [2]. An extension of the Krein-Nagy method was given recently in M. I. Ganzburg [6]. For approximations in different norms we refer to D. Lubinsky [13] and Ganzburg and Lubinsky [7]. For the analogous problem on the torus see T. Ganelius [5].

It is well known (Timan [16, p. 84, 2.12.6]) that if there exist $A \in \mathcal{A}(\eta)$ and $\alpha \in \mathbb{R}$ with the property that

$$\frac{A(x) - f_\nu(x)}{\sin \eta(x-\alpha)} \geq 0 \tag{3}$$

for all real x, then A is a best approximation from $\mathcal{A}(\eta)$ to f_ν.

Let F be a Laguerre-Pólya entire function and let g be the Laplace inverse transformation of $1/F(z)$ in a vertical strip containing the origin (see Section 2); in particular, F has only real zeros. For signed measures ν that are supported on a bounded set and have finite total variation an interpolation method is given in Section 3 that constructs an entire function $G_{\nu,F}$ with the property that

$$f_\nu(z) - G_{\nu,F}(z) = F(z) H_{\nu,F}(z) \tag{4}$$

for all z with $\Re z \neq 0$, where

$$H_{\nu,F}(z) = \begin{cases} \displaystyle\int_{-\infty}^0 e^{-zt}\{g * d\nu\}(t)dt & \text{if } \Re z < 0, \\ \displaystyle -\int_0^\infty e^{-zt}\{g * d\nu\}(t)dt & \text{if } \Re z > 0, \end{cases}$$

here $g * d\nu(t) = \int_{\mathbb{R}} g(t-u)d\nu(u)$.

If $F(z) = \sin \pi\eta(z-\alpha)$ then $G_{\nu,F}$ is in $\mathcal{A}(\eta)$. The factorization (4) reduces investigation of (3) to the problem of finding out if $H_{\nu,F}$ is of one sign on the real line. This in turn is accomplished by an investigation of $g * d\nu$. The function g can be explicitly calculated; if $F(z) = \sin \pi\eta(z-\alpha)$ with $0 < \alpha < 1$, then (cf. Section 2) g_α is given by

$$g_\alpha(t) = e^{\alpha t}(e^t + 1)^{-1}. \tag{5}$$

The function g_α is an example of a so-called *variation diminishing* function (explained in Definition 2.1), for certain ν this property can be used to bound the number of sign changes of $g_\alpha * d\nu$. Section 2 contains results of this kind for certain special measures ν.

For practical purposes the conditions on ν imposed so far are too restrictive. Often, ν has neither finite total variation, nor is its support bounded. In such a case an approximation of ν by measures ν_m that are of finite variation and have bounded support may lead to a sequence of entire functions that converges to the desired best approximation. We consider in Section 4 the following example. Assume that $0 < a < b$. Define

$$f_{a,b}(z) = e^{az}(e^{bz} + 1)^{-1}, \tag{6}$$

which satisfies (2) with

$$\nu_{a,b} = \sum_{n=-\infty}^{\infty} (-1)^{n+1} \delta_{bn-a}. \tag{7}$$

Since ν is a distribution, the interpolation theorems are applied after truncating the series. Let $K_{a,b}$ be given by

$$K_{a,b}(z) = \frac{\sin \pi(z - b^{-1}a)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^{n+1} \frac{f_{a,b}(n + b^{-1}a)}{z - n - b^{-1}a}. \tag{8}$$

We show in Theorem 4.3 that $K_{a,b} = G_{\nu_{a,b},F}$ where F is a translate of $z \mapsto \sin \pi z$. The results from Section 2 and (4) are used to show that $z \mapsto K_{\eta^{-1}a,\eta^{-1}b}(\eta z)$ is the best approximation in $L^1(\mathbb{R})$ to $f_{a,b}$ from $\mathcal{A}(\eta\pi)$, and we compute the error of approximation.

2. Periodic analogues of variation diminishing functions

Definition 2.1. Denote by $S^-[a_1, ..., a_n]$ the number of changes of sign in the real sequence $a_1, ..., a_n$. Zero values do not count as changes of sign, we have $S^-[1, 0, 1] = 0$ and $S^-[1, 0, -1] = 1$. For functions $f : \mathbb{R} \to \mathbb{R}$ we extend this definition to open intervals $I \subseteq \mathbb{R}$ via

$$S_I^-[f] := \sup\{S^-[f(x_1), ..., f(x_n)] \,|\, x_1 < ... < x_n,\ x_i \in I,\ n \in \mathbb{N}\}.$$

For intervals of the form $I = [a, b)$ we define

$$S_I^-[f] = \inf\{S^-_{(a-\varepsilon,b)}[f] \,|\, \varepsilon > 0\},$$

and similarly for left-open and closed intervals. (In particular, $S^-_{\{a\}}[f] \in \{0, 1, \infty\}$.)

A non-negative, integrable function $g : \mathbb{R} \to \mathbb{R}$ is said to be *variation diminishing* if $S^-_{\mathbb{R}}[\varphi * g] \leq S^-_{\mathbb{R}}[\varphi]$ for every bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$.

The next two lemmas collect some simple and useful inequalities for sign change counts.

Lemma 2.2. *Let ψ, ψ_k ($k \in \mathbb{N}$) be real functions defined on $\mathbb{R}$, and let n be a non-negative integer. If $\psi_k \to \psi$ pointwise and $S^-[\psi_k] \leq n$ for all $k \in \mathbb{N}$, then $S^-[\psi] \leq n$.*

Proof. This is Lemma IV.2.1b in [9]. □

Lemma 2.3. *Let I be an interval (possibly infinite). Let $f, g : I \to \mathbb{R}$ be continuous functions. Let m be the infimum of the local maxima of $|f|$ on I. If $m > 0$ and $|f - g| < m$ on I, then $S_I^-[f] \leq S_I^-[g]$.*

Proof. If y is a sign change of f, then there are x_1 and x_2 in I such that $|f(x_i)| \geq m$, $f(x_1)f(x_2) < 0$, and $x_1 < y < x_2$, and no other sign change of f is in (x_1, x_2). The bounds on g imply that g must have a sign change in (x_1, x_2) as well. □

The significance of variation diminishing functions for our topic is the fact that these functions are the Laplace inverse transforms of reciprocals of certain entire functions having only real zeros. The zero sets of such entire functions serve as interpolation nodes in the next section.

We denote by $\mathcal{E}$ the class of Laguerre-Pólya entire functions, that is, all entire functions of the form

$$F(z) = C\exp(-cz^2 + dz)z^\kappa \prod_{k=1}^\infty \Big(1 - \frac{z}{a_k}\Big)\exp(z/a_k), \tag{9}$$

where $c \geq 0$, $\kappa \in \mathbb{N}_0$, d, $a_k (k \in \mathbb{N})$ and C are real, and $\sum_{k=1}^\infty a_k^{-2} < \infty$. We define

$$S_F := \{b \in \mathbb{R} : F(b) = 0\} \cup \{\pm\infty\}. \tag{10}$$

The reciprocals of all elements in $\mathcal{E}$, except the pure exponentials, have representations as Laplace transforms. We omit the discussion of functions $F \in \mathcal{E}$ having none or a single simple root since they are not needed in the later sections.

Lemma 2.4. *Let $F \in \mathcal{E}$ with $F(0) \neq 0$, and assume that F has at least two zeros. There exists an integrable function $g : \mathbb{R} \to \mathbb{R}$ such that*

$$\frac{1}{F(z)} = \int_{-\infty}^\infty e^{-zt} g(t)dt \tag{11}$$

in a vertical strip containing the origin. The integral is absolutely convergent in the largest open vertical strip that contains the origin and no zeros of F. The function g has no sign changes and its sign equals the sign of $F(0)$.

Proof. This is Corollary 5.4 in chapter III of [9]. □

The connection between variation diminishing functions and Laplace inverse transforms of reciprocals of elements in $\mathcal{E}$ mentioned above is as follows.

Lemma 2.5. *An integrable function $G : \mathbb{R} \to \mathbb{R}$ is variation diminishing if and only if $G = g$ a.e., where g satisfies* (11) *in an open strip containing the origin for some $F \in \mathcal{E}$.*

Proof. This is shown by I. J. Schoenberg [14]; see also chapter IV in [9], Theorem 2.1 and Theorem 4.1. □

An important example of an element in $\mathcal{E}$ is the function $z \mapsto \pi \sin \pi(z - \alpha)$. From [9] Chapter III.9 we get for $\alpha \in \mathbb{R}$ that

$$-\frac{\pi}{\sin \pi(z-\alpha)} = \int_{-\infty}^\infty e^{-zt} \frac{e^{\alpha t}}{e^t + 1} dt \quad \text{for} \quad \alpha - 1 < \Re z < \alpha, \tag{12}$$

in particular, $t \mapsto e^{\alpha t}(e^t + 1)^{-1}$ is variation diminishing for $0 < \alpha < 1$.

Let $a < b$ be two consecutive elements in S_F and $c \in (a, b)$. Then $z \mapsto [F(z - c)]^{-1}$ has a representation as in Lemma 2.4. It follows that the reciprocal of F can be represented as a two-sided Laplace transform in $a < \Re z < b$.

Lemma 2.6. *Let $F \in \mathcal{E}$ have at least two zeros. Let $a < b$ be two consecutive elements in S_F (defined in (10)), and let g satisfy (11) for $a < \Re z < b$. If a and b are both finite, then for any $n \in \mathbb{N}_0$ there are polynomials P_n and Q_n such that*

$$g^{(n)}(t) = \mathcal{O}(P_n(t)e^{at}) \text{ as } t \to \infty,$$
$$g^{(n)}(t) = \mathcal{O}(Q_n(t)e^{bt}) \text{ as } t \to -\infty.$$

If $a = -\infty$, then $g^{(n)}(t) = \mathcal{O}(e^{-Kt})$ for all $K > 0$ as $t \to \infty$, and if $b = \infty$, then $g^{(n)}(t) = \mathcal{O}(e^{Kt})$ for all $K > 0$ as $t \to -\infty$.

Proof. This can be found in Theorem 2.1 in chapter V of [9]. □

We estimate now the sign changes of convolutions of certain distributions with variation diminishing functions. Recall that S^- is given in Definition 2.1. We let $\mathcal{S}$ be the space of infinitely differentiable functions that decay faster than any polynomial.

Definition 2.7. Let $0 \le r_1 < \ldots < r_n < 1$, let $a_j \in \mathbb{R}$, and let $\eta_0 := \sum_{j=1}^n a_j \delta_{r_j}$. Define the cyclic sign-changes of η_0 by

$$\mathcal{V}[\eta_0] := S^-[\{a_1, \ldots, a_n, a_1\}].$$

We define a linear functional η with a slight abuse of notation by

$$\int_A \varphi \, d\eta = \sum_{k\in\mathbb{Z}} \int_A \varphi \, d\eta_0(k + A) \qquad (\varphi \in \mathcal{S}),$$

where $k+A = \{k+a \mid a \in A\}$. Let $g : \mathbb{R} \to \mathbb{R}$ such that $g(t) = \mathcal{O}((1+|t|)^{-1-c})$ as $|t| \to \infty$ with some positive c. We define $g * d\eta$ by

$$g * d\eta(x) := \int_{\mathbb{R}} g(x-t)d\eta(t) = \sum_{k\in\mathbb{Z}} \sum_{j=1}^n a_j g(x - k - r_j).$$

Theorem 2.8. *Let $g : \mathbb{R} \to \mathbb{R}$ be variation diminishing and analytic in a horizontal strip containing $\mathbb{R}$. Let η and $g * d\eta$ be as in Definition 2.7. Then $S^-_{[0,1)}[g * d\eta] \le \mathcal{V}[\eta_0]$.*

Proof. The statement is trivially true if $g*d\eta$ is the zero function, so throughout this proof we assume that this is not the case.

Let $\nu = \mathcal{V}[\eta_0]$ and assume to the contrary that $S^-_{[0,1)}[g * d\eta] \ge \nu + 1$. It follows that for every integer ℓ we have

$$S^-_{[-\ell,\ell)}[g * d\eta] \ge 2(\nu + 1)\ell. \tag{13}$$

Consider

$$\gamma_\ell(x) := \sum_{k=-\ell}^{\ell-1} \sum_{j=1}^n a_j g(x - k - r_j).$$

We show the existence of $d > 0$ independent of ℓ so that γ_ℓ has at least $2(\nu + 1)(\ell - d)$ sign changes in $[d - \ell, \ell - d]$. To obtain a contradiction an approximate identity is used to obtain the inequality $S^-_{\mathbb{R}}[\gamma_\ell] \le 2\nu\ell$.

From Lemma 2.5 and Lemma 2.6 we obtain the existence of $c_1 > 0$ and $a > 0$ with $|g(t)| \le c_1 \exp(-a|t|)$ and hence $|g * \eta_0(t)| \le c_2 \exp(-a|t|)$. For $t \le k$ we obtain $|g * d\eta_0(t-k)| \le c_2 e^{a(t-k)}$ and hence for $t \le \ell - 1$

$$\Big| \sum_{k=\ell-1}^{\infty} g * d\eta_0(t-k) \Big| \le c_2 e^{at} e^{-a\ell} (1-e^{-a})^{-1}.$$

Similarly, for $t > -\ell$

$$\Big| \sum_{k=-\infty}^{-\ell} g * d\eta_0(t-k) \Big| \le c_3 e^{-at} e^{-a\ell} (1-e^{-a})^{-1}.$$

There exists therefore $c > 0$ so that for all ℓ and t with $-\ell < t < \ell$

$$|g * d\eta(t) - \gamma_\ell(t)| \le c e^{-a\ell} \cosh(at). \tag{14}$$

Since g is assumed to be analytic in a horizonal strip S containing the real line, $g * d\eta$ is analytic in S as well, hence the derivative of $g * d\eta$ is analytic in S and can have only finitely many zeros in $[0,1]$ (recall that $g * d\eta$ is assumed to be not identically zero). Since $g * d\eta$ is periodic, the minimum m of the local maxima of $|g * d\eta|$ on $[0,1]$ and hence on $\mathbb{R}$ is positive and independent of ℓ.

Inequality (14) implies for $t > 0$ that $|g * d\eta(t) - \gamma_\ell(t)| < m$ holds for $a(t-\ell) < \log(m/c)$, hence with $d := -a^{-1}\log(m/c)$

$$t \le \ell - d.$$

A similar calculation for $t < 0$ gives $|g * d\eta - \gamma_\ell| < m$ for $t > -\ell - d$. Hence (13) implies

$$S^-_{\mathbb{R}}[\gamma_\ell] \ge S^-_{[-\ell-d,\ell+d)}[g * d\eta] \ge 2(\nu+1)(\ell-d). \tag{15}$$

To derive a contradiction, let k be given by $k(t) = (1-|t|)\mathbf{1}_{[-1,1]}(t)$, and define the approximate identity k_ε by $k_\varepsilon(t) = \varepsilon^{-1}k(t/\varepsilon)$ where $\varepsilon > 0$. Let $\varepsilon > 0$ be small enough so that the supports of the functions $t \mapsto k_\varepsilon(t - r_j - k)$ where $0 \le j \le n$ and $-\ell \le k < \ell$ are pairwise disjoint. Let $h_{\ell,\varepsilon}$ be given by

$$h_{\ell,\varepsilon}(t) = \sum_{k=-\ell}^{\ell-1} \sum_{j=0}^{n} a_j k_\varepsilon(t - r_j - k),$$

and note that since g is continuous and bounded, the function $g * h_{\ell,\varepsilon}$ converges (uniformly on compact subsets of $\mathbb{R}$) to γ_ℓ. The function $h_{\ell,\varepsilon}$ satisfies $S^-_{\mathbb{R}}[h_{\ell,\varepsilon}] = 2\ell\nu$ by definition of ν and choice of ε. Since g is variation diminishing, $g * h_{\ell,\varepsilon}$ has at most $2\ell\nu$ changes of sign. Lemma 2.2 implies $S^-_{\mathbb{R}}[\gamma_\ell] \le 2\ell\nu$. For sufficiently large ℓ this is a contradiction to (15). $\square$

With a similar argument the following proposition can be proved:

Proposition 2.9. *Let φ be a continuous 1-periodic function and g as in Proposition 2.8. Then*

$$S^-_{[0,1)}[g * \varphi] \le S^-_{[0,1)}[\varphi].$$

We require the following example in the final section. Recall that for $0 < a < b$ the distribution $\nu_{a,b}$ is defined by

$$\nu_{a,b} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} \delta_{bk-a}. \tag{16}$$

Lemma 2.10. *Let $g \in C^2(\mathbb{R})$ be variation diminishing. If $g * d\nu_{a,b}$ is not identically zero, then*

$$S^-_{[0,2b)}[g * d\nu_{a,b}] = 2,$$

and the two sign changes are the only zeros of $g * d\nu_{a,b}$ on $[0, 2b)$.

Proof. Apply Theorem 2.8 to $t \mapsto g * d\nu_{a,b}(2bt)$ to obtain the inequality

$$S^-_{[0,2b)}[g * d\nu_{a,b}] \leq 2. \tag{17}$$

Since $g * d\nu_{a,b}$ is $2b$-periodic, the number of sign changes in $[0, 2b)$ cannot be odd. If $g * d\nu_{a,b}$ has no sign changes then $g * d\nu_{a,b}$ must be identically equal to zero since $g * d\nu_{a,b}$ has mean value zero on $[0, 2b]$.

Define τ_ε to be the piecewise constant function satisfying $\tau_\varepsilon(t) = \varepsilon^{-2}$ for $-\varepsilon \leq t < 0$, $\tau_\varepsilon(-t) = -\tau_\varepsilon(t)$, and $\tau_\varepsilon(t) = 0$ for $|t| > \varepsilon$. We note that

$$g' * d\nu_{a,b}(t) = \lim_{\varepsilon\to 0+} g * \varphi_\varepsilon(t), \tag{18}$$

where $\varphi_\varepsilon(t) = \sum_k (-1)^{k+1}\tau_\varepsilon(t - bk + a)$.

Since for sufficiently small $\varepsilon > 0$ the function $t \mapsto \tau_\varepsilon(t - a) - \tau_\varepsilon(t + b - a)$ has two sign changes on $[0, 2b)$, the derivative of $g * d\nu_{a,b}$ cannot have more than two sign changes by (18), Lemma 2.2, and Theorem 2.8. Since the assumption that $g * d\nu_{a,b}$ has a (by (17) necessarily even) additional zero would imply that its derivative has more than two sign changes, the two sign changes are the only zeros. □

3. Entire Interpolations

Recall that $f_\nu : \mathbb{R} \to \mathbb{R}$ is given by

$$f_\nu(z) = \begin{cases} -\displaystyle\int_{(-\infty,0)} e^{-sz} d\nu(s) & \text{for } z < 0, \\ \displaystyle\int_{[0,\infty)} e^{-sz} d\nu(s) & \text{for } z > 0 \end{cases} \tag{19}$$

and

$$f_\nu(0) = \frac{1}{2}(f_\nu(0-) + f_\nu(0+)) = \frac{1}{2}\int_{\mathbb{R}} \operatorname{sgn}(s) d\nu(s).$$

Let F be a Laguerre-Polyá entire function. We consider in this section measures ν that have finite total variation and are supported on a bounded set. An entire function $G_{\nu,F}$ is constructed that satisfies (4).

The function $G_{\nu,F}$ is initially defined in (22) in a vertical strip $-c < \Re z < c$. Proposition 3.3 gives the analytic continuations to $\Re z > 0$ and $\Re z < 0$. In Propositions 3.5 and 3.6 the special case $F(z) = \sin \pi(z - a)$ for $a \in \mathbb{R}$

is considered, and it is shown that $G_{\nu,F}$ in these cases is an interpolating series of f_ν.

Definition 3.1. We define $\mathcal{M}_b$ to be the class of signed measures on $\mathbb{R}$ that have finite total variation and are supported on a bounded set. For technical reasons we assume that

$$\nu(\{0\}) = 0. \tag{20}$$

Lemma 3.2. *Let $\nu \in \mathcal{M}_b$, let F be a Laguerre-Pólya entire function, and let g satisfy (11) in an open vertical strip $a < \Re z < b$ containing the origin. Then there exists $c > 0$ so that*

$$g * d|\nu|(t) = \mathcal{O}\left(e^{-c|t|}\right). \tag{21}$$

Proof. Let $0 < c < \min(|a|, |b|)$. Lemma 2.6 implies that $g(t) = \mathcal{O}(e^{-c|t|})$. By assumption, there exists $m > 0$ so that $\operatorname{supp}(\nu) \subseteq [-m, m]$. We have for any real t

$$\begin{aligned} \big|g * d|\nu|(t)\big| = \Big|\int_{-m}^{m} g(t-u) d|\nu|(u)\Big| &\leq \Big(\max_{u \in (t-m, t+m)} g(u)\Big)|\nu|(\mathbb{R}) \\ &= \mathcal{O}\left(e^{c(m-|t|)}\right), \end{aligned}$$

which finishes the proof. □

Proposition 3.3. *Let ν, F, and G satisfy the assumptions of Lemma 3.2. Let $0 < c < \min(|a|, |b|)$ and define for $-c < \Re z < c$*

$$\begin{aligned} G_{\nu,F}(z) := F(z)\Big(\int_{-\infty}^{0} e^{-zt} \int_{0+}^{\infty} g(t-u) d\nu(u) dt \\ + \int_{0}^{\infty} e^{-zt} \int_{-\infty}^{0-} g(t-u) d\nu(u) dt\Big). \end{aligned} \tag{22}$$

Then $G_{\nu,F}$ extends to an entire function, and the analytic continuations to the half planes are given by

$$\begin{aligned} G_{\nu,F}(z) &= f_\nu(z) + F(z) \int_{-\infty}^{0} e^{-zt}\{g * d\nu\}(t)\, dt \qquad (\Re z < 0), \\ G_{\nu,F}(z) &= f_\nu(z) - F(z) \int_{0}^{\infty} e^{-zt}\{g * d\nu\}(t)\, dt \qquad (\Re z > 0). \end{aligned} \tag{23}$$

Proof. The estimate (21) implies that f_ν is analytic in $\Re z < 0$ and in $\Re z > 0$. Throughout this proof we write G_0 for the right hand side of (22), and we write G_1 and G_2 for the expressions on the right in (23). The estimate (21) implies that G_0 is analytic in $-c < \Re z < c$, G_1 is analytic in $\Re z < 0$, and G_2 is analytic in $\Re z > 0$. Equations (22) and (23) claim that

$$G_{\nu,F}(z) = \begin{cases} G_0(z) & \text{if } -c < \Re z < c, \\ G_1(z) & \text{if } \Re z < 0, \\ G_2(z) & \text{if } \Re z > 0. \end{cases}$$

Hence, we have to show that $G_0 = G_1$ in the strip $-c < \Re z < 0$, and $G_0 = G_2$ in the strip $0 < \Re z < c$. Since each of the half-planes in (23) has non-empty intersection with the strip in which (22) holds, and their union is all of $\mathbb{C}$, it will follow that $G_{\nu,F}$ is entire.

Define two signed measures ν_- and ν_+ by

$$\begin{aligned} \nu_+(A) &= \nu(A \cap \{x > 0\}) \\ \nu_-(A) &= -\nu(A \cap \{x < 0\}) \end{aligned}$$

and recall that $\nu(\{0\}) = 0$. Consider z with $-c < \Re z < 0$. Equations (19) and (11) imply the representation

$$\begin{aligned} f_\nu(z) &= -\int_{-\infty}^{0-} e^{-zs} d\nu(s) = \int_{-\infty}^{\infty} e^{-zs} d\nu_-(s) \\ &= F(z) \int_{-\infty}^{\infty} e^{-zt} g(t) dt \int_{-\infty}^{\infty} e^{-zt} d\nu_-(t) \\ &= F(z) \int_{-\infty}^{\infty} e^{-zt} \{g * d\nu_-\}(t)\, dt, \end{aligned} \tag{24}$$

since the assumptions on $g*d|\nu|$ imply that Fubini is applicable to the double integral. Equations (22) and (24) imply for z with $-c < \Re z < 0$

$$\begin{aligned} G_0(z) - f_\nu(z) &= F(z)\Big(\int_{-\infty}^{0} e^{-zt} \{g * d\nu_+\}(t)\, dt + \int_{0}^{\infty} e^{-zt} \{g * d\nu_-\}(t)\, dt \\ &\qquad - \int_{-\infty}^{\infty} e^{-zt} \{g * d\nu_-\}(t)\, dt \Big) \\ &= F(z)\Big(\int_{-\infty}^{0} e^{-zt} \{g * d\nu_+\}(t)\, dt - \int_{-\infty}^{0} e^{-zt} \{g * d\nu_-\}(t)\, dt \Big) \\ &= F(z) \int_{-\infty}^{0} e^{-zt} \{g * d\nu\}(t)\, dt \\ &= G_1(z) - f_\nu(z), \end{aligned}$$

and hence $G_0(z) = G_1(z)$ for all z with $-c < \Re z < 0$.

Similarly, in $0 < \Re z < c$

$$f_\nu(z) = F(z) \int_{-\infty}^{\infty} e^{-zt} \{g * d\nu_+\}(t)\, dt$$

and hence

$$\begin{aligned} G_0(z) - f_\nu(z) &= F(z)\Big(\int_{-\infty}^{0} e^{-zt} \{g * d\nu_+\}(t)\, dt + \int_{0}^{\infty} e^{-zt} \{g * d\nu_-\}(t)\, dt \\ &\qquad - \int_{-\infty}^{\infty} e^{-zt} \{g * d\nu_+\}(t)\, dt \Big) \\ &= -F(z) \int_{0}^{\infty} e^{-zt} \{g * d\nu\}(t)\, dt \\ &= G_2(z) - f_\nu(z), \end{aligned}$$

which implies that $G_0(z) = G_2(z)$ for all z with $0 < \Re z < c$. By the remarks at the beginning of the proof, the proposition is established. □

Lemma 3.4. *Let $\nu \in \mathcal{M}_b$. We have in the region $\Re w < \Re z < 0$*

$$\frac{f_\nu(z) - f_\nu(w)}{z - w} = -\int_{(-\infty,0)} \int_s^0 e^{-\tau z} e^{(\tau - s)w} d\tau d\nu(s)$$

and in the region $\Re z < 0 < \Re w$

$$\frac{f_\nu(z) - f_\nu(w)}{z - w} = \Big(\int_{(-\infty,0)} \int_{-\infty}^s + \int_{(0,\infty)} \int_{-\infty}^0 \Big) e^{-\tau z} e^{(\tau - s)w} d\tau d\nu(s).$$

Proof. The integrals with respect to τ on the right can be evaluated without changing the order of integration. □

The following two propositions establish identities between interpolating series and the entire functions of Proposition 3.3.

Proposition 3.5. *Let $\nu \in \mathcal{M}_b$ and let $0 < \alpha < 1$. We define $F_\alpha(z) = -\pi^{-1} \sin \pi(z - \alpha)$. For all $z \in \mathbb{C}$*

$$G_{\nu,F_\alpha}(z) = F_\alpha(z) \sum_{n \in \mathbb{Z}} (-1)^n \frac{f_\nu(n + \alpha)}{z - n - \alpha}. \tag{25}$$

Proof. Denote by $G(z)$ the right-hand side of (25). The functions $z \mapsto F_\alpha(z)(z - n - \alpha)^{-1}$ are entire functions that are bounded in every compact subset of $\mathbb{C}$, and $f_\nu(n + \alpha)$ decays exponentially as $n \to \pm\infty$ since $\nu \in \mathcal{M}_b$. Hence G is an entire function. By Proposition 3.3 the function $G_{\nu,F}$ is entire as well. In order to prove that $G = G_{\nu,F}$ it suffices to show that $G = G_{\nu,F}$ in a vertical strip to the left of the origin, since analytic continuation implies that the two entire functions agree on $\mathbb{C}$ once they agree on a set with a limit point.

It is possible to show this claim directly using (22), but the calculations are very involved. It is technically easier to use one of the representations of (23). We consider therefore $f_\nu - G$ and show that it equals $f_\nu - G_{\nu,F_\alpha}$ in $\alpha - 1 < \Re z < 0$ where $0 < \alpha < 1$.

The partial fraction expansion [1, (4.3.93) on p. 75]

$$\lim_{N \to \infty} \sum_{n=-2N}^{2N-1} \frac{(-1)^{n+1}}{z - n} = -\frac{\pi}{\sin \pi z} \tag{26}$$

implies after substituting $z - \alpha$ for z

$$\frac{f_\nu(z) - G(z)}{F_\alpha(z)} = \lim_{N \to \infty} \sum_{n=-2N}^{2N-1} (-1)^{n+1} \frac{f_\nu(z) - f_\nu(n + \alpha)}{z - n - \alpha}. \tag{27}$$

Let $\alpha - 1 < \Re z < 0$. If $n \leq -1$ then

$$n + \alpha \leq \alpha - 1 < \Re z < 0,$$

hence Lemma 3.4 gives for $n \leq -1$

$$\frac{f_\nu(z) - f_\nu(n+\alpha)}{z-n-\alpha} = -\iint_{s<\tau<0} e^{(\tau-s)(n+\alpha)} e^{-\tau z} d\tau d\nu(s) \tag{28}$$

and for $n \geq 0$

$$\frac{f_\nu(z) - f_\nu(n+\alpha)}{z-n-\alpha} = \Big(\iint_{\tau<s<0} + \iint_{\tau<0\leq s} \Big) e^{(\tau-s)(n+\alpha)} e^{-\tau z} d\tau d\nu(s). \tag{29}$$

Recall that $g_\alpha(t) = e^{\alpha t}(e^t+1)^{-1}$ and note that

$$(e^{-2N|t|} - 1) g_\alpha(t) = \begin{cases} -\displaystyle\sum_{n=-2N}^{-1} (-1)^{n+1} e^{(n+\alpha)t} \text{ for } t > 0, \\ \displaystyle\sum_{n=0}^{2N-1} (-1)^{n+1} e^{(n+\alpha)t} \text{ for } t < 0. \end{cases} \tag{30}$$

We multiply (28) and (29) by $(-1)^{n+1}$, sum (28) over $-2N \leq n \leq -1$, and sum (29) over $0 \leq n \leq 2N-1$. After moving the finite sums inside the integrals we apply (30) with $t = \tau - s$. In (28) we have $\tau - s > 0$ and in both integrals of (29) we have $\tau - s < 0$. The assumptions of Lebesgue dominated convergence can be checked directly. After taking the limit $N \to \infty$, summation of the resulting three integrals gives in $\alpha - 1 < \Re z < 0$

$$f_\nu(z) - G(z) = -F_\alpha(z) \int_{\mathbb{R}} \int_{-\infty}^0 e^{-\tau z} g_\alpha(\tau - s) d\tau d\nu(s).$$

Since ν has bounded support, $g_\alpha * d|\nu|$ decays exponentially, hence Fubini implies in $\alpha - 1 < \Re z < 0$

$$\begin{aligned} f_\nu(z) - G(z) &= -F_\alpha(z) \int_{-\infty}^0 e^{-zt} \{g_\alpha * d\nu\}(t)\, dt \\ &= f_\nu(z) - G_{\nu, F_\alpha}(z) \end{aligned}$$

by (12), Lemma 2.5, and Proposition 3.3. By the remarks at the beginning of the proof, (25) is shown. $\square$

We encounter a technical difficulty when considering

$$F_0(z) = -\pi^{-1} \sin \pi z.$$

The Laplace inverse transformation of $F_0(z)^{-1}$ is not integrable, and an integration by parts becomes necessary to obtain a representation in the strip $-1 < \Re z < 1$ as a Laplace transform of an integrable function. To set this up, we define

$$h(t) = \begin{cases} -(1+e^t)^{-1} \text{ if } t < 0, \\ (1+e^{-t})^{-1} \text{ if } t \geq 0. \end{cases} \tag{31}$$

We set $h'(0) = h'(0+)$. Since $-(1+e^t)^{-1} = -1+(1+e^{-t})^{-1}$, the function h' is analytic for $|\Im t| < \frac{1}{2}$. We have the Radon-Nikodym decomposition

$$dh(t) = h'(t)dt + d\delta. \tag{32}$$

Since ν is a measure of bounded variation with bounded support and $|h|$ decays exponentially, the function $h * d\nu$ has bounded variation, and (32) implies for any measurable set $A \subseteq \mathbb{R}$

$$\int_A d\{h * d\nu\}(t) = \int_A h'(t)dt + \int_A d\nu(t). \tag{33}$$

Proposition 3.6. *Let $\nu \in \mathcal{M}_b$. Define $E(z) = -(\pi z)^{-1}\sin \pi z$. Then for all $z \in \mathbb{C}$*

$$G_{\nu,E}(z) = F_0(z)\Big(\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} (-1)^{n+1}\frac{f_\nu(n)}{z-n} + \frac{\{h * d\nu\}(0)}{z}\Big) \tag{34}$$

Proof. We define

$$G(z) := F_0(z)\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} (-1)^{n+1}\frac{f_\nu(n)}{z-n}.$$

The partial fraction expansion (26) implies

$$\frac{f_\nu(z) - G(z)}{F_0(z)} = \lim_{N\to\infty}\sum_{\substack{n=-2N\\ n\neq 0}}^{2N} (-1)^{n+1}\frac{f_\nu(z) - f_\nu(n)}{z-n} - \frac{f_\nu(z)}{z}. \tag{35}$$

Let $-1 < \Re z < 0$. We note

$$-(1-e^{-2N|t|})h(t) = \begin{cases} -\displaystyle\sum_{n=-2N}^{-1}(-1)^{n+1}e^{nt} \text{ if } t > 0,\\ \displaystyle\sum_{n=1}^{2N}(-1)^{n+1}e^{nt} \text{ if } t < 0.\end{cases}$$

An expansion of the summands in (35) using Lemma 3.4 with $\alpha = 0$ and an application of dominated convergence gives (recall $\nu(\{0\}) = 0$)

(36)
$$\begin{aligned}
&\frac{f_\nu(z) - G(z)}{F_0(z)} + \frac{f_\nu(z)}{z} \\
&= \lim_{N\to\infty} \left(\sum_{n=-2N}^{-1} (-1)^{n+1} \left(\int_{-\infty}^{0-} \int_{-\infty}^{s} + \int_{0+}^{\infty} \int_{-\infty}^{0} \right) e^{-\tau z} e^{(\tau - s)n} d\tau d\nu(s) \right. \\
&\qquad \left. - \sum_{n=1}^{2N} (-1)^{n+1} \int_{-\infty}^{0-} \int_{s}^{0} e^{-\tau z} e^{(\tau-s)n} d\tau d\nu(s) \right) \\
&= - \int_{\mathbb{R}} \int_{-\infty}^{0} e^{-\tau z} h(t) dt\, d\nu(s).
\end{aligned}$$

An application of Fubini's theorem gives

$$f_\nu(z) - G(z) = -F_0(z) \left(\int_{-\infty}^{0} e^{-zt} \{h * d\nu\}(t)\, dt + \frac{f_\nu(z)}{z} \right) \tag{37}$$

for $-1 < \Re z < 0$. An integration by parts gives

$$\int_{-\infty}^{0} e^{-zt} \{h * d\nu\}(t)\, dt = -\frac{\{h * d\nu\}(0)}{z} + \frac{1}{z} \int_{-\infty}^{0} e^{-zt} d\big[\{h * d\nu\}(t)\big],$$

and (33), (37), and (19) lead to

$$f_\nu(z) - G(z) = -\frac{F_0(z)}{z} \left(\int_{-\infty}^{0} e^{-zt} \{h' * d\nu\}(t)\, dt - \{h * d\nu\}(0) \right). \tag{38}$$

Since $h' = g_0'$ almost everywhere, an integration by parts in (12) implies

$$\frac{z}{F_0(z)} = \int_{-\infty}^{\infty} e^{-zt} h'(t) dt$$

in the strip $-1 < \Re z < 1$. Lemma 2.5 and Proposition 3.3 imply

$$-\frac{F_0(z)}{z} \int_{-\infty}^{0} e^{-zt} \{h' * d\nu\}(t)\, dt = f_\nu(z) - G_{\nu,E}(z), \tag{39}$$

and the right hand side is analytic in $\mathbb{C} \backslash i\mathbb{R}$. Inserting (39) into (38) proves (34). $\square$

4. Best approximations by functions of exponential type

Let $0 < a < b$, and recall that $\nu_{m,a,b}$ is given by

$$\nu_{m,a,b} = \sum_{k=1-2m}^{2m} (-1)^{k+1} \delta_{bk-a}, \tag{40}$$

this signed measure has finite total variation, bounded support, and satisfies $\nu(\{0\}) = 0$. We note that for real x the corresponding $f_{\nu_{m,a,b}}$ is given by

$$f_{\nu_{m,a,b}}(x) = \left(1 - e^{-2m|x|}\right) e^{ax}(1 + e^{bx})^{-1}.$$

The function $f_{\nu_{m,a,b}}$ has an analytic extension to $\Re z < 0$ and to $\Re z > 0$. In particular, $f_{\nu_{m,a,b}}$ converges to

$$f_{a,b}(z) = e^{az}(1 + e^{bz})^{-1} \tag{41}$$

uniformly on compact sets in $\mathbb{C}\backslash i\mathbb{R}$. We use the representations obtained in Section 3 to construct best $L^1(\mathbb{R})$-approximations from the class of entire functions of exponential type $\le \eta$ to $f_{a,b}$.

Let $\nu_{a,b}$ be the distribution given by

$$\nu_{a,b} = \sum_{k\in\mathbb{Z}} (-1)^{k+1} \delta_{bk-a},$$

and note that for $0 < a < b$ (with an abuse of notation for the application of the distribution $d\nu_{a,b}$)

$$\begin{aligned} f_{a,b}(z) &= \begin{cases} \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} e^{-z(bn-a)} & \text{for } \Re z > 0, \\ \displaystyle -\sum_{n=-\infty}^{0} (-1)^{n+1} e^{-z(bn-a)} & \text{for } \Re z < 0 \end{cases} \\ &= \begin{cases} \displaystyle\int_0^{\infty} e^{-zt} d\nu_{a,b}(t) & \text{for } \Re z > 0, \\ \displaystyle -\int_{-\infty}^{0} e^{-zt} d\nu_{a,b}(t) & \text{for } \Re z < 0. \end{cases} \end{aligned} \tag{42}$$

For $z \in \mathbb{C}$ and $0 < a < b$ we let

$$K_{a,b}(z) = \frac{\sin \pi(z - b^{-1}a)}{\pi} \sum_{n=-\infty}^{\infty} (-1)^{n+1} \frac{f_{a,b}(n + b^{-1}a)}{z - n - b^{-1}a}.$$

By construction $K_{a,b}$ is entire and in $L^2(\mathbb{R})$ and has exponential type $\le \pi$. We note that $g_{b^{-1}a} * d\nu_{m,a,b}$ converges pointwise to

$$t \mapsto \{g_{b^{-1}a} * d\nu_{a,b}\}(t) = \sum_{k\in\mathbb{Z}} (-1)^k g_{b^{-1}a}(t - bk + a). \tag{43}$$

We require evaluation of a series related to $g_{b^{-1}a} * d\nu_{a,b}$.

Lemma 4.1. *For every a, b with $0 < a < b$*

$$\sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-ak}}{1 + e^{a-bk}} = 0. \tag{44}$$

Proof. We note that for $0 < a < b$

$$\sum_{n=0}^{\infty} e^{an} \sum_{k=1}^{\infty} e^{-k(a+bn)} = \sum_{n=0}^{\infty} \frac{e^{an}}{e^{a+bn}-1} < \infty,$$

hence $(k,n) \mapsto (-1)^{k+n} e^{-ak+(a-kb)n}$ is in $L^1(\mathbb{N} \times \mathbb{N}_0)$ and Fubini's theorem gives

$$\sum_{k=1}^{\infty} (-1)^k \frac{e^{-ak}}{1+e^{a-bk}} = \sum_{k=1}^{\infty} (-1)^k e^{-ak} \sum_{n=0}^{\infty} (-1)^n e^{(a-bk)n}$$
$$= -\sum_{n=0}^{\infty} (-1)^n \frac{e^{an}}{1+e^{a+bn}}$$

which proves the claim after the substitution $n = -k$ on the right hand side. $\square$

We extend the representation of Proposition 3.5 to the distribution $\nu_{a,b}$.

Lemma 4.2. *Let $0 < a < b$ and recall that $F_\eta(z) = -\pi^{-1} \sin \pi(z-\eta)$. For any $z \in \mathbb{C}$*

$$\lim_{m\to\infty} G_{\nu_{m,a,b}, F_{b^{-1}a}}(z) = K_{a,b}(z) \tag{45}$$

and

(46)
$$-\frac{f_{a,b}(z) - K_{a,b}(z)}{\pi^{-1}\sin \pi(z - b^{-1}a)} = \begin{cases} \displaystyle\int_0^\infty e^{-zt} \{g_{b^{-1}a} * d\nu_{a,b}\}(t)\, dt \text{ for } \Re z > 0 \\ \displaystyle -\int_{-\infty}^0 e^{-zt} \{g_{b^{-1}a} * d\nu_{a,b}\}(t)\, dt \text{ for } \Re z < 0. \end{cases}$$

Proof. From Proposition 3.5 we obtain for fixed $z \in \mathbb{C}\backslash i\mathbb{R}$

$$G_{\nu_{m,a,b}, F_{b^{-1}a}}(z) = F_{b^{-1}a}(z) \sum_{n\in\mathbb{Z}} (-1)^n \frac{f_{\nu_{m,a,b}}(n + b^{-1}a)}{z - n - b^{-1}a},$$

and the right-hand side converges to $K_{a,b}$ in $L^2(\mathbb{R})$ as $m \to \infty$. Since $K_{a,b}$ is entire, (45) follows. To prove the second identity we note that for all real t, $m \in \mathbb{N}$ and $0 < a < b$

$$\{g_{b^{-1}a} * d|\nu_{m,a,b}|\}(t) \le \sum_{k\in\mathbb{Z}} \frac{e^{\frac{a}{b}(t-bk+a)}}{e^{t-bk+a}+1} < \infty. \tag{47}$$

Hence, an application of Lebesgue dominated convergence in (23) for $\Re z \neq 0$ with $d\nu = d\nu_{m,a,b}$, $g = g_{b^{-1}a}$ and $F = F_{b^{-1}a}$ implies (46). $\square$

Theorem 4.3. *Let $0 < a < b$. Then for any entire F of exponential type $\le \pi\eta$*

$$||f_{a,b} - F||_1 \ge \frac{1}{b\eta} \left| \sum_{\mu\in\mathbb{Z}} \frac{e^{-2\pi i \frac{a}{b}(\mu+1/2)}}{(\mu+\frac{1}{2})} \csc\left(\tfrac{\pi\eta}{b}(\tfrac{a}{\eta} + \pi i(2\mu+1))\right) \right|$$

with equality if and only if $x \mapsto F(x) = K_{\eta^{-1}a,\eta^{-1}b}(\eta x)$.

Proof. We consider first $\eta = 1$. We note that $t \mapsto g_{b^{-1}a} * d\nu_{a,b}(t)$ is a $2b$-periodic function. Using this in both integrals in (46) gives

$$f_{a,b}(z) - K_{a,b}(z) = -\frac{\sin \pi(z - b^{-1}a)}{\pi(1 - e^{-2z})} \int_0^{2b} e^{-zt}\{g_{b^{-1}a} * d\nu_{a,b}\}(t)\, dt \tag{48}$$

for all $z \in \mathbb{C}$. (Note that $g_{b^{-1}a} * d\nu_{a,b}$ has mean-value zero on $[0,2]$.) By (44) we have $g_{b^{-1}a} * d\nu_{a,b}(0) = 0$ and hence

$$g_{b^{-1}a} * d\nu_{a,b}(t) = \mathcal{O}(|t|) \tag{49}$$

in a neighborhood of the origin. Equations (49) and (46) imply

$$f_{a,b}(z) - K_{a,b}(z) = \mathcal{O}(|1+z|^{-2}).$$

We show next that $\pi[\sin \pi(z - b^{-1}a)]^{-1}(f_{a,b}(z) - K_{a,b}(z))$ is of one sign on the real line. Lemma 2.10 implies that $S^{-}_{[0,2b)}[g_{b^{-1}a} * d\nu_{a,b}] \le 2$. Since $g_{b^{-1}a} * d\nu_{a,b}(0) = 0$, Lemma 2.10 implies that $t = 0$ is one of the sign changes, and since

$$g_{b^{-1}a} * d\nu_{a,b}(b+t) = -g_{b^{-1}a} * d\nu_{a,b}(t), \tag{50}$$

the other sign change is at $t = b$. The location of the sign changes and (50) imply that the integral in (48) has exactly one sign change on the real line, namely at the origin. Since $z \mapsto 1 - e^{-2z}$ has its only sign change at the origin as well, it follows that

$$\Big| \int_{-\infty}^{\infty} \operatorname{sgn} \sin \pi(x - b^{-1}a)(f_{a,b}(x) - K_{a,b}(x))dx \Big| = ||K_{a,b} - f_{a,b}||_1. \tag{51}$$

Consider an arbitary $F \in L^1(\mathbb{R})$ that has exponential type $\le \pi$. Since the partial sums of the Fourier expansion

$$\text{sgn } \sin \pi x = \frac{2}{\pi i} \lim_{N\to\infty} \sum_{|\mu| \le N} \frac{e^{2\pi i x(\mu+1/2)}}{2\mu + 1}$$

are uniformly bounded, integration and limit may be interchanged in the following calculation. The Paley Wiener theorem implies

$$\int_{-\infty}^{\infty} F(x) \operatorname{sgn}(\sin \pi(x - b^{-1}a))dx = 0.$$

With

$$\widehat{f}_{a,b}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx$$

and noting $\widehat{f}_{a,b}(t) = \frac{\pi}{b}\csc[\frac{\pi}{b}(a - 2\pi i t)]$ we obtain that

$$\begin{aligned}\int_{-\infty}^{\infty} &(f_{a,b}(x) - F(x))\,\mathrm{sgn}\sin\pi(x - b^{-1}a)dx \\ &= \frac{2}{\pi i}\lim_{N\to\infty}\sum_{|\mu|\le N}\frac{e^{-2\pi i\frac{a}{b}(\mu+1/2)}}{2\mu+1}\widehat{f}_{a,b}(-\mu-1/2) \\ &= \frac{2}{ib}\sum_{\mu\in\mathbb{Z}}\frac{e^{-2\pi i\frac{a}{b}(\mu+1/2)}}{2\mu+1}\csc\tfrac{\pi}{b}(a+\pi i(2\mu+1)).\end{aligned} \tag{52}$$

The absolute value of the right-hand side is therefore a lower bound for $||F - f_{a,b}||_1$, and this lower bound is assumed for $F = K_{a,b}$ by (51).

We note that for any $\eta > 0$

$$f_{a,b}(\eta^{-1}x) = f_{\eta^{-1}a,\eta^{-1}b}(x),$$

which implies that for F of exponential type $\le \eta\pi$

$$||f_{a,b} - F||_1 \ge \frac{2}{ib\eta}\sum_{\mu\in\mathbb{Z}}\frac{e^{-2\pi i\frac{a}{b}(\mu+1/2)}}{2\mu+1}\csc\tfrac{\eta\pi}{b}(\tfrac{a}{\eta} + \pi i(2\mu+1))$$

with equality for $z \mapsto F(z) = K_{\eta^{-1}a,\eta^{-1}b}(\eta z)$.

If there exists another F such that $||F - f_{a,b}||_1$ is minimal, then (52) implies that $F(\eta^{-1}(b^{-1}a+m)) = f_{a,b}(\eta^{-1}(b^{-1}a+m)) = K_{\eta^{-1}a,\eta^{-1}b}(\eta^{-1}(b^{-1}a+m))$, and since $x \mapsto F(x) - K_{\eta^{-1}a,\eta^{-1}b}(\eta^{-1}x) \in L^1(\mathbb{R})$, it has to be identically zero by (7.20) in Chapter XVI of Zygmund [18]. □

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