ZEROS OF BERNOULLI-TYPE FUNCTIONS AND BEST APPROXIMATIONS

FRIEDRICH LITTMANN

ABSTRACT. The zero sets of $(D+a)^n g(t)$ with $D = d/dt$ in the (t,a)-plane are investigated for $g(t) = te^{\alpha t}(e^t - 1)^{-1}$ and $g(t) = e^{\theta t}(e^t + 1)^{-1}$.

The results are used to determine entire interpolations to functions $x_+^n e^{-\lambda x}$, which give representations for the best approximation and best one-sided approximation from the class of functions of exponential type $\eta > 0$ to $x_+^n e^{-\lambda x}$.

1. INTRODUCTION

Let $\lambda > 0$ and $n \in \mathbb{N}_0$, and let x_+ be the positive part of x. This article treats the problem of finding the best upper and lower one-sided approximation by entire functions of fixed exponential type η to

$$f_{\lambda,n}(x) := x_+^n e^{-\lambda x}.$$

The term one-sided upper approximation refers to a function A such that $A(x) \geq f(x)$ for all real x, and the approximation error is defined as the $L^1(\mathbb{R})$ norm of $A - f$. Lower approximations are defined analogously. An entire function A is said to be of exponential type $\eta > 0$ if for every $\varepsilon > 0$ exists C_ε such that A satisfies the inequality

$$|A(z)| \leq C_\varepsilon e^{|z|(\eta+\varepsilon)}$$

in the complex plane. The class of these functions will be denoted by $\mathcal{A}(\eta)$.

For $\lambda = 0$ and $n = 0$ this problem is considered by A. Beurling [1], and by A. Selberg in chapter 20 of Selberg [12] in connection with a form of the large sieve inequality. Further applications are given by J. D. Vaaler [13]. S. W. Graham and J. D. Vaaler [2] consider the case $n = 0$ and $\lambda > 0$ in connection with Tauberian theorems (see also Korevaar [7]). The case of several variables is considered in Holt and Vaaler [6]. The case of arbitrary $n \in \mathbb{N}_0$ and $\lambda = 0$ is treated by the author in [8] and [9].

Best approximations are frequently obtained as corollaries of interpolation theorems. The idea is as follows: let F be an entire function that is real on the real line and has only real zeros, then find an entire function G that interpolates f at the zeros of F (with the correct multiplicity) and nowhere else. Best one-sided approximations are obtained by using $F(z) = \sin^2 \pi(z - \alpha)$ with a parameter $\alpha \in \mathbb{R}$.

The interpolation theorems often require F to be a Laguerre-Pólya entire function (see Definition 3.3). By a result of I.J. Schoenberg [11], the reciprocal $F(z)^{-1}$ is representable as a two-sided Laplace transform of some totally positive function g (see Definition 3.2) in some vertical strip in the complex plane. This property of g is helpful when proving that $F(x)$ and $G(x) - f(x)$ have the same sign for all real x.

For given F it is therefore the goal to find an entire function $G_{\lambda,n,F}$ which interpolates $f_{\lambda,n}$ at the zeros of F. It is shown in Lemma 4.2 that under some conditions such an interpolation is given by the formula

$$G_{\lambda,n,F}(z) := \frac{F(z)}{z}\int_{-\infty}^{\infty} e^{-zt} g^{(n+1)}(t-\lambda)dt, \tag{1}$$

provided

$$g^{(n)}(-\lambda) = 0 \tag{2}$$

holds.

It is shown in Section 4 that the techniques of [8] are applicable to this problem. Most of the properties of (1) are corollaries of the interpolation theorems in [8]. The remaining difficulty is to ensure that (2) is satisfied. We consider (2) for the functions

$$g_\alpha(t) := te^{\alpha t}(e^t - 1)^{-1},$$
$$h_\theta(t) := e^{\theta t}(e^t + 1)^{-1}.$$

As a consequence of the interpolation formula (1), a representation for the one-sided best approximation from the set of entire functions of exponential type η to $f_{\lambda,n}$ is obtained. The function F in (1) turns out to be of the form $\sin^2 \pi(z-\alpha)$. As is shown in Section 4, equation (2) needs to hold for g_α and gives an implicit relation that defines the parameter α as a function of λ.

In fact, the best approximation to $f_{\lambda,n}$ (without the one-sided condition) can be obtained as a corollary of (1) as well. In this case (2) needs to hold for h_θ and (2) implicitly defines θ as a function of λ. For $n = 0$, the best approximation was first obtained by U. Haagerup and L. Zsido [4].

2. Results

The zero sets of $g_\alpha^{(n)}(t)$ are plotted in Figure 1 for different n. The behavior in the horizontal strip $0 \le \alpha \le 1$ is of main interest for the interpolation problem described above, and we will restrict our investigation to this region.

We have $g_\alpha(t) \ge 0$ for all α and t. The zero set of $g'_\alpha(t)$ in the (t,α)-plane is parametrized by the curve

$$\alpha = \frac{e^{-t} - 1 + t}{t(1-e^t)}. \tag{3}$$

Proposition 2.1. *Let $n \ge 2$ and recall $g_\alpha(t) = te^{\alpha t}(e^t-1)^{-1}$.*

(1) *The function $\alpha \mapsto g_\alpha^{(n)}(t)$ has two simple and no multiple zeros in $0 \le \alpha < 1$.*

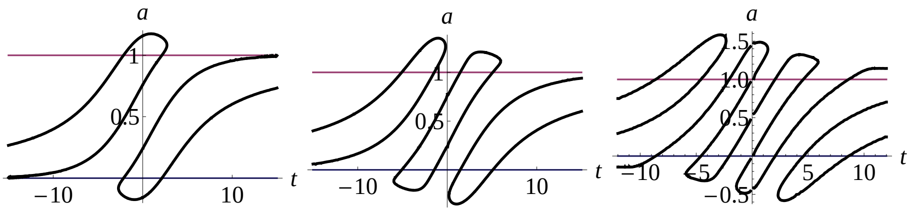

FIGURE 1. The zeros of $(t,\alpha) \mapsto g_\alpha^{(n)}(t)$ for $n \in \{4,6,9\}$

(2) *The function $t \mapsto g_\alpha^{(n)}(t)$ has n simple zeros $t_{1,n}(\alpha) < t_{2,n}(\alpha) < \ldots < t_{n,n}(\alpha)$, and no multiple zeros. Viewed as functions of α, these zeros are continuous and monotonically increasing on $(0,1)$. Moreover,*

$$\lim_{\alpha\to 1-} t_{n,n-1}(\alpha) = \lim_{\alpha\to 1-} t_{n,n}(\alpha) = \infty$$
$$\lim_{\alpha\to 0+} t_{n,1}(\alpha) = \lim_{\alpha\to 0+} t_{n,2}(\alpha) = -\infty,$$

and the limits of the remaining $t_{n,j}$ as $\alpha \to 0+$ and $\alpha \to 1-$ are finite.

The proof of Proposition 2.1 is given in Section 3.

The applications in Section 4 require the change of variable $\lambda = -t$ and inversion of the functions $t_{j,n}$. Hence for $n \geq 2$, denote the two zeros of $\alpha \mapsto g_\alpha^{(n)}(-\lambda)$, $0 \leq \alpha < 1$, by $\alpha_n(\lambda)$ and $\beta_n(\lambda)$, respectively, where $g_{\alpha_n(\lambda)}^{(n+1)}(-\lambda) > 0$ and $g_{\beta_n(\lambda)}^{(n+1)}(-\lambda) < 0$. We obtain after inverting the functions $t_{n,j}$ of Proposition 2.1 and substituting $\lambda = -t$ in the inverted functions:

Corollary 2.2. *Let $n \geq 2$. The functions α_n and β_n are piecewise continuous and monotonically decreasing on their intervals of continuity, and they satisfy*

$$\lim_{\lambda\to -\infty} \alpha_n(\lambda) = \lim_{\lambda\to -\infty} \beta_n(\lambda) = 1,$$
$$\lim_{\lambda\to +\infty} \alpha_n(\lambda) = \lim_{\lambda\to +\infty} \beta_n(\lambda) = 0.$$

Moreover, for any λ such that $\alpha_n(\lambda) = 0$ we have $\alpha_n(\lambda+) = 1$, and the same statement holds for $\beta_n(\lambda)$.

Definition 2.3. We extend the definition of α_n and β_n to $n \in \{0,1\}$ by setting $\alpha_0(\lambda) = 1$, $\beta_0(\lambda) = \alpha_1(\lambda) = 0$, and $\beta_1(\lambda) = (e^\lambda - \lambda - 1)\lambda^{-1}(e^\lambda - 1)^{-1}$.

The corresponding statements for the function $h_\theta(t) = e^{\theta t}(e^t + 1)^{-1}$ are as follows. Examples are plotted in Figure 2.

Proposition 2.4. *Let $n \geq 1$ and recall $h_\theta(t) = e^{\theta t}(e^t + 1)^{-1}$.*

(1) *The function $\theta \mapsto h_\theta^{(n)}(t)$ has one simple and no multiple zeros in $0 \leq \theta < 1$.*

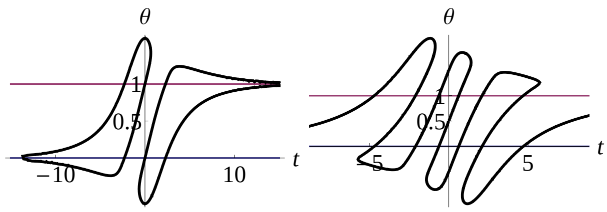

FIGURE 2. The zeros of $(t, \theta) \mapsto h_\theta^{(n)}(t)$ for $n \in \{4, 7\}$

(2) *The function* $t \mapsto h_\theta^{(n)}(t)$ *has* n *simple zeros* $\tau_{1,n}(\theta) < \tau_{2,n}(\theta) < \ldots < \tau_{n,n}(\theta)$*, and no multiple zeros. Viewed as functions of* θ*, these zeros are continuous and monotonically increasing on* $(0, 1)$*. Moreover,*

$$\lim_{\theta \to 1-} \tau_{n,n}(\theta) = \infty$$
$$\lim_{\theta \to 0+} \tau_{n,1}(\theta) = -\infty,$$

and the limits of the remaining $\tau_{n,j}$ *as* $\theta \to 0+$ *and* $\theta \to 1-$ *are finite.*

The proof of Proposition 2.4 is given in Section 3.

Let $\theta_n(\lambda)$ be the zero of $t \mapsto h_\theta^{(n)}(-\lambda)$ in $0 \le \theta < 1$. Proposition 2.4 implies

Corollary 2.5. *Let* $n \ge 1$*. The function* θ_n *is piecewise continuous and monotonically decreasing on its intervals of continuity, and it satisfies*

$$\lim_{\lambda \to -\infty} \theta_n(\lambda) = 1,$$
$$\lim_{\lambda \to +\infty} \theta_n(\lambda) = 0.$$

Moreover, for any λ *such that* $\theta_n(\lambda) = 0$ *we have* $\theta_n(\lambda+) = 1$*.*

We define for $\lambda > 0$, non-negative integers n, and $\alpha \in [0, 1]$,

$$I_{\lambda,n}(\alpha) := \sum_{k=0}^{\infty} (k + \alpha)^n e^{-\lambda(k+\alpha)}, \tag{4}$$

and we set for $\Re z < \alpha$

$$G_{\lambda,n,\alpha}(z) := \frac{\sin^2 \pi(z - \alpha)}{\pi^2 z} \int_{-\infty}^{0} e^{-zt} g_\alpha^{(n+1)}(t - \lambda) dt. \tag{5}$$

Let $\delta > 0$. Corollary 2.2 is used to find the best one-sided approximations to $x_+^n e^{-\lambda x}$ from the class of entire functions of exponential type δ. The case $n = 0$ was first proved by S.W. Graham and J.D. Vaaler [2]. In order to avoid excessive notation, Theorem 2.6 considers $\delta = 2\pi$. The general case is given in Corollary 2.7.

Theorem 2.6. *Let $\mathcal{A}(\delta)$ be the class of entire functions of exponential type δ. Let $n \in \mathbb{N}_0$ and $\lambda > 0$. The inequality*

$$G_{\lambda,n,\alpha_n(\lambda)}(x) \le x_+^n e^{-\lambda x} \le G_{\lambda,n,\beta_n(\lambda)}(x) \tag{6}$$

holds for all $x \in \mathbb{R}$. Moreover,

(i) *for any $A^+ \in \mathcal{A}(2\pi)$ satisfying $A^+(x) \ge x_+^n e^{-\lambda x}$*

$$\int_{-\infty}^{\infty} A^+(x)dx \ge I_{\lambda,n}(\beta_n(\lambda)) \tag{7}$$

with equality if and only if $A^+ = G_{\lambda,n,\beta_n(\lambda)}$.

(ii) *for any $A^- \in \mathcal{A}(2\pi)$ satisfying $A^-(x) \le x_+^n e^{-\lambda x}$*

$$\int_{-\infty}^{\infty} A^-(x)dx \le I_{\lambda,n}(\alpha_n(\lambda)) \tag{8}$$

with equality if and only if $A^- = G_{\lambda,n,\alpha_n(\lambda)}$.

The proof of the general version is given in Section 4. A scaling argument gives

Corollary 2.7. *The unique best lower and upper one-sided $L^1(\mathbb{R})$ – approximation from $\mathcal{A}(2\pi\delta)$ to $x_+^n e^{-\lambda x}$ is given by*

$$\delta^{-n} G_{\delta^{-1}\lambda,n,\alpha_n(\delta^{-1}\lambda)}(\delta x) \le x_+^n e^{-\lambda x} \le \delta^{-n} G_{\delta^{-1}\lambda,n,\beta_n(\delta^{-1}\lambda)}(\delta x).$$

Let $\theta_0(\lambda) = 0$ for all λ. We define for $n \in \mathbb{N}_0$

$$H_{\lambda,n,\theta}(z) := \frac{\sin \pi(z-\theta)}{\pi z} \int_{-\infty}^{0} e^{-zt} h_\theta^{(n+1)}(t-\lambda)dt \tag{9}$$

and the Fourier series

$$J_{\lambda,n}(x) = \frac{n!}{\pi} \sum_{\mu \in \mathbb{Z}} \frac{1}{(\mu+\frac{1}{2})(\lambda - 2\pi i(\mu+\frac{1}{2}))^{n+1}} e^{-\pi i x(2\mu+1)}. \tag{10}$$

It is shown in Section 4 that Corollary 2.5 can be used to give the best approximation from $\mathcal{A}(\delta)$ to $x_+^n e^{-\lambda x}$ in the $L^1(\mathbb{R})$–norm. The following theorem gives the statement for $\delta = \pi$, the general case can be obtained with a scaling argument. The case $n = 0$ of Theorem 2.8 was solved with a different method by U. Haagerup and L. Zsido [4].

Theorem 2.8. *Let $n \in \mathbb{N}_0$ and $\lambda > 0$. The inequality*

$$\int_{-\infty}^{\infty} |A(x) - x_+^n e^{-\lambda x}|\, dx \ge |J_{\lambda,n}(\theta_n(\lambda))| \tag{11}$$

holds for all $A \in \mathcal{A}(\pi)$ with equality if and only if $A = H_{\lambda,n,\theta_n(\lambda)}$.

3. Zeros of Generalized Bernoulli Functions

Recall that $g_\alpha(t) = te^{\alpha t}(e^t-1)^{-1}$. To analyze the behavior of $g_\alpha^{(n)}(t)$ for fixed t it is convenient to define auxiliary functions

$$B_n(\alpha, t) := e^{-\alpha t} g_\alpha^{(n)}(t).$$

A power series expansion of $g_\alpha(z+t)$ in z implies the identity

$$(z+t)e^{\alpha z}(e^{z+t}-1)^{-1} = \sum_{n=0}^{\infty} \frac{B_n(\alpha,t)}{n!} z^n \quad (|z+t| < 2\pi). \tag{12}$$

Differentiating (12) with respect to α and comparing coefficients for $n \geq 1$ gives

$$nB_{n-1}(\alpha, t) = \frac{d}{d\alpha} B_n(\alpha, t), \tag{13}$$

and integrating (12) gives for $n \geq 1$

$$\int_0^1 e^{\alpha t} B_n(\alpha, t) d\alpha = 0. \tag{14}$$

The classical Bernoulli polynomials are given by $\alpha \mapsto B_n(\alpha, 0)$, and it is shown in the proof of the following lemma that for every real t the function $B_n(\alpha, t)$ is a polynomial of degree n in α.

Lemma 3.1. *Let $n \geq 2$. We have $e^t B_n(1,t) = B_n(0,t)$. For fixed $t \in \mathbb{R}$ the function $\alpha \mapsto B_n(\alpha, t)$ has two simple and no multiple zeros in $[0,1)$.*

Proof. We start with

$$\frac{d}{d\alpha}\left[e^{t\alpha} B_n(\alpha,t)\right] = e^{t\alpha}\Big(\frac{d}{d\alpha} B_n(\alpha,t) + tB_n(\alpha,t)\Big)$$

and integrate from $\alpha = 0$ to $\alpha = 1$. This gives with (13)

$$e^t B_n(1,t) - B_n(0,t) = n\int_0^1 e^{t\alpha} B_{n-1}(\alpha,t) d\alpha + t\int_0^1 e^{t\alpha} B_n(\alpha,t) d\alpha,$$

and the first statement of the lemma follows from (14).

By definition $B_0(\alpha, t) = e^{-\alpha t} g_\alpha(t) = t(e^t-1)^{-1}$. Since

$$B_n(\alpha, t) = n\int_0^\alpha B_{n-1}(x,t) dx + C_n,$$

we know in particular that B_1 is a polynomial of degree 1 in α. (In fact, (13) implies that $B_n(\alpha, t)$ is a polynomial of degree n in α since B_0 is constant in α.)

The mean value condition (14) implies that $B_1(0,t)$ and $B_1(1,t)$ have opposite sign. An induction over n is used to prove that $\alpha \mapsto B_n(\alpha, t)$ has two simple and no multiple zeros for $n \geq 2$ and $0 \leq \alpha < 1$. The proof uses (13), (14), and $e^t B_n(1,t) = B_n(0,t)$. Since it is essentially the proof for the Bernoulli polynomials (cf. Nörlund [10], p. 22), the steps are omitted here. □

The next step is the investigation of the zeros of $t \mapsto g_\alpha^{(n)}(t)$ for fixed $\alpha \in (0,1)$. The main ingredient is a structural property of the functions g_α called *total positivity*. Denote by $S^-[a_1, ..., a_n]$ the number of sign changes of the sequence $a_1, ..., a_n$. (Zero values do not count as changes.) For $f : \mathbb{R} \to \mathbb{R}$, we set

$$S^-[f] = \sup\{S^-[f(x_1, ..., x_n)] \mid -\infty < x_1 < ... < x_n < \infty, n \in \mathbb{N}\}.$$

Definition 3.2. A non-negative, integrable function $f : \mathbb{R} \to \mathbb{R}$ is called totally positive (or variation diminishing) if $S^-[f * \varphi] \leq S^-[\varphi]$ for every bounded continuous function $\varphi : \mathbb{R} \to \mathbb{R}$.

Definition 3.3. An entire function F is called a Laguerre-Pólya entire function if it has the form

$$F(z) = C \exp(-cz^2 + dz) z^k \prod_{k=1}^{\infty} \left(1 - \frac{z}{a_k}\right) \exp(z/a_k),$$

where $c \geq 0$, $k \in \mathbb{N}_0$, d, a_k $(k \in \mathbb{N})$, and C are real, and $\sum a_k^{-2} < \infty$.

I.J. Schoenberg [11] gives the following characterziation for totally positive functions (see also Hirschman and Widder [5]):

Lemma 3.4. *An integrable function $g : \mathbb{R} \to \mathbb{R}$ is totally positive if and only if for some Laguerre-Pólya entire function F the identity*

$$F(z)^{-1} = \int_{-\infty}^{\infty} e^{zt} g(t) dt$$

holds in an open vertical strip containing the origin.

Proof. This is IV 2.1 and IV 4.1 in [5]. □

For future reference we record two particular Laguerre-Pólya entire functions. These expansions imply in particular that h_θ and g_α are totally positive functions.

Lemma 3.5. *Let $\alpha \in \mathbb{R}$. In the strip $\alpha - 1 < \Re z < \alpha$ we have the expansions*

$$-\frac{\pi}{\sin \pi(z-\alpha)} = \int_{-\infty}^{\infty} e^{-zt} h_\alpha(t) dt, \tag{15}$$

$$\frac{\pi^2}{\sin^2 \pi(z-\alpha)} = \int_{-\infty}^{\infty} e^{-zt} g_\alpha(t) dt. \tag{16}$$

Proof. Equation (15) for $\alpha = 1/2$ can be found on page 69 of [5]. Equation (16) for $\alpha = 0$ can be found in III.9.7 on page 72 of [5]. The statements for arbitrary α follow with a translation. □

Lemma 3.6. *Let g be a totally positive function which is analytic on a set containing the real line, and let $n \in \mathbb{N}$. The derivative $g^{(n)}$ has n simple and no multiple zeros on the real line.*

Proof. From Theorem 5.3 in Chapter IV of [5] we obtain that $g^{(n)}$ has n sign changes and that these sign changes are simple zeros. If n is the least non-negative integer with the property that $g^{(n)}$ has an additional zero of even multiplicity, then $g^{(n+1)}$ would have $\geq n+3$ changes of sign. $\square$

Let $\alpha \in (0,1)$. As mentioned above, Lemma 3.5 implies that g_α is totally positive. Since $g_\alpha^{(n)}(t)$ and $B_n(\alpha,t)$ have the same zero sets, Lemma 3.6 implies that the function $t \mapsto B_n(\alpha,t)$ has n simple and no multiple zeros. We label the zeros in increasing order,

$$t_{n,1}(\alpha) < t_{n,2}(\alpha) < ... < t_{n,n}(\alpha).$$

If n is fixed, we suppress the first subscript and write t_j for $t_{n,j}$.

Lemma 3.7. *Let $(\alpha,t) \in (0,1)\times\mathbb{R}$. If there exists a sequence α_k with $\alpha_k \to \alpha$ and $\eta > 0$ such that the union of line segments $\{\alpha_k\}\times[t-\eta,t+\eta]$ contains no zero of B_n, then (α,t) is not a zero of B_n.*

Proof. By assumption, B_n is either negative or positive on the union of the sets $\{\alpha_k\}\times[t-\eta,t+\eta]$. By continuity, B_n is either non-positive or non-negative on $\{\alpha\}\times[t-\varepsilon,t+\varepsilon]$, hence $B_n(\alpha,t)=0$ would contradict Lemma 3.6. $\square$

Lemma 3.8. *Let $0<\alpha<1$ and $B_n(\alpha,t)=0$. For every sequence $\alpha_k \to \alpha$ there exists $\eta_k \to 0$ and $k_0 \in \mathbb{N}$ such that $\{\alpha_k\}\times[t-\eta_k,t+\eta_k]$ contains exactly one zero of B_n.*

Proof. There exists a sequence of line segments with length converging to zero so that each segment contains at least one zero by Lemma 3.7. If a sequence of such line segment existed where each segment contained two zeros of B_n, then by the pidgeonhole principle there would exist (α,s) satisfying the assumptions, but not the conclusion of Lemma 3.7. $\square$

Proof of Proposition 2.1. It only remains to show (2), since (1) follows from Lemma 3.6. The continuity is an immediate consequence of Lemma 3.8, since by the pidgeonhole principle the zero approximating $t_j(\alpha)$ will be $t_j(\alpha_k)$ for $\alpha_k \to \alpha$ from Lemma 3.8.

Assume that $t_j(=t_{n,j})$ has a minimum at $\alpha_0 \in (0,1)$. Since $t_{j-1}(\alpha_0) < t_j(\alpha_0) < t_{j+1}(\alpha_0)$, there exists a neighborhood U of $(\alpha_0, t_j(\alpha_0))$ in $\mathbb{R}^2$ which contains only zeros $(\alpha, t_j(\alpha))$. Since $(\alpha,t) \mapsto g_\alpha^{(n)}(t)$ is of one sign on $U \cap \{(\alpha,t) : \alpha < \alpha_0\}$, the function $\alpha \to B_n(\alpha, t_j(\alpha_0))$ would have an even order zero at $\alpha=\alpha_0$, which contradicts Lemma 3.1. Similarly, t_j cannot have a maximum for $\alpha \in (0,1)$. It follows that t_j is monotonic.

The zero of B_1 is given by

$$\alpha(t) = \frac{e^{-t}-(1-t)}{te^{-t}(e^t-1)},$$

hence its inverse $t_{1,1}$ is increasing. Equation (13) implies that the two zeros of $\alpha \mapsto B_n(\alpha,t)$ in $[0,1)$ are separated by one of the zeros of $\alpha \mapsto B_{n-1}(\alpha,t)$, hence since $t_{1,1}$ is increasing, all $t_{n,j}$ are increasing. By Lemma 3.1 the function $\alpha \mapsto B_n(\alpha, t)$ has two zeros for $\alpha \in [0, 1)$. Together with the monotonicity behavior of $\lambda_j(\alpha)$, it follows that

$$t_2(0+) = t_1(0+) = -\infty,$$
$$t_{n-1}(1-) = t_n(1-) = \infty,$$

and the remaining limits are finite.

Lemma 3.5 implies after two integration by parts that the Laplace transforms of g_0'' and g_1'' represent $(\pi x)^{-2} \csc^2 \pi x$ and $(\pi(x-1))^{-2} \csc^2 \pi x$, respectively. Hence Lemma 3.6 implies that $g_0^{(n)}$ and $g_1^{(n)}$ have $n-2$ simple and no multiple zeros for $n \geq 2$. By continuity, these zeros are the limits of the functions t_j at $\alpha = 0$ and $\alpha = 1$. $\square$

The argumentation for the function $h(x) = e^{\theta t}(e^t+1)^{-1}$ goes along similar lines as above, so we sketch the outline, but leave the steps to the reader. The auxiliary functions in this case are given by

$$E_n(\theta, t) := e^{-\theta t} h_\theta^{(n)}(t),$$

and a power series expansion of $h_\theta(z+t)$ in z gives

$$e^{\theta z}(1 + e^{z+t})^{-1} = \sum_{n=0}^{\infty} E_n(\theta, t) \frac{z^n}{n!}, \tag{17}$$

where $|z + t| < 2\pi$. For $\theta = 0$ we obtain $E_n(\theta, t) = 2^{-1} E_n(t)$ with the Euler polynomials E_n. We have

$$\frac{d}{d\theta} E_n(\theta, t) = n E_{n-1}(\theta, t).$$

Lemma 3.9. *The identity $-E_n(0, t) = e^t E_n(1, t)$ holds for all real t. For fixed $t \in \mathbb{R}$, the function $\theta \to E_n(\theta, t)$ has a simple and no multiple zeros for $0 \leq \theta < 1$.*

Proof. From the identity

$$u \int_0^1 e^{\theta u}(e^u + 1)^{-1} d\theta = 1 - 2(e^u + 1)^{-1}$$

we obtain after substituting $u = t + z$, replacing the generating functions by their power series expansions and a comparison of coefficients of z^n,

$$\begin{aligned} -2E_n(0, t) &= t \int_0^1 e^{t\theta} E_n(\theta, t) d\theta + n \int_0^1 e^{t\theta} E_{n-1}(\theta, t) d\theta \\ &= \int_0^1 \frac{d}{d\theta}[h_\theta^{(n)}(t)] d\theta = h_1^{(n)}(t) - h_0^{(n)}(t). \end{aligned}$$

Since $h_0^{(n)}(t) = E_n(0, t)$ and $h_1^{(n)}(t) = e^t E_n(1, t)$, we obtain $-E_n(0, t) = e^t E_n(1, t)$.

For $n = 1$ it can be checked directly that $E_1(\theta, t)$ is a polynomial of degree 1 in θ with a zero for $0 < \theta < 1$. The identities $-E_n(0, t) = e^t E_n(1, t)$ and

$nE_{n-1}(\theta, t) = \frac{d}{d\theta} E_n(\theta, t)$ can be used in an induction over n to show the second statement of the lemma. □

Lemma 3.5 implies that h_θ is totally positive. Lemma 3.6 implies that $t \mapsto E_n(\theta, t)$ has n simple and no multiple zeros for all $\theta \in [0, 1)$. The statements of Lemma 3.7 and Lemma 3.8 hold for $E_n(\theta, t)$, the proofs are essentially unchanged. The proof of Proposition 2.4 is analogous to the proof of Proposition 2.1 and is omitted.

4. Approximations to truncated exponential functions

We use the notation

$$\mathcal{L}[g](z) := \int_{-\infty}^{\infty} e^{-zt} g(t) dt.$$

Let $n \in \mathbb{N}_0$ and $\lambda > 0$. Recall that x_+^n equals x^n for $x \geq 0$ and 0 for $x < 0$. In this section we consider the problem of finding best one-sided approximations to $x_+^n e^{-\lambda x}$ from the class $\mathcal{A}(\delta)$ of entire functions of exponential type δ. The approximations are obtained from the results of Section 3 using techniques from [8].

Definition 4.1. Let F be a Laguerre-Pólya entire function which satisfies $F(z)^{-1} = \mathcal{L}[g](z)$ for $g : \mathbb{R} \to \mathbb{R}$ in some strip $a < \Re z < b$. The numbers a and b are two consecutive zeros of F. If g is analytic on a set containing the real line, then we call (F, g) an admissible pair on (a, b).

Let (F, g) be an admissible pair on (a, b) with $a \leq 0 \leq b$. For $\Re z < b$ we define

$$G_{\lambda,n,F}(z) := \frac{F(z)}{z} \int_{-\infty}^{0} e^{-zt} g^{(n+1)}(t - \lambda) dt. \tag{18}$$

Lemma 4.2. *Let $\lambda \geq 0$, and let (F, g) be an admissible pair on (a, b) with $a \leq 0 \leq b$. If at least one of the conditions $F(0) = 0$ or $g^{(n)}(-\lambda) = 0$ is satisfied, then $G_{\lambda,n,F}$ has an analytic continuation to the entire complex plane, and the estimate*

$$|G_{\lambda,n,F}(x) - x_+^n e^{-\lambda x}| \ll |x|^{-2} |F(x)| \tag{19}$$

holds for all real x with $|x| \geq (a + b)/2$. Moreover,

$$|G_{\lambda,n,F}(z)| \ll 1 + |z|^n + |F(z)| \tag{20}$$

for every $z \in \mathbb{C}$.

Proof. If $F(z)^{-1} = \mathcal{L}[g](z)$ in a vertical strip whose closure contains the origin, then $(e^{\lambda z} F(z))^{-1} = \mathcal{L}[g(. - \lambda)](z)$. From Theorem 4.3 of [8] we obtain that

$$T_{\lambda,n,F}(z) := \frac{e^{\lambda z} F(z)}{z} \int_{-\infty}^{0} e^{-zt} g^{(n+1)}(t - \lambda) dt$$

has an entire continuation, provided $F(0) = 0$ or $g^{(n)}(-\lambda) = 0$. Moreover, $|T_{\lambda,n,F}(z) - x_+^n| \le c|x|^{-2}|e^{\lambda x}F(x)|$, hence (19) follows after multiplication with $e^{-\lambda x}$. Equation (20) follows from investigating the growth of $G = T_{\lambda,n,F}$ in (4.5) and (4.7) of [8] after multiplication by $e^{-\lambda z}$. $\square$

Define

$$R_{\lambda,n,F}(z) := \begin{cases} -z^{-1}\displaystyle\int_0^\infty g^{(n+1)}(t-\lambda)e^{-zt}dt & \text{for } \Re z > 0, \\ z^{-1}\displaystyle\int_{-\infty}^0 g^{(n+1)}(t-\lambda)e^{-zt}dt & \text{for } \Re z < 0. \end{cases} \tag{21}$$

We obtain from equation (4.9) of [8] that the identity

$$G_{\lambda,n,F}(z) - z_+^n e^{-\lambda z} = F(z)R_{\lambda,n,F}(z) \tag{22}$$

holds in $\mathbb{C}\backslash\{z : \Re z = 0\}$.

Lemma 4.3. *Let (F, g) be an admissible pair on (a, b).*

(1) *If $0 \in \{a, b\}$, then*
$$g'(-\lambda)R_{\lambda,0,F}(x) < 0$$
for all real $x \neq 0$.

(2) *If $0 \in \{a, b\}$ such that the zero of F at the origin has multiplicity at least two, then*
$$g''(-\lambda)R_{\lambda,1,F}(x) < 0$$
for all real $x \neq 0$.

(3) *Assume either $0 \in (a, b)$, or $0 \in \{a, b\}$ and F has a zero of order two at the origin. If $g^{(n)}(-\lambda) = 0$, then*
$$g^{(n+1)}(-\lambda)R_{\lambda,n,F}(x) < 0$$
for all real $x \neq 0$.

Proof. These statements are Propositions 4.4, 4.5 and 4.7 in [8] applied to the pair $(e^{\lambda z}F(z), g(t - \lambda))$. $\square$

We define for $\lambda \ge 0$ and $\alpha \in \mathbb{R}$

$$F_\alpha(z) := \pi^{-2}\sin^2 \pi(z - \alpha), \tag{23}$$

and recall $g_\alpha(t) = te^{\alpha t}(e^t - 1)^{-1}$. Lemma 3.5 implies after a translation that these functions are connected by the formula

$$e^{\lambda z}F_\alpha^{-1}(z) = \int_{-\infty}^\infty e^{-zt}g_\alpha(t - \lambda)dt \quad \text{for } \alpha - 1 < \Re z < \alpha.$$

We write $G_{\lambda,n,\alpha}$ instead of G_{λ,n,F_α}, i.e., for $\Re z < \alpha$

$$G_{\lambda,n,\alpha}(z) = \frac{F_\alpha(z)}{z}\int_{-\infty}^0 e^{-zt}g_\alpha^{(n+1)}(t - \lambda)dt. \tag{24}$$

In the following, $\alpha_n(\lambda)$ and $\beta_n(\lambda)$ have the same meaning as in Section 3 and Definition 2.3.

Proof of Theorem 2.6. Lemma 4.2 implies that the integrands in (7) and (8) are integrable, and that $G_{\lambda,n,\alpha_n(\lambda)}, G_{\lambda,n,\beta_n(\lambda)} \in \mathcal{A}(2\pi)$ for all $n \in \mathbb{N}_0$.

Next, we show (6). An application of Lemma 4.3 (1) together with

$$g_0'(-\lambda) = e^{-\lambda}(e^{-\lambda} - 1)^{-2}\big(1 + \lambda - e^{\lambda}\big) < 0,$$
$$g_1'(-\lambda) = e^{-\lambda}(e^{-\lambda} - 1)^{-2}\big(e^{-\lambda} - 1 + \lambda\big) > 0$$

for all $\lambda > 0$ implies (6) for $n = 0$ since $\beta_0(\lambda) = 0$ for all λ by Definition 2.3.

For $n = 1$, the lower best approximation follows from Lemma 4.3 (2) with

$$g_0''(-\lambda) = -e^{-\lambda}(e^{-\lambda} - 1)^{-3}\big((2 + \lambda)e^{-\lambda} + \lambda - 2\big) > 0,$$

whereas the upper approximation follows with aid of Lemma 4.3 (3); note that $0 < \beta_1(\lambda) < 1$ for $\lambda \geq 0$. Hence to finish the proof in this case, we need to show that $g'_{\beta_1(\lambda)}(-\lambda) = 0$ and that $g''_{\beta_1(\lambda)}(-\lambda) < 0$.

The first part follows immediately, since

$$g_\alpha'(-\lambda) = e^{\lambda}(1 - \alpha\lambda - e^{\lambda} + \alpha\lambda e^{\lambda} + \lambda)(1 - e^{\lambda})^{-2}$$

is a linear polynomial in α with zero $\alpha = \beta_1(\lambda)$.

For all $\lambda \geq 0$, the function $t \mapsto g_{\beta_1(\lambda)}(t - \lambda)$ is a variation diminishing function which is analytic on a set containing the real line, hence each of its derivatives can have only simple zeros (Lemma 3.6). Since $g'_{\beta_1(\lambda)}(-\lambda) = 0$, we must have $g''_{\beta_1(\lambda)}(-\lambda) \neq 0$ for all λ. Since $g''_{1/2}(0) = B_2(1/2) = -1/12 < 0$ and $\lambda \mapsto g''_{\beta_1(\lambda)}(-\lambda)$ is continuous, the statement $g''_{\beta_1(\lambda)}(-\lambda) < 0$ follows for all $\lambda \geq 0$.

For $n \geq 2$, (6) follows with aid of Lemma 4.3 (3) and the results about α_n and β_n from Section 3.

The integral values in (7) and (8) are obtained similarly to equations (7.6) and (7.7) in [8]. By construction, for $\alpha \in \{\alpha_n(\lambda), \beta_n(\lambda)\}$ the functions $G_{\lambda,n,\alpha}$ interpolate $e^{-\lambda x}x_+^n$ at $\alpha + \mathbb{Z}$ and an application of Poisson summation with the Paley-Wiener theorem gives

$$\int_{-\infty}^{\infty} G_{\lambda,n,\alpha}(x)dx = \widehat{G_{\lambda,n,\alpha}}(0) = \sum_{k\in\mathbb{Z}} G_{\lambda,n,\alpha}(k + \alpha) = \sum_{k\in\mathbb{Z}} (k + \alpha)_+^n e^{-\lambda(k+\alpha)}. \tag{25}$$

It remains to prove uniqueness of the one-sided approximations. Assume that $A \in \mathcal{A}(2\pi)$ is a lower best approximation to $f_{\lambda,n}(x) = x_+^n e^{-\lambda x}$. Since the integrals of A and $G_{\lambda,n,\alpha_n(\lambda)}$ have to agree, it follows from (25) and the Paley-Wiener theorem that

$$\begin{aligned} 0 = \widehat{G_{\lambda,n,\alpha_n(\lambda)}}(0) - \widehat{A}(0) &= \sum_{k\in\mathbb{Z}} [G_{\lambda,n,\alpha_n(\lambda)}(k + \alpha_n(\lambda)) - A(k + \alpha_n(\lambda)] \\ &= \sum_{k\in\mathbb{Z}} [f_{\lambda,n}(k + \alpha_n(\lambda)) - A(k + \alpha_n(\lambda))]. \end{aligned} \tag{26}$$

Since $A \leq f_{\lambda,n}$ on the real line, it follows from (26) that A equals $f_{\lambda,n}$ on $\alpha_n(\lambda) + \mathbb{Z}$, and if $n \geq 2$ or $0 \notin \alpha_n(\lambda) + \mathbb{Z}$, it follows also that A' equals $f'_{\lambda,n}$

on $\alpha_n(\lambda)+\mathbb{Z}$. The same statement is true for $G_{\lambda,n,\alpha_n(\lambda)}$, hence by Theorem 9 of [13] we obtain $A = G_{\lambda,n,\alpha_n(\lambda)}$. In the remaining case (when $f'_{\lambda,n}(0)$ does not exist) we obtain from Theorem 9 of [13] that

$$A(x) - G_{\lambda,n,\alpha_n(\lambda)}(x) = (A'(0) - G'_{\lambda,n,\alpha_n(\lambda)}(0))\frac{\sin^2(\pi x)}{\pi^2 x}$$

and since the left-hand side is integrable, the right-hand side has to be integrable as well. We obtain $A'(0) = G'_{\lambda,n,\alpha_n(\lambda)}(0)$, and hence $A = G_{\lambda,n,\alpha_n(\lambda)}$. □

Proof of Theorem 2.8. By Lemma 4.2 the function $H_{\lambda,0,0}$ is entire and the difference $H_{\lambda,0,0}(x) - x_+^0 e^{-\lambda x}$ is integrable. By construction $g^{(n)}_{\theta_n(\lambda)}(-\lambda) = 0$ for $n \geq 1$, hence Lemma 4.2 implies for these n that $H_{\lambda,n,\theta_n(\lambda)}$ is entire and $H_{\lambda,n,\theta_n(\lambda)}(x) - x_+^n e^{-\lambda x}$ is integrable, and that $H_{\lambda,n,\theta_n(\lambda)} \in \mathcal{A}(\pi)$.

As in the proof of Theorem 6.2 of [8], we define for $A \in \mathcal{A}(\pi)$ the difference $\varphi_{\lambda,n,A}(x) := A(x) - x_+^n e^{-\lambda x}$ and note that

$$\int_{\mathbb{R}} |\varphi_{\lambda,n,A}(x)| dx \geq \Big| \int_{\mathbb{R}} \operatorname{sgn} \sin \pi(x - \theta_n(\lambda)) \varphi_{\lambda,n,A}(x) dx \Big|,$$

with equality for any A for which

$$|\varphi_{\lambda,n,A}(x)| = \operatorname{sgn} \sin \pi(x - \theta_n(\lambda)) \varphi_{\lambda,n,A}(x) \tag{27}$$

for all real x holds. Lemma 4.3 implies that this is the case for $A = H_{\lambda,n,\theta_n(\lambda)}$.

To compute the value of the integral we define $\psi(x) = e^{-\pi x^2}$. From Theorem 8.36 in Folland [3] we have

$$\sigma_\varepsilon(x) := \frac{2}{\pi i} \sum_{\mu\in\mathbb{Z}} \frac{1}{2\mu+1} \psi(\varepsilon\mu) e^{2\pi i x(\mu+1/2)} \to \operatorname{sgn} \sin \pi x \text{ in } L^1([0,2)).$$

The series is absolutely convergent for $\varepsilon > 0$, hence we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \sigma_\varepsilon(x - \theta_n(\lambda)) \varphi_{\lambda,n,A}(x) dx \\
&\qquad = \int_{\mathbb{R}} \frac{2}{\pi i} \sum_{\mu\in\mathbb{Z}} \frac{e^{2\pi i (x - \theta_n(\lambda)(\mu+1/2)}}{2\mu+1} \psi(\varepsilon\mu) \varphi_{\lambda,n,A}(x) dx \\
&\qquad = \frac{1}{\pi i} \sum_{\mu} \frac{\psi(\varepsilon\mu)\widehat{\varphi_{\lambda,n,A}}(-\mu-1/2)}{\mu+1/2} e^{-2\pi i \theta_n(\lambda)(\mu+1/2)} \\
&\qquad = \frac{1}{\pi i} \sum_{\mu} \frac{\psi(\varepsilon\mu)}{(\lambda - 2\pi i(\mu+\frac{1}{2}))^{n+1}(\mu+\frac{1}{2})} e^{-\pi i \theta_n(\lambda)(2\mu+1)},
\end{aligned}$$

where we used that by the Paley-Wiener theorem the value of $\widehat{\varphi_{\lambda,n,A}}(t)$ for $|t| \geq 1/2$ equals the value of the transform of $x_+^n e^{-\lambda x}$. Since the last series converges absolutely for every $\varepsilon \geq 0$, we may take the limit $\varepsilon \to 0+$ in the first and the last term to obtain

$$||\varphi_{\lambda,n,A}||_1 = \Big|\frac{1}{\pi}\sum_{\mu}\frac{n!}{(\lambda - 2\pi i(\mu+\frac{1}{2}))^{n+1}(\mu+\frac{1}{2})}e^{-\pi i\theta_n(\lambda)(2\mu+1)}\Big|.$$

It should be emphasized that this calculation is valid for every function $A \in \mathcal{A}(\pi)$ which satisfies (27). It remains to show uniqueness of the best approximation. If equality were to hold for another function $A^* \in \mathcal{A}(\pi)$, equation (27) would imply that A^* and $H_{\lambda,n,\theta_n(\lambda)}$ would agree on the set $\theta_n(\lambda) + \mathbb{Z}$. Since A^* and $H_{\lambda,n,\theta_n(\lambda)}$ are best approximations to the same function we obtain

$$A^* - H_{\lambda,n,\theta_n(\lambda)} \in L^1(\mathbb{R}),$$

and since the transform of this difference has bounded support, the difference is in $L^2(\mathbb{R})$ as well. From (7.19) in Chapter XVI of [14] we obtain $A^* - H_{\lambda,n,\theta_n(\lambda)} = 0$. □

References

[1] A. Beurling, *On functions with a spectral gap*, Seminar on Harmonic Analysis, Univ. of Uppsala, March 24, 1942. Published in *Collected Works of Arne Beurling: Vol II, Harmonic Analysis*, Birkhäuser, Boston, 1989, 359–364.

[2] S. W. Graham and J. D. Vaaler, *A class of extremal functions for the Fourier transform*, Trans. Amer. Math. Soc. **265** (1981), 283–302.

[3] G. B. Folland, Real Analysis, John Wiley and Sons, Inc., New York 1999.

[4] U. Haagerup and L. Zsido, *Resolvent estimate for Hermitean operators and a related minimal extrapolation problem*, Acta Math. (Szeged) **59** (1994), 503–524.

[5] I. I. Hirschman and D. V. Widder, *The Convolution Transform*, Princeton University Press, Princeton, New Jersey 1955.

[6] J. J. Holt and J. D. Vaaler, *The Beurling-Selberg extremal functions for a ball in Euclidean space*, Duke Math. J. **83** (April 1996), No. 1, 203–247.

[7] J. Korevaar, *A century of complex Tauberian theory*, Bull. Amer. Math. Soc. **39** (Oct. 2002), 475–532.

[8] F. Littmann, *Entire approximations to the truncated powers*, Constr. Approx. **22** No.2 (2005), 273–295.

[9] F. Littmann, *Entire Majorants via Euler-Maclaurin summation*, Trans. Amer. Math. Soc. **358** No. 7 (2006), 2821-2836.

[10] N. E. Nörlund, Differenzenrechnung, Springer, Berlin 1924.

[11] I. J. Schoenberg, *On Pólya Frequency Functions. I. The totally positive functions and their Laplace transforms*, J. Analyse Math. **1**, (1951), 331–374, (I. J. Schoenberg, *Selected Papers 2*, Birkhäuser Verlag 1988.)

[12] A. Selberg, *Lectures on sieves*, Collected Papers II, Springer-Verlag, New York, 1991.

[13] J. D. Vaaler, *Some extremal functions in Fourier analysis*, Bull. Amer. Math. Soc. **12** (1985), 183–216.

[14] A. Zygmund, Trigonometric Series, Vol. II. Cambridge University Press, 1968.

DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, FARGO, ND 58105, USA, [1.5EX]EMAIL: FRIEDRICH.LITTMANN@NDSU.EDU, PHONE: 701 231 8095