Interpolation and Approximation by Entire Functions

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Abstract. In this note we study the connection between best approximation and interpolation by entire functions on the real line. A general representation for entire interpolants is outlined. As an illustration, best upper and lower approximations from the class of functions of fixed exponential type to the Gaussian are constructed.

§1. Approximation Background

The Fourier transform of $\varphi \in L^1(\mathbb{R})$ is defined by

$$\widehat{\varphi}(t) = \int_{-\infty}^{\infty} \varphi(x) e^{-2\pi i x t} dx.$$

This article considers the following problem: given a function $f : \mathbb{R} \to \mathbb{R}$ and a sequence of points $\{a_j\}_{j\in\mathbb{N}}$ from $\mathbb{R}$ (not necessarily distinct), find entire functions A and F such that F has its real zeros at the values a_j and such that the equality

$$f(x) - A(x) = F(x)H(x) \tag{1}$$

holds with a function H which is real-valued and of one sign on the real line.

Classically, this is often done by defining A with aid of an interpolation series and then estimating the difference on the left-hand side of (1). Investigations by Holt and Vaaler [5], Li and Vaaler [7], and the author [8] suggest that a generalization of this approach is needed which allows for non-equally spaced node sets.

The starting point is a relation between F and a function g given by

$$1 = F(x) \int_{-\infty}^{\infty} e^{xt} g(t) dt, \tag{2}$$

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valid in some vertical strip in $\mathbb{C}$. For almost every $x \in \mathbb{R}$ we construct a function $t \mapsto f(D)\{g\}_x(t)$ with the property that

$$f(x) = F(x) \int_{-\infty}^{\infty} e^{xt} f(D)\{g\}_x(t)dt \tag{3}$$

in a neighborhood of x, and we define the interpolation A using the Laplace transform of the "diagonal" $t \mapsto f(D)\{g\}_t(t)$:

$$A(x) := F(x) \int_{-\infty}^{\infty} e^{xt} f(D)\{g\}_t(t)dt. \tag{4}$$

The motivation for this investigation arises from the problem of finding best approximations of fixed exponential type in $L^1(\mathbb{R})$–norm. We give a short account of the principles governing best approximation and one-sided best approximation in $L^1(\mathbb{R})$–norm to show where the interpolation method of this article can be useful.

Definition 1. *Let $\eta \geq 0$. An entire function $A : \mathbb{C} \to \mathbb{C}$ is said to be of exponential type η if and only if for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $z \in \mathbb{C}$ the growth estimate*

$$|A(z)| \leq C_\varepsilon e^{|z|(\eta+\varepsilon)}$$

holds. We define the set

$$\mathcal{A}(\eta) := \{A : \mathbb{C} \to \mathbb{C} \,|\, A \text{ is entire and of exponential type } \eta\}.$$

A function $A^* \in \mathcal{A}(\eta)$ is called a best approximation in $L^1(\mathbb{R})$-norm to $f : \mathbb{R} \to \mathbb{R}$, if

$$||f - A^*||_1 = \inf_{A \in \mathcal{A}(\eta)} ||f - A||_1. \tag{5}$$

In general, a best approximation is neither unique nor does it necessarily exist. A best upper one-sided approximation from $\mathcal{A}(\eta)$ satisfies (5) and the additional requirement $A \geq f$ (which requires implicitly the additional condition that A is real-valued on $\mathbb{R}$). Best lower approximations are defined analogously.

It should be emphasized that this note is not concerned with the problem of finding bounds for the error of best approximations for function classes. The focus of this paper are explicitly constructed best approximations via interpolations, and the essential similarities of the constructions for the problem of best approximation and best one-sided approximation.

As was obtained (essentially independently) in investigations of A. Beurling [1], A. Selberg [10], and S.W. Graham and J.D. Vaaler [3], if $A \geq f$ with $A \in \mathcal{A}(\eta)$, then an application of the Poisson summation formula gives a lower bound independent of A which is assumed if A interpolates f at a translate of $2\pi\eta^{-1}\mathbb{Z}$:

Theorem A (cf. [3]). *If $A^* \geq f$ with $A^* \in \mathcal{A}(\eta)$ and $||A^* - f||_1 < \infty$, and there exists $\alpha \in \mathbb{R}$ such that $A^*(2\pi\eta^{-1}n + \alpha) = f(2\pi\eta^{-1}n + \alpha)$ for all $n \in \mathbb{Z}$, then A^* is a best one-sided upper approximation to f.*

The statement of Theorem A addresses only the situation when the best approximation interpolates f (and f') at a translate of the integers. Explicitly computable best one-sided approximations are known that interpolate at sets other than the integers. This occurs in the case of approximation in $L^1(\mathbb{R}, d\mu)$-norms, most notably for $d\mu(x) = |x|^\alpha dx$ where $\alpha > -1$. This problem was considered for $f(x) = \operatorname{sgn}(x)$ by J. J. Holt and J. D. Vaaler [5]; the interpolation nodes in this case are zeros of Bessel functions.

Finally, for some characteristic functions f of intervals, best one-sided approximations from $\mathcal{A}(\eta)$ to f are known (in $L^1(\mathbb{R})$-norm) that do not interpolate at a translate of the integers. This is implicit in Lemma 4 of Donoho and Logan [2].

A similar phenomenon occurs in the problem of best approximation (in the sense of (5)) from $\mathcal{A}(\eta)$ in $L^1(\mathbb{R})$-norm. An early application can be found in the approximation theorems of M.E. Krein [6] and Sz.-Nagy [12]. Accounts can be found in the books of H. Shapiro [11] and A. Timan [14].

Theorem B (*cf. 4.2.1 and 4.2.2 and Chapter 7 in Shapiro [11]*). *Let $\eta > 0$. Let $\psi : \mathbb{R} \to \mathbb{C}$ satisfy $|\psi(x)| = 1$ for almost all $x \in \mathbb{R}$, and assume*

$$\int_{\mathbb{R}} \psi(x)B(x)dx = 0 \tag{6}$$

for all $B \in \mathcal{A}(\eta) \cap L^1(\mathbb{R})$. If there exists $A^* \in \mathcal{A}(\eta)$ with $||A^* - f||_1 < \infty$ so that ψ is the argument of $A^* - f$ then A^* is a best approximation from $\mathcal{A}(\eta)$ to f.

A (distributional) Fourier inversion gives an equivalent characterization for signatures ψ satisfying (6), namely, (6) is equivalent to the property that the distributional Fourier transform of ψ has no support inside $(-\eta, \eta)$.

The simplest signatures with this property are of the form $\psi(x) = \operatorname{sgn} \sin \eta(x - \alpha)$ where α can be any parameter in $\mathbb{R}$. Hence, for functions f with nice properties it can be expected that the best approximation from $\mathcal{A}(\eta)$ to f in $L^1(\mathbb{R})$-norm interpolates f at a translate of the integers and nowhere else on the real line. In particular, for nice functions f the signature ψ will assume only values from $\{\pm 1\}$.

In Section 2 the method of constructing the interpolation (4) is outlined. To illustrate the application, in Section 3 the best upper approximation to e^{-x^2} from $\mathcal{A}(2\pi)$ is constructed.

§2. Interpolation Principles

For $g : \mathbb{R} \to \mathbb{C}$ we will use the notation

$$\mathcal{L}[g](z) = \int_{-\infty}^{\infty} e^{zt} g(t) dt$$

for all $z \in \mathbb{C}$ for which the integral converges absolutely.

Let $\{a_j\}_{j\in\mathbb{N}}$ with $a_j \in \mathbb{R}$ be the nodes for the interpolation. We impose the restriction that

$$\sum_{j=1}^{\infty} a_j^{-2} < \infty, \tag{7}$$

which implies in particular that

$$F(z) := \prod_{j=1}^{\infty} \Big(1 - \frac{z}{a_j}\Big) e^{z/a_j} \tag{8}$$

converges absolutely for all $z \in \mathbb{C}$ and hence defines an entire function in the complex plane. The a_j are not required to be distinct, in fact, for one-sided approximations each node needs to occur an even number of times among the a_j.

The function F defined in (8) has the property that its reciprocal is representable as a two-sided Laplace transform in every vertical strip $\Re z \in I = (a, b)$ containing none of the elements of $\{a_j\}$ (cf. Schoenberg [13] and the book by Hirschman and Widder [4]). The subscript I in the representation $F(z)^{-1} = \mathcal{L}[g_I](z)$ will be omitted when it is clear from the context which interval is considered.

We require a few notions from distribution theory. We define $\mathcal{S}$ to be the testing space of all functions $\theta : \mathbb{R} \to \mathbb{C}$ that are infinitely smooth and are such that, as $|t| \to \infty$, they and all their derivatives decrease to zero faster than any power of $|t|^{-1}$. We denote by $\mathcal{S}'$ the space of continuous linear functionals (or tempered distributions) on $\mathcal{S}$, and we write $\langle T, \theta\rangle$ for the value of $T \in \mathcal{S}'$ at $\theta \in \mathcal{S}$.

We need the notion of a Laplace transform of a tempered distribution. The following construction is a special case of a method outlined in Chapter 8 of Zemanian [15].

Let $\mathcal{S}_1' \subseteq \mathcal{S}'$ be the set of tempered distributions with support bounded to one side. (Since the support of $T \in \mathcal{S}_1'$ can be bounded to the right or to the left, $\mathcal{S}_1'$ is not a vector space.) For $T \in \mathcal{S}_1'$ exists $\lambda \in C^\infty(\mathbb{R})$ such that $\lambda(t) = 1$ for all t in a neighborhood of $\operatorname{supp}(T)$ and $\lambda(t) = 0$ on a half-line starting at some $t_0 \in \mathbb{R}$, and hence there exists an open half-plane $H \subseteq \mathbb{C}$ (depending on T) such that for any $s \in H$ the function $\theta_s(t) = e^{st}\lambda(t)$ is an element of $\mathcal{S}$.

Definition 2. *The Laplace transform of $T \in \mathcal{S}_1'$ is defined by*

$$\mathcal{L}[T](s) := \langle T, \theta_s \rangle,$$

where θ_s is constructed as above, and we set

$$\mathcal{L}_{\mathcal{S}_1'} := \{\mathcal{L}[T] \,|\, T \in \mathcal{S}_1'\}.$$

By construction $\mathcal{L}[T]$ does not depend on λ. By assumption $\mathcal{L}[T]$ is analytic on the associated half-plane H defined above, and hence any $F \in \mathcal{L}_{\mathcal{S}_1'}$ is an analytic function in some half-plane.

Our goal is now to define an operator $f(D)$ (with $D = d/dx$). As we will see in the examples below, it is not possible to view $f(D)$ as an operator on $\mathcal{S}_1'$. The next definitions introduce a space Γ_x on which $f(D)$ can be defined.

Definition 3. *Let $g : \mathbb{R} \to \mathbb{C}$ and $x \in \mathbb{R}$. If there exists an open ball $U \subseteq \mathbb{C}$ with x as its center and $T \in \mathcal{S}_1'$ such that $g(u) = \mathcal{L}[T](u)$ on U, then we set*

$$\{g\}_x := (g, T).$$

We write $\{g\}_x(s)$ for the value of the analytic continuation of $\mathcal{L}[T]$ from U to the (largest) open star body with star point x.

Locally (namely at least in U), we will have $\{g\}_x(s) = g(s)$. We give two examples. First consider

$$h(t) = \begin{cases} 1 & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

We have for $x \neq 0$ that $\{h\}_x = (h, \delta)$ (where δ denotes the Dirac delta) and $\{h\}_x(s) = 1$ on $\mathbb{C}$, and $\{h\}_0$ does not exist.

As a second example consider $g_n(t) = -te^{-nt}(e^{-t} - 1)^{-1}$ where $n \in \mathbb{Z}$ and $t \in \mathbb{R}$. We have

$$\begin{aligned} g_n(t) &= \begin{cases} -t \displaystyle\sum_{k=1}^{\infty} e^{(k-n)t} & \text{for } t < 0, \\ t \displaystyle\sum_{k=-\infty}^{0} e^{(k-n)t} & \text{for } t > 0 \end{cases} \\ &= \begin{cases} \mathcal{L}[T_n^-](t) & \text{for } t < 0, \\ \mathcal{L}[T_n^+](t) & \text{for } t > 0, \end{cases} \end{aligned}$$

where the distributions T_n^- and T_n^+ are defined by the identities $\langle T_n^-, \theta\rangle = -\sum_{k=-n+1}^{\infty} \theta'(k)$ and $\langle T_n^+, \theta\rangle = \sum_{k=-\infty}^{-n} \theta'(k)$ for all $\theta \in \mathcal{S}$. Hence for $x < 0$ we have $\{g_n\}_x = (g_n, T_n^-)$ and for $x > 0$ we have $\{g_n\}_x = (g_n, T_n^+)$. In this example we have $\{g_n\}_x(s) = \{g_n\}_y(s) = -se^{-ns}(e^{-s} - 1)^{-1}$ for all non-zero x and y and all complex s with $|\Im s| < 2\pi$, but nonetheless $\{g_n\}_x \neq \{g_n\}_y$ for $xy < 0$.

Definition 4. *Let $x \in \mathbb{R}$. We define*

$$\Gamma_x := \big\{\{g\}_x \,|\, g \in \mathcal{L}_{\mathcal{S}_1'} \text{ such that } \{g\}_x \text{ exists}\big\}.$$

For certain functions f and distributions T we require a notion of the pointwise product Tf.

Definition 5. *Let $T \in \mathcal{S}_1'$, and let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function satisfying $f(x) = \mathcal{O}(1+|x|^n)$ for some $n \in \mathbb{N}$ and all $x \in \mathbb{R}$, which is C^∞ on a neighborhood of* $\mathrm{supp}(T)$. *Let $\lambda \in \mathcal{S}$ be such that $\lambda(t) = 1$ for all t in the support of T and $\lambda(t) = 0$ for all t where f is not C^∞. The multiplication of T and f is defined by*

$$\langle Tf, \theta\rangle = \langle T, f\lambda\theta\rangle$$

and gives an element $Tf \in \mathcal{S}_1'$.

It should be remarked that the definition does not depend on the choice of λ. The operator $f(D)$ can now be defined on a subset of Γ_x.

Definition 6. *Let $x \in \mathbb{R}$. For T and f as in Definition 5, the operator $f(D)$ applied to $\{g\}_x = (g, T) \in \Gamma_x$ is defined by*

$$f(D)\{g\}_x := \mathcal{L}[Tf].$$

We continue with the example $g_n(t) = -te^{-nt}(e^{-t} - 1)^{-1}$ from above. We compute e^{-D^2} applied to g_n. We obtain

$$e^{-D^2}\{g_n\}_x(t) = \begin{cases} -\displaystyle\sum_{\ell=-n+1}^{\infty} (t-2\ell)e^{\ell t-\ell^2} & \text{for } x < 0, t \in \mathbb{C}, \\ \displaystyle\sum_{\ell=-\infty}^{-n} (t-2\ell)e^{\ell t-\ell^2} & \text{for } x > 0, t \in \mathbb{C}. \end{cases} \tag{9}$$

Comparing the two series shows that $e^{-D^2}\{g_n\}_x(t) \neq e^{-D^2}\{g_n\}_y(t)$ for almost all t and all x and y with $xy < 0$, even though $\{g_n\}_x(t) = \{g_n\}_y(t)$ for all t and all non-zero x and y.

The meromorphic structure of g_n is not sufficient to define $e^{-D^2}g_n$ in a unique way; we obtain two different entire functions depending on whether we view g_n as an analytic extension of $\mathcal{L}[T_n^+]$ or of $\mathcal{L}[T_n^-]$. The application of e^{-D^2} to $\{g_n\}_x$ avoids this ambiguity since the latter term keeps track of the distribution of which it is a Laplace transform near x.

Let F be a Polya-Laguerre entire function. Let g be defined by

$$F(z)^{-1} = \mathcal{L}[g](z). \tag{10}$$

We assume that $\{g\}_x$ exists for almost all $x \in \mathbb{R}$, or in other words, that for almost all real x exists $T_x \in \mathcal{S}_1'$ with $g = \mathcal{L}[T_x]$ in a neighborhood of x. In particular, g is required to be piecewise analytic on a neighborhood of the real line.

We will not address the question here for which sequences $\{a_j\}$ the reciprocal of the corresponding F in (8) can be represented as such a function g. We note however that any polynomial F has the property that its reciprocal can be represented in such a way, hence for arbitrary F given by (8), the function g can be pointwise approximated by such Laplace transforms (cf. [4]).

We return to the general structure of the interpolation problem.

1. Let $F(0) \neq 0$, hence

$$F(z)^{-1} = \int_{-\infty}^{\infty} e^{zt} g(t)dt$$

   converges absolutely in a vertical open strip in the complex plane containing the origin. (In particular, $g(t)$ decays exponentially as $t \to \pm\infty$ by Theorem 2.1 in Chapter 5 of [4].)
2. Assume a regularity condition about $\{a_j\}_{j\in\mathbb{N}}$, namely, $\{g\}_x = \{g\}_y$ if and only if $xy > 0$.
3. Let f be analytic in the half-planes $\Re z > 0$ and $\Re z < 0$.
4. Let f be represented by the formula

$$f(x) = F(x) \int_{-\infty}^{\infty} e^{xt} f(D)\{g\}_x(t)dt$$

   for all x in the strip of convergence of the integral. This implies in particular that $t \mapsto f(D)\{g\}_x(t)$ decays exponentially as $t \to \pm\infty$.

Theorem 1. *Under assumptions 1.-4. above, the function*

$$A(z) := F(z) \int_{-\infty}^{\infty} e^{zt} f(D)\{g\}_t(t)dt$$

extends to an entire function with the property that

$$f(z) - A(z) = F(z) \int_{-\infty}^{\infty} e^{zt}[f(D)\{g\}_z(t) - f(D)\{g\}_t(t)]dt$$

is absolutely convergent for all z with $\Re z \neq 0$.

Proof: By assumption, the integral defining A is absolutely convergent only in the strip in which the Laplace representation of $F(z)^{-1}$ converges.

We have the assumptions $f(x) = F(x)\int_{-\infty}^{\infty} e^{xt} f(D)\{g\}_x(t)dt$ in the strip of convergence and $\{g\}_x = \{g\}_y$ if and only if $xy > 0$, hence it follows initially in the strip of convergence that (with $x = \Re z$)

$$\begin{aligned} f(z) - A(z) &= F(z)\int_{-\infty}^{\infty} e^{zt}[f(D)\{g\}_x(t) - f(D)\{g\}_t(t)]dt \\ &= \begin{cases} F(z)\displaystyle\int_{-\infty}^{0} e^{zt}[f(D)\{g\}_x(t) - f(D)\{g\}_t(t)]dt & \text{if } \Re z > 0, \\ F(z)\displaystyle\int_{0}^{\infty} e^{zt}[f(D)\{g\}_x(t) - f(D)\{g\}_t(t)]dt & \text{if } \Re z < 0, \end{cases} \end{aligned}$$

and the latter representations are absolutely convergent not only in the initial strip but also in the respective indicated half-planes. Since f is analytic in $\Re z > 0$ and $\Re z < 0$, this representation defines the analytic continuation of A to $\mathbb{C}$. □

As we will see in the examples below, the assumptions about f are too restrictive to be used in practice. In general, it would be desirable to have statements about f (for a given g) which imply the existence of the representation that we have assumed here.

The assumptions about the representation of $F(z)^{-1}$ are done here for the purpose of shortening the statements. There is an extensive theory about the existence of such representations and their properties in Hirschman and Widder [4], but it would have gone beyond the frame of this article to list them here.

In order to prove statements of best approximation, an investigation of $f(D)\{g\}_x(t) - f(D)\{g\}_t(t)$ for real x and t is necessary. At this point, no general statement about this difference is known, the estimates are done for each example separately. It is not clear which framework to use to combine the arguments for the different examples into one general theory.

We sketch now a few examples. If $f(x) = x_+^n e^{-\lambda x}$, then for all real t,

$$f(D)\{g\}_x(t) = \begin{cases} g^{(n)}(t-\lambda) & \text{for } x > 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Indeed, in [8] and [9] it is shown that, if $1 = F(z)\mathcal{L}[g](z)$ with either $F(0) = 0$ or $g^{(n)}(0) = 0$, then

$$A_{n,F}(z) = F(z)\int_0^{\infty} e^{zt} g^{(n)}(t-\lambda)dt$$

is an analytic function in $\Re z < 0$ that extends to an entire function that interpolates $x_+^n e^{-\lambda x}$ at the zeros of F and nowhere else. In this case, there is an exhaustive description of the sign change properties of $f(D)\{g\}_x(t) - f(D)\{g\}_t(t)$ which depends crucially on the fact that the function g is a so-called totally positive function. The case $\lambda = 0$ was treated by the author in [8], the case $\lambda > 0$ is treated in connection with certain Tauberian theorems [9].

§3. One-sided approximation to the Gaussian

In this section the details of the one-sided best approximation from $\mathcal{A}(2\pi)$ to e^{-x^2} are worked out. Integral convolution is denoted by $*$, i.e., $(f * g)(t) := \int_{\mathbb{R}} f(t-u)g(u)du$. Recall that for $n \in \mathbb{Z}$ we have $g_n(t) = -te^{-nt}(e^{-t} - 1)^{-1}$.

Lemma 1. *Let $n \in \mathbb{Z}$. For all $t \neq 0$ we have*

$$g_n(t) = (k * e^{-D^2}\{g_n\}_t)(t),$$

where $k(u) = \frac{1}{2\sqrt{\pi}} e^{-u^2/4}$. (We regard $e^{-D^2}\{g_n\}_t(u)$ as a function of u with a fixed parameter t.)

Proof: We have for $t < 0$

$$\begin{aligned}
(k * e^{-D^2}\{g_n\}_t)(t) &= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-u)^2/4} e^{-D^2}\{g_n\}_t(u)du \\
&= -\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-u)^2/4} \sum_{\ell=-n+1}^{\infty} (u - 2\ell)e^{\ell u - \ell^2} du \\
&= -\sum_{\ell=-n+1}^{\infty} \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-u)^2/4}(u - 2\ell)e^{\ell u - \ell^2} du \\
&= -\sum_{\ell=-n+1}^{\infty} te^{\ell t} = g_n(t).
\end{aligned}$$

In this computation we differentiated the representation

$$e^{\ell t} = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(t-u)^2/4} e^{\ell u - \ell^2} du \tag{11}$$

with respect to ℓ. Representation (11) can be found, e.g., in (1) on page 178 of [4]. For $t > 0$ the claim follows with an analogous computation. □

To simplify notation we define

$$\mathrm{Tr}[e^{-D^2} g_n](t) := e^{-D^2}\{g_n\}_t(t) \tag{12}$$

with $t \in \mathbb{R}$.

Theorem 2. *Let $n \in \mathbb{Z}$. For $n-1 < \Re z < n$ the best upper $L^1(\mathbb{R})$–approximation to $\exp(-x^2)$ from $\mathcal{A}(2\pi)$ has representation*

$$G(z) := \frac{\sin^2 \pi z}{\pi^2} \int_{-\infty}^{\infty} e^{zt} \operatorname{Tr}[e^{-D^2} g_n](t)dt, \tag{13}$$

note that this integral exists for the given z from the calculations in the previous section.

Remark 1. *The lower best approximation from $\mathcal{A}(2\pi)$ to e^{-x^2} has the same structure, but instead of interpolating at the integers, it interpolates at $1/2+\mathbb{Z}$, hence the representation of the lower approximation G_- is obtained with $\sin^2 \pi(x-1/2)$ and $e^{-t/2} g_n(t)$.*

Remark 2. *The best approximation (without the one-sided condition) to the Gaussian from $\mathcal{A}(\pi)$ can be obtained by using $\sin \pi(x-1/2)$ and the Laplace inverse transform of its reciprocal; this involves the generating function of the Euler polynomials.*

Proof: Let $n-1 < x < n$. We would like to express e^{-x^2} similar to (13) with integrand $e^{xt} e^{-D^2}\{g_n\}_x(t)$, but this integral would diverge everywhere. To avoid this problem, the integral is expanded with a kernel of fast decay (which can be taken as the Gaussian again).

The representation of e^{x^2} as a Laplace integral implies for $n-1 < x < n$

$$G(x) = e^{-x^2} \frac{\sin^2 \pi x}{\pi^2} \int_{-\infty}^{\infty} e^{xt} (k * \operatorname{Tr}[e^{-D^2} g_n])(t)dt,$$

where k is defined in Lemma 1. The identity $1 = \pi^{-2} \sin^2(\pi x) \mathcal{L}[g_n](x)$, valid for $n-1 < x < n$, and Lemma 1 imply for $n-1 < x < n$

$$1 = \frac{\sin^2 \pi x}{\pi^2} \int_{-\infty}^{\infty} e^{-xt} g_n(t)dt = \frac{\sin^2 \pi x}{\pi^2} \int_{-\infty}^{\infty} e^{-xt} (k * e^{-D^2}\{g_n\}_t)(t)dt,$$

which gives with $h_t(u) := e^{-D^2}\{g_n\}_u(u) - e^{-D^2}\{g_n\}_t(u)$ the representation

$$G(x) - e^{-x^2} = e^{-x^2} \frac{\sin^2 \pi x}{\pi^2} \int_{-\infty}^{\infty} e^{-xt} (k * h_t)(t)dt,$$

recall that $k * h_t(t) = \int_{\mathbb{R}} k(t-u) h_t(u)du$. (Convergence in $n-1 < x < n$ can be established with (9).) Since

$$h_t(u) = \begin{cases} 0 & \text{if } tu > 0 \\ e^{-D^2}\{g_n\}_{-t}(u) - e^{-D^2}\{g_n\}_t(u) & \text{if } tu < 0, \end{cases}$$

we obtain

$$G(x) - e^{-x^2} \tag{14}$$
$$= e^{-x^2} \frac{\sin^2 \pi x}{\pi^2} \int_{tu<0} e^{xt} k(t-u)(e^{-D^2}\{g_n\}_{-t} - e^{-D^2}\{g_n\}_t)(u)\,du\,dt,$$

where $\int_{tu<0}$ stands for $\int_{t=0}^{\infty}\int_{u=-\infty}^{0} + \int_{t=-\infty}^{0}\int_{u=0}^{\infty}$. Considering $t > 0$ (and $u < 0$), we have

$$e^{-D^2}\{g_n\}_{-t}(u) - e^{-D^2}\{g_n\}_t(u) = -\sum_{\ell=-\infty}^{\infty} (u - 2\ell)e^{u\ell-\ell^2},$$

hence the inner integral of $\int_{t=0}^{\infty}\int_{u=-\infty}^{0}$ becomes

$$\begin{aligned}
I_1(x,t) :&= -\int_{-\infty}^{0} e^{xt} k(t-u) \sum_{\ell=-\infty}^{\infty} (u-2\ell)e^{u\ell-\ell^2} du \\
&= \frac{1}{2\sqrt{\pi}} e^{xt} e^{-t^2/4} \int_{-\infty}^{0} e^{tu/2} \Big(-\sum_{\ell=-\infty}^{\infty} (u-2\ell) e^{-(\frac{u}{2}-\ell)^2} \Big) du \\
&= \frac{1}{2\sqrt{\pi}} e^{xt} e^{-t^2/4} \int_{-\infty}^{0} e^{tu/2} \frac{d}{du} \Big[\sum_{\ell=-\infty}^{\infty} (e^{-(\frac{u}{2}-\ell)^2} - e^{-\ell^2}) \Big] du \\
&= -\frac{1}{4\sqrt{\pi}} t e^{xt-t^2/4} \int_{-\infty}^{0} e^{tu/2} \sum_{\ell=-\infty}^{\infty} (e^{-(\frac{u}{2}-\ell)^2} - e^{-\ell^2}) du \\
&\geq 0,
\end{aligned}$$

since the 1–periodic function $p(y) := \sum_{\ell} e^{-(y-\ell)^2}$ has non-negative Fourier coefficients and satisfies therefore $p(y) \leq |p(y)| \leq p(0)$. It is worth emphasizing that I_1 does not depend on n anymore. The integral $\int_0^{\infty} I_1(x,t)dt$ is absolutely convergent for every $x \in \mathbb{C}$ and defines an entire function. (Slight care is necessary since the integral over u does not converge for $t = 0$, but the factor t in the last representation of $I_1(x,t)$ rectifies this.)

The second integral of the sum, call it I_2, is dealt with in an analogous way. Since e^{-x^2} is entire, equation (13) together with the fact that I_1 and I_2 are entire, shows that the representations in (13) extend to a single entire function G.

We show now that the difference $G(x) - e^{-x^2}$ is absolutely integrable. This follows if we can show that $\int_{\mathbb{R}} I_j(x,t)dt \in L^1(\mathbb{R})$ for $j = 1, 2$. We prove this for $j = 1$ and leave the case $j = 2$ to the reader. Consider

$$J(t,u,x) := \frac{\sin^2 \pi x}{\pi^2} e^{xt-t^2/4-x^2} e^{tu/2}\, t \sum_{\ell=-\infty}^{\infty} (e^{-(\frac{u}{2}-\ell)^2} - e^{-\ell^2}).$$

We have

$$\int_0^\infty \int_{\mathbb{R}} |J(t,u,x)|dxdt = \mathcal{O}\Big(\frac{1}{u^2}\sum_{\ell=-\infty}^{\infty}(e^{-(\frac{u}{2}-\ell)^2} - e^{-\ell^2})\Big),$$

and since the series is bounded as $u \to -\infty$ and asymptotically equal to u^2 as $u \to 0$, the integral over $u \in (-\infty, 0]$ is absolutely convergent. An application of Fubini's theorem finishes the proof.

It remains to prove that G has exponential type 2π. This is done by investigating the growth of G in $\mathbb{C}$ and applying the Paley-Wiener theorem. G is bounded on the real line, since $G(x) - \exp(-x^2)$ is integrable. We investigate now the contribution of the integral in the representation (13).

Let $t_+^0 = 1$ for $t > 0$ and $t_+^0 = 0$ for $t < 0$. Splitting off the first term in (9) of the series for $e^{-D^2}\{g_n\}_t$ for $t > 0$ gives for all $t \in \mathbb{R}$

$$e^{-D^2}\{g_n\}_t(t) = h(t) - t_+^0(t+2n)e^{-nt-n^2},$$

and the Laplace transform of h exists in the vertical strip $n - 1 < \Re z < n + 1$. Investigation of the growth of the two series defining h shows that in the strip $n - 3/4 < \Re z < n + 3/4$ we may bound $\mathcal{L}[h]$ by a polynomial in n (with absolute constants). The strips overlap, hence G is bounded by a polynomial factor times $e^{2\pi|z|}$ in $\mathbb{C}$, and the Paley-Wiener theorem implies that G is of exponential type 2π.

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