QUADRATURE AND EXTREMAL BANDLIMITED FUNCTIONS

FRIEDRICH LITTMANN

Abstract. Let $\mathcal{A}(\delta)$ be the class of functions of exponential type $\delta > 0$. We prove that for integrable $F \in \mathcal{A}(2\pi\delta)$

$$\int_{-\infty}^{\infty} F(x)dx = \delta^{-1} \sum_{\xi \in \mathcal{T}_{\gamma,r}} \left(1 - \frac{\gamma}{\pi(\xi^2+\gamma^2)}\right) F(\delta^{-1}\xi)$$

where $\mathcal{T}_{\gamma,r}$ is the set of zeros of $B_{\gamma,r}(z) = z\sin\pi(z+r) - \gamma\cos\pi(z+r)$.

Let $a > (2\delta)^{-1}$. It is shown that for any Polya-Laguerre entire function E with $E(\pm a) = 0$ there exist two integrable functions $G_-, G_+ \in \mathcal{A}(2\pi\delta)$ such that for all real x

$$E(x)\{G_-(x) - \chi_{[-a,a]}(x)\} \le 0,$$
$$E(x)\{G_+(x) - \chi_{[-a,a]}(x)\} \ge 0.$$

Combining these results we find the minimal value of $||S-T||_1$ where $S, T \in \mathcal{A}(2\pi\delta)$ satisfy

$$S(x) \le \chi_{[-a,a]}(x) \le T(x)$$

for all real x. We determine extremal functions for which the minimal value is assumed. As an application we give an explicit expression for

$$C(\delta,\alpha) = \inf_{g\in\mathcal{A}_2(\delta)} \sup_{x\in[-\alpha,\alpha]} \frac{||g||_2^2}{|g(x)|^2}$$

where $\mathcal{A}_2(\delta)$ is the set of square integrable functions in $\mathcal{A}(\delta)$. This constant occurs in work of Donoho and Logan regarding reconstruction of bandlimited functions.

1. Introduction

An entire function f is said to be of exponential type $\eta \ge 0$ if for every ε there exists $A_\varepsilon > 0$ so that

$$|f(z)| \le A_\varepsilon e^{(\eta+\varepsilon)|z|} \tag{1.1}$$

for all complex z. We denote by $\mathcal{A}(\eta)$ the class of entire functions of exponential type η. For a test function φ (i.e., $\varphi \in C^\infty(\mathbb{R})$ that decays faster than any polynomial) we define the Fourier transform $\widehat{\varphi}$ by

$$\widehat{\varphi}(t) = \int_{-\infty}^{\infty} e^{-2\pi i x t}\varphi(x)dx, \tag{1.2}$$

2010 *Mathematics Subject Classification.* primary 42A05, secondary 30D15, 26A51.

Key words and phrases. entire functions of exponential type, totally positive functions, extremal majorants, quadrature, de Branges spaces.

and for a tempered distribution T we define the Fourier transform $\widehat{T}$ by $\widehat{T}(\varphi) = T(\widehat{\varphi})$ where φ is any test function. By the Paley-Wiener theorem for distributions (cf. [4, Theorem 7.3.1]), functions in $\mathcal{A}(\eta)$ have distributional Fourier transforms supported in $[-\eta/(2\pi), \eta/(2\pi)]$.

We consider the problem of finding the closest $f^* \in \mathcal{A}(\eta)$ with respect to $L^1(\mathbb{R})$-norm to given $f : \mathbb{R} \to \mathbb{R}$ with the additional constraint that $f^* \in \mathcal{A}(\eta)$ satisfies $f^* \geq f$ or $f^* \leq f$. The function f^* is called the best upper (or lower) approximation to f from $\mathcal{A}(\eta)$.

This kind of approximation with constraints occurred previously in connection with inequalities from analytic number theory and signal processing (cf. [1, 3, 13] and the references therein). In problems of this type particular special functions f are considered. Explicit knowledge of the error for such functions translates into optimal constants in certain inequalities.

We consider here the case that f is the characteristic function of an interval. For the case $\alpha\eta \in \pi\mathbb{Z}$ the solution was obtained by A. Selberg (see the account in [13]). An extension of Selberg's solution in different norms and higher dimensions was obtained by J.J. Holt and J.D. Vaaler [6].

Donoho and Logan [3, Lemma 10] found the optimal upper approximation for $\alpha\eta < \pi$. In our terminology they showed that there exists $f^* \in \mathcal{A}(\eta)$ with $f* \geq \mathbf{1}_{[-\alpha,\alpha]}$ on the real line such that

$$\int_{\infty}^{\infty} f^*(x)dx = 4\pi \left(\eta + \frac{\sin \eta\alpha}{\alpha} \right)^{-1}, \tag{1.3}$$

and they showed that this value is best possible. Logan announced a solution of the best upper approximation for any $\alpha > 0$ in [11], but his proof has not been published.

In this article we find for $\alpha\eta > \pi$ the best upper and lower approximation to $\mathbf{1}_{[-\alpha,\alpha]}$. We give expressions for the errors of the onesided approximations as certain finite sums that involve solutions of equations with transcendental and algebraic terms. In general, there does not seem to be a simple expression for these errors, however, the L^1-norm of the difference of upper and lower best approximation has a very simple form.

2. Results

For technial reasons we consider $\eta = 2\pi\delta$. Let $\alpha > (2\delta)^{-1}$. Assume that $S, T \in \mathcal{A}(2\pi\delta)$ satisfy

$$S(x) \leq \mathbf{1}_{[-\alpha,\alpha]}(x) \leq T(x) \tag{2.1}$$

for all real x. We prove in Theorem 5.2 that the inequality

$$\int_{-\infty}^{\infty} \{T(x) - S(x)\}dx \geq \frac{2}{\delta} \left(1 + \left| \frac{\sin 2\pi\alpha\delta}{2\pi\alpha\delta} \right| \right)^{-1} \tag{2.2}$$

holds. Furthermore (Theorem 5.3) there exist $T^*, S^* \in \mathcal{A}(2\pi)$ satisfying (2.1) such that there is equality in (2.2).

Inequality (2.2) is established in Section 5 using a quadrature formula for integrable functions in $\mathcal{A}(2\pi\delta)$. Let $\gamma > 0$ and let $\mathcal{T}_{\gamma,r}$ be the zeros of $B_{\gamma,r}(z) = z\sin\pi(z+r) - \gamma\cos\pi(z+r)$. We prove in Theorem 3.1 that if $F \in \mathcal{A}(2\pi\delta)$ is real valued on the real line and integrable, then

$$\int_{-\infty}^{\infty} F(x)dx = \delta^{-1} \sum_{\xi\in\mathcal{T}_{\gamma,r}} F(\delta^{-1}\xi)\left(1 - \frac{\gamma}{\pi(\xi^2+\gamma^2)+\gamma}\right). \tag{2.3}$$

It is evident that for any integrable, real-valued $F \in \mathcal{A}(2\pi\delta)$ with $F \geq \mathbf{1}_{[-\alpha,\alpha]}$ the inequality

$$\int_{-\infty}^{\infty} F(x)dx \geq \delta^{-1} \sum_{\substack{\xi\in\mathcal{T}_{\gamma,r}\\ |\xi|\leq\delta\alpha}} \left(1 - \frac{\gamma}{\pi(\xi^2+\gamma^2)+\gamma}\right) \tag{2.4}$$

follows. The importance of (2.3) lies in the fact that the integral of a function in $\mathcal{A}(2\pi\delta)$ is determined by its values at the zeros of a function in $\mathcal{A}(\pi\delta)$. (In this sense, (2.3) is a Gaussian quadrature.) Combined with an interpolation formula that controls not just the values of F at the nodes of (2.3), but of F' as well, this turns out to be crucial to show that (2.4) is sharp.

Identity (2.3) is proved by working in $L^2(\mathbb{R}, (\gamma^2+x^2)^{-1}dx) \cap \mathcal{A}(\pi)$, which is a de Branges Hilbert space generated by the Hermite-Biehler function $D_{\gamma,r}(z) = e^{-\pi i(z+r)}(z+i\gamma)$. Section 3 contains the details of this construction.

De Branges space were used previously by Holt and Vaaler [6] when solving an extremal problem for the signum function. Analogous polynomial spaces were used by Li and Vaaler [10]. The necessary conditions in their proofs are based on the fact that the value of a function in a de Branges space can be used to give a lower bound for its square norm. We use instead the representation of the square norm via an infinite series from [2, Theorem 22].

Formula (3.2) suggests that an entire function of exponential type $2\pi\delta$ that interpolates $\mathbf{1}_{[-\alpha,\alpha]}$ at the zeros of $B_{\gamma,r}$ and has derivatives equal to zero at the zeros of $B_{\gamma,r}$ with the exception of $\pm\alpha$ is a good candidate for a function T that satisfies (2.1) and (2.4).

We give a general method based on ideas from [9] that allows to construct entire functions interpolating the characteristic function of an interval at essentially arbitrary prescribed values of a (discrete) set $\mathcal{T}$. The interpolation procedure given in Section 4 has the property that under very mild conditions the set $\mathcal{T}$ already equals the set of points where interpolating function and $\mathbf{1}_{[-\alpha,\alpha]}$ agree.

We prove in Theorem 4.14 that for any symmetric Laguerre-Pólya entire function E with $E(\pm\alpha) = 0$ there exists entire $G_{E,\alpha}$ whose growth is essentially the growth of E and $\varepsilon \in \{\pm 1\}$ such that

$$\varepsilon E(x)\{G_{E,\alpha}(x) - \mathbf{1}_{[-\alpha,\alpha]}(x)\} \geq 0 \tag{2.5}$$

for all real x. In Section 5 we show that the choice

$$E(z) = [z \sin \pi(z + r) - \gamma \cos \pi(z + r)]^2 \tag{2.6}$$

in (2.5) gives best upper and lower approximation to $\mathbf{1}_{[-\alpha,\alpha]}$ from $\mathcal{A}(2\pi)$ in $L^1(\mathbb{R})$-norm. For fixed $\alpha > 0$ there are two different choices of r and γ so that $E(\pm\alpha) = 0$ which lead to the best upper and lower approximation.

We set $\mathcal{A}_2(\delta) = \mathcal{A}(\delta) \cap L^2(\mathbb{R})$. As an application of the above results we consider the constant $C(\delta, \alpha)$ defined by

$$C(\delta, \alpha) = \inf_{g \in \mathcal{A}_2(\delta)} \sup_{x \in [-\alpha,\alpha]} \frac{||g||_2^2}{|g(x)|^2}. \tag{2.7}$$

Let $f \in \mathcal{A}_2(\alpha)$ and assume that $T \subset \mathbb{R}$ is measurable. Logan and Donoho [3] proved

$$\int_T |f(x)|^2 dx \le C(\delta/2, \alpha) \left\{ \sup_{t \in \mathbb{R}} \ell([t, t + \delta] \cap T) \right\} \int_{-\infty}^{\infty} |f(x)|^2 dx$$

where $\ell(S)$ denotes Lebesgue measure of a set S. It is shown in [3] that bounding the ratio of the L^2-norm of f restricted to a set T and the L^2-norm of f is of importance in questions of reconstruction of bandlimited functions from a noisy sample. We give in Section 6 an expression for this constant and prove an asymptotic for fixed δ and $\alpha \to \infty$.

3. Gaussian-type quadrature for functions of exponential type

It is the goal of this section to prove

Theorem 3.1. *Let $\gamma > 0$ and let $\mathcal{T}_{\gamma,r}$ be the zeros of $B_{\gamma,r}$ given by*

$$B_{\gamma,r}(z) = z \sin \pi(z + r) - \gamma \cos \pi(z + r). \tag{3.1}$$

Let $F \in \mathcal{A}(2\pi\delta)$ be integrable. We have

$$\int_{-\infty}^{\infty} F(x) dx = \delta^{-1} \sum_{\xi \in \mathcal{T}_{\gamma,r}} F(\delta^{-1}\xi) \left(1 - \frac{\gamma}{\pi(\xi^2 + \gamma^2) + \gamma}\right). \tag{3.2}$$

Let $\gamma > 0$, $r \in \mathbb{R}$, and define

$$D_{\gamma,r}(z) = e^{-\pi i(z+r)}(z + i\gamma). \tag{3.3}$$

We note that for $z = x + iy$ with $y > 0$ the inequality

$$|D_{\gamma,r}(x + iy)| > |D_{\gamma,r}(x - iy)|$$

holds for all real x. Here $D^*_{\gamma,r}$ is defined by $D^*_{\gamma,r}(z) = \overline{D_{\gamma,r}(\overline{z})}$. The function $D_{\gamma,r}$ is clearly an element of $\mathcal{A}(\pi)$, hence (cf. [2, Problem 37]) the function $D_{\gamma,r}$ has bounded type in the upper half plane, and its mean type ([2, p. 26]) equals π.

We define $\mathcal{H}(D_{\gamma,r})$ to be the complex vector space of all entire functions $F \in \mathcal{A}(\pi)$ such that

$$\int_{-\infty}^{\infty} \left| \frac{F(x)}{D_{\gamma,r}(x)} \right|^2 dx < \infty,$$

and we define the scalar product

$$\langle F, G\rangle = \int_{-\infty}^{\infty} F(x)\overline{G(x)}|D_{\gamma,r}(x)|^{-2}dx. \tag{3.4}$$

Let $F \in \mathcal{H}(D_{\gamma,r})$. Since $D_{\gamma,r}$ has exponential type π, [6, Lemma 12 (iii)] implies that $F/D_{\gamma,r}$ and $F/D^*_{\gamma,r}$ have bounded type (and non-positive mean type) in the upper half plane. Hence, $\mathcal{H}(D_{\gamma,r})$ is a reproducing kernel Hilbert space (cf. [2, Ch. 2, section 19]).

To calculate the reproducing kernel we note that $D_{\gamma,r} = A_{\gamma,r} - iB_{\gamma,r}$ where $B_{\gamma,r} = (i/2)(D_{\gamma,r} - D^*_{\gamma,r})$ is given by (3.1), and $A_{\gamma,r}$ is defined by

$$A_{\gamma,r}(z) = \frac{1}{2}(D_{\gamma,r}(z) + D^*_{\gamma,r}(z)) = z\cos\pi(z+r) + \gamma\sin\pi(z+r). \tag{3.5}$$

Hence

$$\begin{aligned} K_{\gamma,r}(w,z) &= \frac{B_{\gamma,r}(z)A_{\gamma,r}(\overline{w}) - A_{\gamma,r}(z)B_{\gamma,r}(\overline{w})}{\pi(z-\overline{w})} \\ &= \frac{\gamma(\overline{w}-z)\cos\pi(z-\overline{w}) - (\gamma^2 + z\overline{w})\sin\pi(z-\overline{w})}{\pi(\overline{w}-z)} \end{aligned} \tag{3.6}$$

for $z \neq \overline{w}$, and

$$\begin{aligned} K_{\gamma,r}(\overline{z},z) &= \pi^{-1}(B'_{\gamma,r}(z)A_{\gamma,r}(z) - A'_{\gamma,r}(z)B_{\gamma,r}(z)) \\ &= z^2 + \gamma^2 + \pi^{-1}\gamma. \end{aligned} \tag{3.7}$$

Within this framework it is now straightforward to give the

Proof of Theorem 3.1. We consider first $F \in \mathcal{A}(2\pi)$ such that F is non-negative and integrable on the real line. We denote by $\mathcal{C}$ the class of entire functions F satisfying

$$\sum_{k=1}^{\infty} |\Im(a_k^{-1})| < \infty,$$

where a_k are the zeros of F. We define G_γ by

$$G_\gamma(z) = F(z)(\gamma^2 + z^2). \tag{3.8}$$

for all complex z. Since F is integrable we have

$$\int_{-\infty}^{\infty} \frac{\log^+ |G_\gamma(x)|}{1+x^2}dx < \infty,$$

and by [8, Theorem 7 on p. 243], it follows that $G_\gamma \in \mathcal{C}$. Since G_γ is non-negative on the real line and has exponential type 2π, [8, Theorem 1 on p. 437] implies that

$$G_\gamma = SS^*,$$

where S is of type π and has no zeros in the upper half plane. We note that

$$\int_{-\infty}^{\infty} \frac{|S(x)|^2}{|D_{\gamma,r}(x)|^2}dx = \int_{-\infty}^{\infty} \frac{G_\gamma(x)}{x^2+\gamma^2}dx = ||F||_1 < \infty,$$

hence by [6, Lemma 12 (iii)], S is an element of the de Branges space $\mathcal{H}(D_{\gamma,r})$. Since $B_{\gamma,r} \notin \mathcal{H}(D_{\gamma,r})$, [2, Theorem 22 and Problem 48] imply

$$\int_{-\infty}^{\infty} F(x)dx = ||S||_{\mathcal{H}(D_{\gamma,r})} = \sum_{\xi\in\mathcal{T}_{\gamma,r}} \frac{G_\gamma(\xi)}{K_{\gamma,r}(\xi,\xi)}, \tag{3.9}$$

and (3.7) gives (3.2) for non-negative F. Since $\gamma > 0$, the weights in (3.2) are positive.

Let now $F \in \mathcal{A}(2\pi)$ be real valued on the real line and integrable. We define $f : \mathbb{R} \to \mathbb{R}_0^+$ by

$$f(x) = \max(F(x), 0) \tag{3.10}$$

for all real x. Since F' is integrable, the function f has finite total variation. By [13, Corollary 12], there exists integrable $F^+ \in \mathcal{A}(2\pi)$ such that $F^+(x) \geq f(x)$ for all real x. We may therefore write

$$F = F^+ - (F^+ - F)$$

as a difference of two non-negative, integrable functions in $\mathcal{A}(2\pi)$ to each of which we may apply (3.9), and (3.2) follows for real and integrable $F \in \mathcal{A}(2\pi)$. For $F \in \mathcal{A}(2\pi)$ that is integrable but not necessarily real valued on the real line, we write

$$F(z) = \frac{F(z) + F^*(z)}{2} + i\frac{F(z) - F^*(z)}{2i},$$

and we note that $2^{-1}(F + F^*)$ and $2^{-1}i(F - F^*)$ are real entire, integrable functions in $\mathcal{A}(2\pi)$ to which (3.2) applies. □

4. Interpolation at zeros of Laguerre-Pólya functions

In this section we prove the general interpolation formula mentioned in the introduction. We consider nodes of an interpolation formula that are given as zeros of so-called Laguerre-Pólya functions. An entire function E is said to be a Laguerre-Pólya function if it has the form

$$E(z) = e^{-cz^2+bz} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\xi_k}\right) e^{z/\xi_k} \tag{4.1}$$

where $c \geq 0$, b is real, ξ_k are real, and

$$\sum_{k=1}^{\infty} \xi_k^{-2} < \infty. \tag{4.2}$$

The elements of this class are exactly those entire functions that are uniform limits on compact subsets of $\mathbb{C}$ of polynomials having only real roots (Theorem 3.2 and Theorem 3.3 on page 42 of [5]). In order to simplify some of the statements we restrict attention to those Laguerre-Pólya functions that have a zero set that is unbounded above and below.

Definition 4.1. The class $\mathcal{E}$ consists of all entire functions E satisfying (4.1) such that the set of zeros of E is unbounded above and below. We denote by $\mathcal{T}_E$ the set of zeros of E. When no confusion can arise we omit the subscript. We denote by $\mathcal{B}$ the subclass of even functions in $\mathcal{E}$.

Let $E \in \mathcal{E}$. For $c \notin \mathcal{T}$ we define a function g_c by

$$g_c(t) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{c-iT}^{c+iT} \frac{e^{ts}}{B(s)} ds, \tag{4.3}$$

where the integral is a complex line integral along a vertical line segment. By Corollary 5.4 on page 53 of [5] applied to $z \mapsto E(z+c)$ we have

$$\frac{1}{E(z)} = \int_{-\infty}^{\infty} e^{-zu} g_c(u) du \text{ for } \ell_c < \Re z < r_c, \tag{4.4}$$

where ℓ_c is the largest element in $\mathcal{T}$ with $\ell_c < c$ and $r_c \in \mathcal{T}$ is the smallest element with $c < r_c$. If $\xi \in \mathcal{T}$, we define $g_{\xi-}$ by (4.3) with $c = \xi - \varepsilon$ so that (4.4) holds in $\ell_\xi < \Re z < \xi$. Analogously, $g_{\xi+}$ denotes the function for which (4.4) holds in $\xi < \Re z < r_\xi$. If $c = 0$ then g_0 has an intrinsic characterization that is crucial for the inequalities of this section.

Lemma 4.2. *Let $E \in \mathcal{E}$ with $E(0) \neq 0$, and let g_0 be defined by (4.3) with $c = 0$. If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and has n sign changes on the real line, then the integral convolution $g * \varphi$ given by*

$$g * \varphi(t) = \int_{-\infty}^{\infty} g(t-u)\varphi(u) du \tag{4.5}$$

has no more than n changes of sign on the real line.

Proof. This is Theorem 2.1 in chapter IV of [5]. □

This 'sign-reducing' property is in fact a necessary and sufficient characterization of g_0. This was first shown by I. J. Schoenberg [12], see also [5, 7]. For our purpose we only require

Corollary 4.3. *Let $E \in \mathcal{E}$ with zero set $\mathcal{T}$. If $c \notin \mathcal{T}$, then*

$$E(c) g_c(t) \geq 0 \tag{4.6}$$

for all real t.

Proof. If $E \in \mathcal{E}$, then any translation of E along the real line is also an element of $\mathcal{E}$. Such a translation corresponds to multiplication of g_c by an exponential which does not change the sign of g_c.

We may therefore assume that $c = 0$ in which case the statement follows from Lemma 4.2, and the sign is obtained from (4.4). □

Differentiation and an application of the residue theorem in (4.3) give

Lemma 4.4. *Let $E \in \mathcal{E}$ and $c \notin \mathcal{T}$. Let g_c satisfy* (4.4) *and let $n \in \mathbb{N}_0$. Then there exist polynomials P_n and Q_n so that*

$$g_c^{(n)}(t) \le \begin{cases} P_n(t)e^{\ell_c t} & \text{as } t \to \infty, \\ Q_n(t)e^{r_c t} & \text{as } t \to -\infty. \end{cases} \tag{4.7}$$

Proof. For $c = 0$ this is Theorem 2.1 in chapter VI of [5]. A translation of E gives the result for any $c \notin \mathcal{T}$. □

We define for $\alpha \in \mathbb{R}$ functions

$$f_{\alpha+}(t) = \begin{cases} e^{\alpha t} \text{ if } t > 0, \\ 0 \text{ else}, \end{cases} \qquad f_{\alpha-}(t) = \begin{cases} 0 \text{ if } t > 0, \\ -e^{\alpha t} \text{ else}, \end{cases} \tag{4.8}$$

and we note the Laplace transforms of both functions equal $(z-\alpha)^{-1}$ with convergence region $\Re z > \alpha$ for $f_{\alpha+}$ and $\Re z < \alpha$ for $f_{\alpha-}$.

We recall that the (two-sided) Laplace transform of the convolution $f * g$ is absolutely convergent and equals the product of the Laplace transforms of f and g for all z in the largest vertical open strip where the two-sided Laplace transforms of f and g both converge absolutely.

Lemma 4.5. *Let $E \in \mathcal{E}$, $c \notin \mathcal{T}$, and g_c given by* (4.3)*. If $\xi \in \mathcal{T}$ then*

$$\frac{z-\xi}{E(z)} = \int_{-\infty}^{\infty} e^{-zt}(D-\xi)g_c(t)dt \tag{4.9}$$

in $\ell_c < \Re z < r_c$. If ξ is a simple zero of E and equal to r_c or ℓ_c, the strip of convergence includes ξ. (D stands for the differential operator d/dt.)

If $\alpha < c$, then

$$\frac{1}{(z-\alpha)E(z)} = \int_{-\infty}^{\infty} e^{-zt}[f_{\alpha+} * g_c](t)dt \tag{4.10}$$

in $\max(\ell_c, \alpha) < \Re z < r_c$. If $\alpha > c$, the analogous representation with $f_{\alpha-}$ instead of $f_{\alpha+}$ holds in $\ell_c < \Re z < \min(\alpha, r_c)$.

Proof. An integration by parts in (4.4) gives (4.9) in $\ell_c < \Re z < r_c$. To prove (4.10), note that the Laplace transform of $f_{\alpha+}$ represents $(z-\alpha)^{-1}$ in $\Re z > \alpha$, and since $\alpha < c$ by assumption, we obtain (4.10) in $\max(t_{c-}, \alpha) < \Re z < t_{c+}$. □

In the next lemma we investigate the effect on the derivatives of g_c if a zero of E is moved 'to the right'.

Lemma 4.6. *Let $k \in \mathbb{N}_0$, and let $g \in C^k(\mathbb{R})$. Assume $g^{(k)}(x) > 0$ for $x \le x_0$. Then for $\alpha < \beta$ the function h defined by*

$$h = f_{\beta+} * [(D-\alpha)g]$$

satisfies $h^{(k)}(x) > 0$ for all $x \le x_0$.

Proof. Define $\delta > 0$ by $\delta = \beta - \alpha$. We have

$$\begin{aligned} h(x) &= \int_{-\infty}^{x} e^{\beta(x-u)}\{g'(u) - \alpha g(u)\}du \\ &= e^{\beta x}\int_{-\infty}^{x} e^{-\beta u}\{g'(u) - \alpha g(u)\}du \\ &= e^{\beta x}\int_{-\infty}^{x} e^{-\delta u}[e^{-\alpha u}g(u)]'du \\ &= e^{\beta x}\left\{e^{-\beta x}g(x) + \delta\int_{-\infty}^{x} e^{-\beta u}g(u)du\right\} \\ &= g(x) + \delta\int_{-\infty}^{x} e^{\beta(x-u)}g(u)du \\ &= g(x) + \delta\{f_{\beta+} * g\}(x). \end{aligned}$$

It follows that $h^{(k)} = g^{(k)} + \delta\{f_{\beta+} * g^{(k)}\}$, and since

$$f_{\beta+} * g^{(k)}(x) = \int_{-\infty}^{x} e^{\beta(x-u)}g^{(k)}(u)du,$$

the result follows. □

Lemma 4.7. *Let $E \in \mathcal{E}$ and $\alpha, \xi_1, \xi_2 \in \mathcal{T}$ with $\alpha > 0$ and $0 \le \xi_1 \le \xi_2 < \alpha$. If $\xi_1 = \xi_2$, assume that ξ_1 has multiplicity at least two. Define g by*

$$\frac{z-\alpha}{E(z)} = \int_{-\infty}^{\infty} e^{-zt}g(t)dt \text{ for } \ell_\alpha < \Re z < \alpha. \tag{4.11}$$

Then g, g', and g'' are all of one sign on $\mathbb{R}$, and their sign equals the sign of $(z-\alpha)^{-1}E(z)$ in the interval (ℓ_α, α).

Proof. We note that

$$z \mapsto \frac{z^2E(z)}{(z-\alpha)(z-\xi_1)(z-\xi_2)} \in \mathcal{E}. \tag{4.12}$$

We define h by (4.3) for the function in (4.12), i.e.,

$$\frac{(z-\alpha)(z-\xi_1)(z-\xi_2)}{z^2E(z)} = \int_{-\infty}^{\infty} e^{-zt}h(t)dt \tag{4.13}$$

holds in $\ell_\alpha < \Re z < \alpha$. Two integrations by parts give

$$\frac{(z-\alpha)(z-\xi_1)(z-\xi_2)}{z^{2-j}E(z)} = \int_{-\infty}^{\infty} e^{-zt}h''(t)dt \qquad (j \in \{0,1,2\}) \tag{4.14}$$

and the functions on the left are reciprocals of elements in $\mathcal{E}$. By Corollary 4.3 the functions h, h' and h'' have no sign changes on the real line, and their signs equal the sign of $(z-\alpha)^{-1}E(z)$ in (ℓ_α, α). Since

$$g = f_{\xi_1+} * f_{\xi_2+} * h'' = f_{\xi_1+} * [f_{\xi_2+} * h']'$$

the result follows with two applications of Lemma 4.6. □

Definition 4.8. Let $E \in \mathcal{E}$ and $\alpha \in \mathcal{T}$. Define for all real t functions $h_{\alpha-}$ and $h_{\alpha+}$ by

$$h_{\alpha\pm}(t) = g_{\alpha\pm}(t) - \frac{e^{\alpha t}}{\alpha} g'_{\alpha\pm}(0). \tag{4.15}$$

The functions $h_{\alpha\pm}$ will be used in Theorem 4.14 to construct entire functions that interpolate the characteristic function of $[-\alpha, \alpha]$ with sign changes at the zeros of E. We prove first integral representations for the derivatives of $h_{\alpha+}$ and $h_{\alpha-}$.

Lemma 4.9. *Let $E \in \mathcal{E}$, and let $\alpha \in \mathcal{T}$ with $\alpha > 0$. Let g_- be given by*

$$g_-(t) = \int_{c-i\infty}^{c+i\infty} e^{st} \frac{s-\alpha}{E(s)} ds \tag{4.16}$$

with $\ell_\alpha < c < \alpha$, and define g_+ analogously with c satisfying now $\alpha < c < r_\alpha$. Then

$$g'_{\alpha\pm}(t) - g'_{\alpha\pm}(0) e^{\alpha t} = \int_0^t e^{\alpha(t-u)} g'_\pm(u) du \tag{4.17}$$

for all real t.

Proof. We prove the claim for g_-. We note that $z \mapsto (z-\alpha)^{-1}E(z)$ is in $\mathcal{E}$. Equation (4.4) gives

$$\frac{z-\alpha}{E(z)} = \int_{-\infty}^{\infty} e^{-zt} g_-(t) dt \tag{4.18}$$

in $\ell_\alpha < \Re z < \alpha$, and Lemma 4.5 implies

$$(D-\alpha) g'_{\alpha-} = g'_-.$$

It follows that

$$\frac{d}{dt}[e^{-\alpha t} g'_{\alpha-}(t)] = e^{-\alpha t} g'_-(t) \tag{4.19}$$

holds for all real t, and hence

$$e^{-\alpha t} g'_{\alpha-}(t) - g'_{\alpha-}(0) = \int_0^t e^{-\alpha u} g'_-(u) du. \tag{4.20}$$

This implies (4.17). The proof for $g_{\alpha+}$ proceeds analogously. □

The next proposition gives inequalities for $h_{\alpha\pm}$ that are central for the interpolation properties later on.

Proposition 4.10. *Let $E \in \mathcal{E}$ and $\xi_1, \xi_2, \alpha \in \mathcal{T}$ with $\alpha > 0$ and $0 \le \xi_1 \le \xi_2 < \alpha$ (multiplicities as in Lemma 4.7). Let $\varepsilon_{\alpha-}$ and $\varepsilon_{\alpha+}$ be the sign of E in the interval (ℓ_α, α) and (α, r_α), respectively. We have*

$$\varepsilon_{\alpha+}[h_{\alpha+}(t) - h_{\alpha+}(-t)] \ge 0 \text{ for } t > 0, \tag{4.21}$$

$$\varepsilon_{\alpha-}[h_{\alpha-}(t) - h_{\alpha-}(-t)] \le 0 \text{ for } t > 0, \tag{4.22}$$

$$\varepsilon_{\alpha+} h_{\alpha+}(t) \le 0 \text{ for } t < 0, \tag{4.23}$$

$$\varepsilon_{\alpha-} h_{\alpha-}(t) \ge 0 \text{ for } t < 0. \tag{4.24}$$

Proof. Let g_+ and g_- be defined by (4.16). Consider first $t > 0$. We claim that

$$\begin{aligned} h_{\alpha\pm}(t) - h_{\alpha\pm}(-t) \\ = \frac{1}{\alpha}\int_0^t \Big\{ g_\pm'(u)(e^{\alpha(t-u)} - 1)du + g_\pm'(-u)(e^{-\alpha(t-u)} - 1) \Big\}\, du. \end{aligned} \tag{4.25}$$

To see this, differentiate both sides in (4.25) with respect to t and use (4.17) to show that the derivatives agree. Furthermore, both sides equal zero at $t = 0$. This proves (4.25).

We consider first g_+ when $\varepsilon_{\alpha+} > 0$. Lemma 4.7 implies that g_+, g_+', g_+'' are all positive on the real line. In particular g_+' is increasing and positive, hence for positive u

$$g_+'(u) \geq g_+'(-u) \geq 0.$$

It follows from (4.25) that

$$h_{\alpha+}(t) - h_{\alpha+}(-t) \geq 2\int_0^t g_+'(-u)\{\cosh(\alpha(t-u)) - 1\}\, du \geq 0. \tag{4.26}$$

If $\varepsilon_{\alpha+} < 0$, the inequalities reverse. This establishes (4.21).

Consider now g_-. If $\varepsilon_{\alpha-} < 0$ then g_- is positive since the Laplace transform of g_- is the reciprocal of $z \mapsto (z-\alpha)^{-1}E(z)$. The functions g_-' and g_-'' are positive by Lemma 4.7. In this case inequality (4.22) follows as in (4.26). The sign of g_- reverses if $\varepsilon_{\alpha-}$ is positive.

Consider $t < 0$. The identity

$$-h_{\alpha\pm}(t) = \frac{1}{\alpha}\left(\int_{-\infty}^t g_\pm'(u)du + \int_t^0 e^{\alpha(t-u)} g_\pm'(u)du\right) \tag{4.27}$$

can be established by differentiating both sides with respect to t and applying (4.17) to show that the derivatives agree. Note also that $e^{-\alpha u}g'(u)$ has at most polynomial growth as $u \to -\infty$, hence both sides in (4.27) converge to zero as $t \to -\infty$. If $\varepsilon_{\alpha+} > 0$ then g_+' is positive, otherwise it is negative. This proves (4.23). Inequality (4.24) is shown in the same way. □

The next lemma is an auxiliary result that shall allow us to prove that the interpolations in Theorem 4.14 below are entire functions.

Lemma 4.11. *Let $E \in \mathcal{E}$ and $\alpha \in \mathcal{T}$. The functions H_- and H_+ defined by*

$$H_\pm(z) = E(z)\int_0^\infty e^{-zt} g_{\alpha\pm}(t)dt \tag{4.28}$$

in the region $\Re z > \alpha$ extend to entire functions in the complex plane. Moreover,

$$H_-(\xi) = \begin{cases} 0 \text{ for } \xi \in \mathcal{T} \text{ with } \xi \geq \alpha, \\ 1 \text{ for } \xi \in \mathcal{T} \text{ with } \xi < \alpha, \end{cases} \tag{4.29}$$

and

$$H_+(\xi) = \begin{cases} 0 \text{ for } \xi \in \mathcal{T} \text{ with } \xi > \alpha, \\ 1 \text{ for } \xi \in \mathcal{T} \text{ with } \xi \le \alpha. \end{cases} \tag{4.30}$$

Proof. We prove Lemma 4.11 for H_-. Recall that ℓ_α is the largest zero of E strictly less than α. We note that by Lemma 4.7, $g_{\alpha-}(t) \lesssim e^{\ell_\alpha t}$ for $t \to \infty$. The integral defining H_- represents therefore an analytic function in the half plane $\Re z > \ell_\alpha$. Moreover, absolute convergence of the integral in this region implies $H_-(\xi) = 0$ for $\xi \in \mathcal{T}$ with $\xi > \ell_\alpha$.

For $\ell_\alpha < \Re z < \alpha$ we have with (4.4)

$$H_-(z) = 1 - E(z) \int_{-\infty}^{0} e^{-zt} g_{\alpha-}(t) dt, \tag{4.31}$$

and the right hand side is analytic in the half plane $\Re z < \alpha$, providing the analytic continuation of H_- to $\mathbb{C}$. The evaluations $H_-(\xi) = 1$ for $\xi \in \mathcal{T}$ with $\xi < \alpha$ follow as above. The proof for H_+ is analogous; the half planes are now $\Re z > \alpha$ and $\Re z < r_\alpha$. □

Recall that $\mathcal{B}$ is the subclass of $\mathcal{E}$ consisting of even functions in $\mathcal{E}$.

Definition 4.12. Let $E \in \mathcal{B}$ and $\alpha \in \mathcal{T}$. Define $G^+_{E,\alpha}$ and $G^-_{E,\alpha}$ by

$$G^{\pm}_{E,\alpha}(z) = H_\pm(z) + H_\pm(-z) - 2g'_{\alpha\pm}(0)\frac{E(z)}{z^2 - \alpha^2} - 1. \tag{4.32}$$

Lemma 4.11 implies that $G^+_{E,\alpha}$ and $G^-_{E,\alpha}$ are entire functions, and they are plainly even. We write $G^\pm_\alpha$ if no confusion can arise. The following lemma provides an integral representation of $G^\pm_\alpha - \mathbf{1}_{[-\alpha,\alpha]}$ that allows an application of Proposition 4.10 to control the sign changes of this difference.

Lemma 4.13. *Let $E \in \mathcal{B}$ and $\alpha \in \mathcal{T}$ with $\alpha > 0$. We have*

$$G^{\pm}_{E,\alpha}(z) = \begin{cases} E(z) \int_0^\infty e^{-zt}\{h_{\alpha\pm}(t) - h_{\alpha\pm}(-t)\} dt & \text{for } \Re z > \alpha, \\ 1 - E(z) \int_{-\infty}^0 (e^{-zt} + e^{zt}) h_{\alpha\pm}(t) dt & \text{for } -\alpha < \Re z < \alpha, \\ E(z) \int_0^\infty e^{zt}\{h_{\alpha\pm}(t) - h_{\alpha\pm}(-t)\} dt & \text{for } \Re z < -\alpha. \end{cases} \tag{4.33}$$

Proof. We consider G^-_α. Let $\Re z > \alpha$. Then (4.31) and the fact that E is even imply

$$\begin{aligned} H_-(-z) - 1 &= -E(-z) \int_{-\infty}^{0} e^{zt} g_{\alpha-}(t) dt \\ &= -E(z) \int_0^\infty e^{-zt} g_{\alpha-}(-t) dt, \end{aligned} \tag{4.34}$$

and combining this with (4.28) gives

$$H_-(z) + \{H_-(-z) - 1\} = E(z) \int_0^\infty e^{-zt}\{g_{\alpha-}(t) - g_{\alpha-}(-t)\}dt.$$

Equation (4.32) and

$$\frac{2\alpha}{z^2 - \alpha^2} = \int_0^\infty e^{-zt}(e^{\alpha t} - e^{-\alpha t})dt$$

imply (4.33) in $\Re z > \alpha$.

In $-\alpha < \Re z < \alpha$ we have

$$\begin{aligned} H_-(z) - 1 - g'_{\alpha-}(0)\frac{E(z)}{\alpha(z-\alpha)} &= -E(z)\int_{-\infty}^0 e^{-zt}\left(g_{\alpha-}(t) - \frac{e^{\alpha t}}{\alpha}\right)dt \\ &= -E(z)\int_{-\infty}^0 e^{-zt}h_{\alpha-}(t)dt \end{aligned}$$

which implies (4.33) in this region. The statement for $\Re z < \alpha$ follows by symmetry. The proof for G_α^+ is essentially the same. □

Theorem 4.14. *Let $E \in \mathcal{B}$ with zero set $\mathcal{T}$. Let $\xi_1, \xi_2, \alpha \in \mathcal{T}$ with $0 \le \xi_1 \le \xi_2 < \alpha$ (in the sense of Lemma 4.7). Let $\varepsilon_{\alpha+}$ and $\varepsilon_{\alpha-}$ be the signs of E in (α, r_α) and (ℓ_α, α), respectively. The entire functions defined in (4.32) satisfy*

$$\pm\frac{G_\alpha^\pm(x) - \mathbf{1}_{[-\alpha,\alpha]}(x)}{\varepsilon_{\alpha\pm}E(x)} \ge 0 \tag{4.35}$$

for all real x (with $\pm$ equal to 1 for G_α^+ and -1 for G_α^-). Moreover,

$$|G_\alpha^\pm(z)| \lesssim \frac{|E(z)|}{1 + |\Re z|^4} \tag{4.36}$$

for all complex z.

Proof. We consider G_α^-. Inequality (4.35) follows for $|\Re z| > \alpha$ from (4.33) and (4.22), and for $|\Re z| < \alpha$ from (4.33) and (4.24).

We note that h_- defined by $h_-(t) = h_{\alpha-}(t) - h_{\alpha-}(-t)$ satisfies

$$h_-(0) = h'_-(0) = h''_-(0) = 0 \tag{4.37}$$

and (4.33) implies for $\Re z > \alpha$ that

$$|G_\alpha^-(z)| \lesssim |E(z)| \int_0^\infty e^{-t\Re z}t^3 dt \lesssim \frac{|E(z)|}{1 + |\Re z|^4}. \tag{4.38}$$

Equation (4.15) shows that this estimate extends to $\Re z > \ell_\alpha$.

Estimate (4.36) holds for $\Re z < -\alpha$ by symmetry. Finally, in the range $-\alpha < \Re z < \alpha$ equation (4.33) implies the bound due to the fact that

$$z \mapsto \int_{-\infty}^0 (e^{-zt} + e^{zt})h_{\alpha-}(t)dt$$

remains bounded for all z in $|\Re z| \le \frac{1}{2}(\alpha + \ell_\alpha)$. The argumentation for $h_{\alpha+}$ is similar. □

5. One sided approximation

To set the interpolation of $\mathbf{1}_{[-\alpha,\alpha]}$ up, we define

$$\gamma(\alpha) = \begin{cases} \alpha \tan \pi\alpha \text{ if } k < \alpha < k + \frac{1}{2}, \\ -\alpha \cot \pi\alpha \text{ if } k + \frac{1}{2} < \alpha < k + 1 \end{cases} \tag{5.1}$$

where $k \in \mathbb{Z}$.

Lemma 5.1. *Let $k \in \mathbb{Z}$ and $\alpha > 0$. Then $\gamma(\alpha) > 0$ for all α with $2\alpha \notin \mathbb{Z}$, $B_{\gamma(\alpha),0}(\alpha) = 0$ if $k < \alpha < k + \frac{1}{2}$, $B_{\gamma(\alpha),\frac{1}{2}}(\alpha) = 0$ if $k + \frac{1}{2} < \alpha < k$, and*

$$\alpha^2 + \gamma^2(\alpha) = \begin{cases} \alpha^2 \sec^2 \pi\alpha \text{ if } k < \alpha < k + \frac{1}{2}, \\ \alpha^2 \csc^2 \pi\alpha \text{ if } k + \frac{1}{2} < \alpha < k + 1. \end{cases} \tag{5.2}$$

Proof. These properties may be verified by direct calculation. □

We shall use Theorem 4.14 with $E = B^2_{\gamma(\alpha),0}$ and $E = B^2_{\gamma(\alpha),\frac{1}{2}}$. To simplify notation we define $\mathcal{G}^+_\alpha$ by

$$\mathcal{G}^+_\alpha(z) = \begin{cases} G^+_{B^2_{\gamma(\alpha),0},\alpha}(z) \text{ if } k < \alpha < k + \frac{1}{2}, \\ G^+_{B^2_{\gamma(\alpha),\frac{1}{2}},\alpha}(z) \text{ if } k + \frac{1}{2} < \alpha < k, \end{cases} \tag{5.3}$$

where $k \in \mathbb{Z}$ and $z \in \mathbb{C}$. The function $\mathcal{G}^-_\alpha$ is defined analogously. We denote

$$\mathcal{T}_{\gamma,r} = \{z \in \mathbb{C} \mid B_{\gamma,r}(z) = 0\}. \tag{5.4}$$

The following theorems collect the properties of $\mathcal{G}^+_\alpha$ and $\mathcal{G}^-_\alpha$.

Theorem 5.2. *Let $\alpha > \frac{1}{2}$ with $2\alpha \notin \mathbb{Z}$. Then $\mathcal{G}^+_\alpha$ and $\mathcal{G}^-_\alpha$ are in $\mathcal{A}(2\pi)$ and satisfy*

$$\mathcal{G}^-_\alpha(x) \le \mathbf{1}_{[-\alpha,\alpha]}(x) \le \mathcal{G}^+_\alpha(x) \tag{5.5}$$

for all real x and

$$\int_{-\infty}^{\infty} \mathcal{G}^+_\alpha(x)dx = \sum_{\substack{\xi \in \mathcal{T}_{\gamma(\alpha),r(\alpha)} \\ |\xi| \le \alpha}} \left(1 - \frac{\gamma(\alpha)}{\pi(\xi^2 + \gamma(\alpha)^2) + \gamma(\alpha)}\right) \tag{5.6}$$

$$\int_{-\infty}^{\infty} \mathcal{G}^-_\alpha(x)dx = \sum_{\substack{\xi \in \mathcal{T}_{\gamma(\alpha),r(\alpha)} \\ |\xi| < \alpha}} \left(1 - \frac{\gamma(\alpha)}{\pi(\xi^2 + \gamma(\alpha)^2) + \gamma(\alpha)}\right) \tag{5.7}$$

where $r(\alpha) = 0$ if $k < \alpha < k + \frac{1}{2}$ and $r(\alpha) = \frac{1}{2}$ if $k + \frac{1}{2} < \alpha < k + 1$ for $k \in \mathbb{Z}$. In particular

$$\int_{-\infty}^{\infty} [\mathcal{G}^+_\alpha(x) - \mathcal{G}^-_\alpha(x)]dx = 2\left(1 + \left|\frac{\sin 2\pi\alpha}{2\pi\alpha}\right|\right)^{-1}. \tag{5.8}$$

Proof. The assumption $\alpha > 1/2$ implies that $B^2_{\gamma(\alpha),r(\alpha)}$ has (at least) one double zero in $[0,\alpha)$. The properties in Lemma 5.1 can be used to show that the assumptions of Theorem 4.14 with

$$E = B^2_{\gamma(\alpha),r(\alpha)}$$

are satisfied. Since $B_{\gamma,r} \in \mathcal{A}(2\pi)$, inequality (4.36) and the Paley-Wiener theorem imply $\mathcal{G}_\alpha^-, \mathcal{G}_\alpha^+ \in \mathcal{A}(2\pi)$. We obtain from $B^2_{\gamma,r}(x) \lesssim |x|^2$ on $\mathbb{R}$ that $\mathcal{G}_\alpha^+$ and $\mathcal{G}_\alpha^-$ are integrable. Inequality (4.35) gives (5.5). Lemma 4.11 and (4.32) imply

$$\begin{aligned}&\mathcal{G}_\alpha^\pm(\xi) = \mathbf{1}_{[-\alpha,\alpha]}(\xi) \text{ for } \xi \in \mathcal{T}_{\gamma(\alpha),r(\alpha)}\backslash\{\pm\alpha\},\\ &\mathcal{G}_\alpha^+(\pm\alpha) = 1, \qquad \mathcal{G}_\alpha^-(\pm\alpha) = 0,\end{aligned} \tag{5.9}$$

and hence equations (5.6) and (5.7) follow from (3.2). The identity

$$1 - \frac{\gamma(\alpha)}{\pi(\alpha^2+\gamma(\alpha)^2)+\gamma(\alpha)} = \left(1 + \left|\frac{\sin 2\pi\alpha}{2\pi\alpha}\right|\right)^{-1}$$

implies (5.8). ∎

Theorem 5.3. *Let $\alpha > (2\delta)^{-1}$ with $2\alpha\delta \notin \mathbb{Z}$. Let $S, T \in \mathcal{A}(2\pi\delta)$ with*

$$S(x) \le \mathbf{1}_{[-\alpha,\alpha]}(x) \le T(x) \tag{5.10}$$

for all real x, then

$$\int_{-\infty}^{\infty} [T(x) - S(x)]dx \ge \frac{2}{\delta}\left(1 + \left|\frac{\sin 2\pi\alpha\delta}{2\pi\alpha\delta}\right|\right)^{-1}, \tag{5.11}$$

with equality if $S(z) = \mathcal{G}^-_{\delta\alpha}(\delta z)$ and $T(z) = \mathcal{G}^+_{\delta\alpha}(\delta z)$ on $\mathbb{C}$.

Proof. We prove the theorem for $\delta = 1$ and note that the general case follows with a straightforward scaling argument. We consider α with $k + \frac{1}{2} < \alpha < k+1$ for some $k \in \mathbb{Z}$. Let $T \in \mathcal{A}(2\pi)$ with $T \ge \mathbf{1}_{[-\alpha,\alpha]}$ on the real line. We may assume that T is integrable. We note that the weights in (3.2) are positive, hence

$$\begin{aligned}\int_{-\infty}^{\infty} T(x)dx &= \sum_{\xi\in\mathcal{T}_{\gamma(\alpha),\frac{1}{2}}} T(\xi)\frac{\xi^2+\gamma^2(\alpha)}{\xi^2+\gamma^2(\alpha)+\pi^{-1}\gamma(\alpha)}\\ &\ge \sum_{\substack{|\xi|\le\alpha\\ \xi\in\mathcal{T}_{\gamma(\alpha),\frac{1}{2}}}} \frac{\xi^2+\gamma^2(\alpha)}{\xi^2+\gamma^2(\alpha)+\pi^{-1}\gamma(\alpha)} = \int_{-\infty}^{\infty}\mathcal{G}_\alpha^+(x)dx\end{aligned} \tag{5.12}$$

by (5.6). An identical argument gives an analogous inequality for S and $\mathcal{G}_\alpha^-$ which implies (5.11). Theorem 5.2 gives equality for $T = \mathcal{G}_\alpha^+$ and $S = \mathcal{G}_\alpha^-$. ∎

6. Determination of $C(\delta,\alpha)$

Recall the definitions $\mathcal{A}_2(\delta) = \mathcal{A}(\delta) \cap L^2(\mathbb{R})$ and

$$C(\delta,\alpha) = \inf_{g\in\mathcal{A}_2(\delta)} \sup_{x\in[-\alpha,\alpha]} \frac{||g||_2^2}{|g(x)|^2}. \tag{6.1}$$

We note that the inequality $C(\alpha,\delta) \le 2\alpha + 2/\delta$ may also be obtained from [3]. We prove

Corollary 6.1. *For $\delta > 0$ and $\alpha > (2\delta)^{-1}$,*

$$C(\pi\delta,\alpha) = \frac{1}{\delta} \sum_{\xi\in\mathcal{T}_{\gamma(\alpha\delta),r(\alpha\delta)}} \left(1 - \frac{\gamma(\alpha\delta)}{\pi(\xi^2+\gamma^2(\alpha\delta))+\gamma(\alpha\delta)}\right), \tag{6.2}$$

and in particular,

$$\lim_{\alpha\to\infty} (C(\delta,\alpha) - 2\alpha) = \frac{2}{\delta}$$

Proof. Denote by $D(\delta,\alpha)$ the expression on the right of (6.2). Let $\delta = 1$. We show first that $C(\pi,\alpha) \le D(1,\alpha)$. Since $\mathcal{G}_\alpha^+ \in \mathcal{A}(2\pi)$ is integrable we obtain that

$$\int_{-\infty}^{\infty} \log^+ |\mathcal{G}_\alpha^+(x)| dx < \infty,$$

and as in the proof of Theorem 3.1 it follows that there exists $S \in \mathcal{A}(\pi)$ with

$$\mathcal{G}_\alpha^+(z) = S(z)S^*(z).$$

We note that $|S(x)| \ge 1$ for $-\alpha \le x \le \alpha$ with equality for $x = \alpha$ (and possibly other values in $[-\alpha,\alpha]$). By construction

$$\int_{-\infty}^{\infty} |S(x)|^2 dx = \int_{-\infty}^{\infty} \mathcal{G}_\alpha^+(x) dx$$

which implies that S satisfies

$$\begin{aligned} ||S||_2^2 &= \sum_{\xi\in\mathcal{T}_{\gamma(\alpha),r(\alpha)}} \left(1 - \frac{\gamma(\alpha)}{\pi(\xi^2+\gamma^2(\alpha))+\gamma(\alpha)}\right) \\ &= \inf_{x\in[-\alpha,\alpha]} |S(x)|^2 D(1,\alpha) \end{aligned} \tag{6.3}$$

and hence $C(\pi,\alpha) \le D(1,\alpha)$. Scaling gives the result for general δ.

For the other direction we note that the argument is essentially reversible. For $\varepsilon > 0$ let $g \in \mathcal{A}_2(\pi\delta)$ so that

$$\sup_{x\in[-\alpha,\alpha]} \frac{||g||_2}{|g(x)|^2} \le C(\pi\delta,\alpha) + \varepsilon$$

holds, and since $C(\pi\delta,\alpha) < \infty$, there exists c such that $|g(x)| \ge c > 0$ for all $x \in [-\alpha,\alpha]$. After multiplication of g by a constant we may take $c = 1$. Define $T = gg^*$. Then $T \in \mathcal{A}(2\pi\delta)$ is integrable and non-negative on $\mathbb{R}$, and we have

$$T(x) \ge 1 \text{ for } -\alpha \le x \le \alpha.$$

It follows from (5.12) and (5.6) that

$$C(\pi\delta,\alpha)+\varepsilon \geq \int_{-\infty}^{\infty}|g(x)|^2dx = \int_{-\infty}^{\infty}T(x)dx \geq D(\delta,\alpha)$$

which was to be shown. For the second statement we note from (5.11)

$$\begin{aligned}|C(\pi\delta,\alpha)-2\alpha| &= \int_{-\infty}^{\infty}\{\mathcal{G}^+_{\delta\alpha}(\delta x)-\chi_{[-\alpha,\alpha]}(x)\}dx\\ &\leq \int_{-\infty}^{\infty}\{\mathcal{G}^+_{\delta\alpha}(\delta x)-\mathcal{G}^-_{\delta\alpha}(\delta x)\}dx\\ &= \frac{2}{\delta}\left(1+\left|\frac{\sin 2\pi\alpha\delta}{2\pi\alpha\delta}\right|\right)^{-1}\end{aligned}$$

which finishes the proof. □

References

[1] E. Carneiro, F. Littmann, J. D. Vaaler. *Gaussian subordination for the Beurling-Selberg extremal problem.* Trans. Amer. Math. Soc., to appear.

[2] L. de Branges. *Hilbert spaces of entire functions.* Prentice Hall, Englewood Cliffs, NJ, 1968.

[3] D. L. Donoho, B. F. Logan. *Signal recovery and the large sieve.* SIAM J. Appl. Math. **52** (1992), no. 2, 577 – 591.

[4] L. Hörmander. *The analysis of linear partial differential operators I. Distribution theory and Fourier analysis.* 2nd ed., Springer, Berlin 1990.

[5] I. I. Hirschman, D. V. Widder. *The convolution transform.* Princeton University Press, Princeton, NJ 1955.

[6] J. J. Holt, J. D. Vaaler, *The Beurling-Selberg extremal functions for a ball in Euclidean space.* Duke Math. J. **83** (1996), no. 1, 202 – 248.

[7] S. Karlin. *Total Positivity I.* Stanford University Press, 1968.

[8] B. Ja. Levin. *Distribution of zeros of entire functions.* Translations of Mathematical Monographs 5. American Mathematical Society, Providence, RI, 1980.

[9] F. Littmann. *Entire approximations to the truncated powers.* Constr. Approx. **22** (2005), 273 – 295.

[10] X.-J. Li and J.D. Vaaler. *Some trigonometric extremal functions and the Erdös-Turán type inequalities.* Indiana Univ. J. Math. **48** no. 1 (1999), 183 – 236.

[11] B. F. Logan. *Bandlimited functions bounded below over an interval.* Notices Amer. Math. Soc. **24** (1977), A – 331.

[12] I. J. Schoenberg. *On Pólya frequency functions. I. The totally positive functions and their Laplace transforms.* J. Analyse Math. **1** (1951), 331 – 374.

[13] J. D. Vaaler, *Some extremal functions in Fourier analysis.* Bull. Amer. Math. Soc. **12** (1985), no. 2, 183 – 216.

Friedrich Littmann, Mathematics Department 2750, North Dakota State University, Fargo ND 58108

E-mail address: Friedrich.Littmann@ndsu.edu