1 Analytic continuation

Let \( f : G \to \mathbb{C} \) be analytic. Assume that \( G \) is contained in an open connected set \( G_1 \). When is it possible to extend \( f \) to an analytic function on \( G_1 \)?

We recall that a function \( f \) is analytic on \( G_1 \) if and only if every point in \( G_1 \) is a center of a disk on which \( f \) can be expanded into a convergent power series. So we try to cover \( G_1 \) by overlapping disks. In every disk whose center is in \( G \) we define \( f \) by its power series. If we are lucky, the series will converge on the larger disk. If so, we now have “defined” \( f \) on a set that properly contains \( G \), and we continue this process.

This process will fail if the boundary of \( G \) consists of singularities of \( f \), and in this case there is nothing further that can be done. However, even if this is not the case, this process is actually not well defined as the example of the logarithm shows. Analytic continuation deals with the latter situation.

We define

\[
G^* = \{ z \in \mathbb{C} : z \in G \}.
\]

Note that \( G \cap G^* \neq \emptyset \) and \( G \cap G^* = \emptyset \) may both occur. We set \( G_+ = \{ z \in G : \Re z > 0 \} \), \( G_- = \{ z \in G : \Re z < 0 \} \) and \( G_0 = \{ z \in G : \Re G = 0 \} \). The following theorem is called the Schwarz reflection principle.

**Theorem 1.** Let \( G \) be an open connected set with \( G = G^* \). If \( f : G_+ \cup G_0 \to \mathbb{C} \) is a continuous function that is analytic on \( G_+ \), then there exists an analytic function \( g : G \to \mathbb{C} \) with \( g = f \) on \( G_+ \cup G_0 \).

**Proof.** We set \( f^*(z) = \overline{f(z)} \). We define

\[
g(z) = \begin{cases} 
  f(z) & \text{if } z \in G_+ \cup G_0 \\
  f^*(z) & \text{if } z \in G_-.
\end{cases}
\]

The function \( g \) is analytic on \( G_+ \) and \( G_- \), and it is continuous on \( G \), hence analytic on \( G \). \[\square\]
Note that reflection principles can be defined for other curves as well. For example, \( f^\#(z) = f(1/z) \) can be used to define a reflection principle across the unit circle. (Final exam question.)

In order to deal with situations like the logarithm, we have to keep track not just of the domain of \( f \), but of the path that we took to get the analytic continuation.

**Definition 1.** A function element is a pair \((f,G)\) where \(G\) is an open connected set, and \(f : G \to \mathbb{C}\) is analytic. For given \((f,G)\) we denote by \([f]_a\) the collection of all function elements \((g,D)\) such that

1. \( g : D \to \mathbb{C} \) is analytic
2. \( a \in D \)
3. \( f = g \) in some neighborhood of \( a \).

Example: \( f(z) = \log(z + z) \) in \( G = \{ \Re z > -1 \} \) and \( g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^n / n \) in \( D(0,1) \) are two different function elements. \((g,D(0,1)) \in [f]_0\), and \((f,G) \in [g]_0\).

It is worth emphasizing that \([f]_a\) is not a function element! Note also that

\[(g,D) \in [f]_a \iff (f,G) \in [g]_a.\]

Note also that this definition does not yet distinguish between different analytic continuations, e.g., principal branch and branch with negative imaginary axis cut are in each others function germs.

Note: If \( a \neq b \), then \([f]_a \neq [f]_b\), since \([f]_a\) will contain function elements \((g,D)\) where \( D \) is a disk not containing \( b \), so that the collections can never be equal.

On the other hand, \([f]_a = [g]_a\) is possible even if the functions \( f \) and \( g \) are distinct: if \( f(z) = \log(1 + z) \) and \( g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} z^n / n \), then \([f]_0 = [g]_0\). (Compare with the function element statement above!) These distinctions appear to be trivial, but they are needed to be able to define analytic continuations later on.

**Lemma 1.** If \( D_1, D_2 \subseteq \mathbb{C} \) with \( \in D_1 \cap D_2 \neq \emptyset \), and \( f : D_1 \to \mathbb{C}, g : D_2 \to \mathbb{C} \) with \( f = g \) on \( D_1 \cap D_2 \), then \([f]_a = [g]_a\) for all \( a \in D_1 \cap D_2 \).

**Proof.** Let \((h,D) \in [f]_a\). This means that \( h : D \to \mathbb{C} \) is analytic, \( a \in D \) and \( f = h \) in some neighborhood \( N \) of \( a \). We may assume that \( N \subseteq D_1 \cap D_2 \), since this intersection is open. It follows that \( g = h \) on \( N \), and hence \((h,D) \in [g]_a\). \(\square\)
Intuitively speaking, the collection \([f]_a\) is built to keep track of the situation in this lemma without having to refer to the assumptions of this lemma explicitly. A useful triviality in this sense:

**Lemma 2.** If \(f : B \to \mathbb{C}, g : C \to \mathbb{C}\) analytic and \([f]_a = [g]_a\) for \(a \in B \cap C\), then \(f = g\) on \(B \cap C\).

**Definition 2.** Let \(\gamma : [0, 1] \to \mathbb{C}\) be a path, and suppose that for each \(0 \leq t \leq 1\) there is a function element \((f_t, D_t)\) so that

1. \(\gamma(t) \in D_t\),

2. there is a \(\delta = \delta(t) > 0\) so that \(|s - t| < \delta\) implies: \(\gamma(s) \in D_t\) and \([f_s]_{\gamma(s)} = [f_t]_{\gamma(t)}\).

Then \((f_1, D_1)\) is called the analytic continuation of \((f_0, D_0)\) along the path \(\gamma\).

Consider the sets \(D_t\) to be disks with center at \(\gamma(t)\). For the second condition note the following: \(\gamma(s) \in D_t\) means that \(D_t \cap D_s \neq \emptyset\). The germs being equal means in particular that \(f_s = f_t\) on this intersection!

**Proposition 1.** Let \(\gamma\) be a path from \(a\) to \(b\) and \((f_t, D_t)\), \((g_t, B_t)\) be two analytic continuations along \(\gamma\). If \([f_0]_a = [g_0]_a\) for two analytic continuations along a path, then \([f_1]_b = [g_1]_b\).

**Proof.** Let \(T = \{t \in [0, 1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}. One needs to show that \(T\) is open and closed in \([0, 1]\), which then implies that \(T = [0, 1]\).

To see that \(T\) is open, fix \(t \in T\) and show that \((t - \delta, t + \delta) \subseteq T\) for some \(\delta > 0\). (Omitted; left as an uncollected exercise. Use the second condition in the definition.)

To see that \(T\) is closed, let \(t \in [0, 1]\) be a limit point of \(T\) (\(t\) is not necessarily in \(T\)). Choose \(\delta > 0\) so that \(|s - t| < \delta\) implies that \(\gamma(s) \in D_t \cap B_t\) and \([f_s]_{\gamma(s)} = [f_t]_{\gamma(t)}\) and \([g_s]_{\gamma(s)} = [g_t]_{\gamma(t)}\).

Note that there exists such \(s\) with \(s \in T\), since \(t\) is a limit point of \(T\)! Let \(G\) be an open connected set in \(D_t \cap B_t\) that contains \(\gamma((t - \delta, t + \delta))\), and note that \(s \in G\). Then \(f_t = f_s\) on \(G\) and \(g_t = g_s\) on \(G\). But \(s \in T \cap G\), so \(f_s = g_s\)! It follows that \(f_t = f_s = g_s = g_t\) on \(G\).

This identity holds now on \(B_t \cap D_t\), since \(G\) has a limit point in this intersection, and hence \([f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\) by the lemma.

**Definition 3.** If \(\gamma\) is a path from \(a\) to \(b\) and \(\{(f_t, D_t)\}\) is an analytic continuation along \(\gamma\), then we call the germ \([f_1]_b\) the analytic continuation of \([f_0]_a\) along \(\gamma\).
If \((f, G)\) is a function element, then the complete analytic function obtained from \((f, G)\) is the collection \(F\) of all germs \([g]_b\) for which there is \(a \in G\) and a path \(\gamma\) from \(a\) to \(b\) such that \([g]_b\) is the analytic continuation of \([f]_a\) along \(\gamma\).

Why is this a function? Let
\[
R = \{(z, [f]_z); [f]_z \in F\}.
\]

Define \(G : R \to \mathbb{C}\) by
\[
G(z, [f]_z) = f(z).
\]

We identify \(G\) and \(F\).

2 Monodromy theorem

Let \(a\) and \(b\) be two points in \(\mathbb{C}\) and let \(\gamma\) and \(\sigma\) be two paths from \(a\) to \(b\). If \((f_t, D_t)\) are analytic continuations along \(\gamma\), and \((g_t, B_t)\) analytic continuations along \(\sigma\) with \([f_0]_a = [g_0]_a\), under which conditions does it follow that \([f_1]_b = [g_1]_b\)? (We will see that \(\gamma \sim \sigma\) in some suitable domain is a sufficient criterion.)

How does the radius of convergence behave along an analytic continuation? For a given analytic continuation \((f_t, D_t)\) along \(\gamma\), define

\[ R(t) = \text{Convergence radius of the power series of } f_t \text{ about } \gamma(t) \]

**Lemma 3.** \(R(t)\) is either identically \(\infty\), or it defines a continuous function on \([0,1]\).

**Proof.** If \(R(s) = \infty\) for some \(s \in [0,1]\), then \(f_s\) is entire. But then all \(f_t\) are entire as well (since they are equal to \(f_s\) in this case).

Assume \(R(t) < \infty\), all \(t\). Set \(\sigma = \gamma(s)\) and \(\tau = \gamma(t)\) where \(s\) is chosen as follows. The point \(s\) is so close to \(t\) (within distance \(\delta\) so that \(\sigma \in D(\tau, R(t))\)). Since \(D(\sigma, R(t) - |\tau - \sigma|) \subseteq D(\tau, R(t))\) (make a sketch!), it follows that

\[ R(s) \geq R(t) - |\tau - \sigma|. \]

The inequality with \(R(s)\) and \(R(t)\) follows by symmetry, and hence

\[ |R(s) - R(t)| \leq |\gamma(t) - \gamma(s)| \]

for sufficiently close \(s\) and \(t\). Hence, \(R\) is continuous since \(\gamma\) is continuous. \(\square\)
For a fixed path \( \gamma \) it follows that if an analytic continuation \((f_t, D_t)\) along \( \gamma \) exists, then there exists \( \varepsilon > 0 \) such that \( R(t) \geq \varepsilon > 0 \) for all \( t \in [0, 1] \). From the previous, it follows that \( R \) is uniformly continuous, hence assumes its minimum on \([0, 1]\). Note that by assumption the radius of convergence of \( f_t \) at \( \gamma(t) \) is positive for all \( t \), hence the minimum of \( R(t) \) must be positive as well.

Lemma 4. Let \( \gamma \) be a path from \( a \) to \( b \). Let \((f_t, D_t)\) be an analytic continuation along \( \gamma \). Then there exists \( \varepsilon > 0 \) with the following property:

If \( \sigma \) is any path from \( a \) to \( b \) with \(|\gamma(t) - \sigma(t)| < \varepsilon \) for all \( t \in [0, 1] \) and \((g_t, B_t)\) is an analytic continuation along \( \sigma \) with \([g_0]_a = [f_0]_a\), then \([g_1]_b = [f_1]_b\).

Proof. Let \( R = R_\gamma \) be the convergence radius from above for \( \gamma \). If \( R \equiv \infty \) then all functions involved are entire, and the claim becomes trivial. So assume that \( R \) is finite on \([0, 1]\).

Choose \( \varepsilon \) to be half of the minimum of \( R(t) \) on \([0, 1]\). We assume in addition that \( D_t \) is a disk about \( \gamma(t) \) of radius \( R(t) \) (we may always change to this function element; note that it is an element of \([f_t]_{\gamma(t)}\), and that \( B_t \) is a disk about \( \sigma(t) \).

We note that \( \sigma(t) \in D_t \) by choice of \( \varepsilon \) (and \( \sigma(t) \) is the center of \( B_t \)). Thus, \( f_t \) and \( g_t \) are both analytic on \( B_t \cap D_t \), and this intersection is not empty. We need to show that \( f_1 = g_1 \) on \( D_1 \cap B_1 \).

As in the previous proof, we define \( T = \{ t : f_t = g_t \text{ on } B_t \cap D_t \} \). We show that \( T \) is open and closed in \([0, 1]\). We prove that \( T \) is open. \( T \) is not empty since it contains 0. Fix \( t \in T \).

Choose \( \delta > 0 \) so that for \( s \in [0, 1] \) with \(|s-t| < \delta \) the following properties hold.

1. \(|\gamma(s) - \gamma(t)| < \varepsilon \) (continuity of \( \gamma \)) and \([f_s]_{\gamma(s)} = [f_t]_{\gamma(t)}\) (second property of a.c.),
2. \(|\sigma(s) - \sigma(t)| < \varepsilon \) and \([g_s]_{\sigma(s)} = [g_t]_{\sigma(s)}\)
3. \( \sigma(s) \in B_t \).

Claim: \( \sigma(s) \in B_s \cap B_t \cap D_s \cap D_t =: G \).

\( B_s \) has center \( \sigma(s) \), \( B_t \) by construction. By definition of \( \varepsilon \) we have \(|\sigma(s) - \gamma(s)| < \varepsilon \), and \( \gamma(s) \) is the center of \( D_s \) (radius \( \geq 2\varepsilon \)). Hence \( \sigma(s) \in D_s \). Use triangle inequality to show that \(|\sigma(s) - \gamma(t)| < 2\varepsilon \), hence \( \sigma(s) \in D_t \).

Since \( f_t = g_t \) we have that \( f_t = g_t \) on \( G \); but these agree with \( f_s \) and \( g_s \), respectively, on \( G \). We get \( f_s = g_s \), i.e., \( s \in T \) for all \( s \in (t-\delta, t+\delta) \).
Use a similar argumentation to show that $f_{t_0} = g_{t_0}$ on some neighborhood of a limit point $t_0$ of $T$. \hfill \square

Let $(f, D)$ be a function element and $G$ a region with $D \subseteq G$. Then $(f, D)$ admits unrestricted analytic continuation along $G$, if for any path $\gamma$ in $G$ with starting point in $D$ there is some analytic continuation of $(f, D)$ along $\gamma$.

**Theorem 2.** Let $\sigma$ and $\gamma$ be two paths in $G$ with common starting point in $D$ and common endpoint. Assume that $(f_t, D_t)$ and $(g_t, B_t)$ are analytic continuations of $(f, D)$ along $\sigma$ and $\gamma$, respectively. If $\gamma$ and $\sigma$ are FEP homotopic in $G$, then $[f_1]_{\gamma(1)} = [g_1]_{\sigma(1)}$.

**Proof.** Let $\Gamma : [0,1]^2 \to G$ such that $\Gamma(t,0) = \gamma(t)$, $\Gamma(t,1) = \sigma(t)$, and $\Gamma(0,u) = a$, $\Gamma(1,u) = b$.

Define $\gamma_u(t) = \Gamma(t,u)$. By assumption some analytic continuation $(h_{t,u}, D_{t,u})$ along $\gamma_u$ exists. Need to show: $[h_{1,0}]_b = [h_{1,1}]_b$. Define

$$U = \{u \in [0,1] : [h_{1,u}]_b = [h_{1,0}]_b\}$$

and show that $U$ is non-empty, and open and closed in $[0,1]$.

We have that $\Gamma$ is uniformly continuous. Hence for every $\varepsilon > 0$ there exists $\delta > 0$ so that $|u - v| < \delta$ implies $|\gamma_u(t) - \gamma_v(t)| < \varepsilon$. With the result from last time it follows that for $u \in U$ there exists $\delta > 0$ so that $(u - \delta, u + \delta) \subseteq U$. It follows that $U$ is open.

If $u$ is a limit point of $U$, then for every $\delta > 0$ can find $v \in U$ with $|u - v| < \delta$. Apply uniform continuity and the result from last time again to conclude that $[h_{1,v}]_b = [h_{1,u}]_b$, hence $u \in U$. It follows that $U$ is closed. \hfill \square