LIE ALGEBRAS IN CLASSICAL AND QUANTUM MECHANICS

by

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ABSTRACT

A nonrelativistic quadratic Hamiltonian constitutes the simplest class of completely solvable problems in classical and quantum mechanics. Such a Hamiltonian is a sum of terms, each of which is a quadratic combination of positions and momenta. Physical systems governed by quadratic Hamiltonians include the n-dimensional harmonic oscillator and a particle in a constant magnetic field. These simple systems can be used effectively to model more complicated physical systems. They also form a starting point for approximate treatments of many phenomena in condensed matter and statistical physics. Thus, it is important to develop as complete an understanding as possible of simple quadratic models. The purpose of this work is to present two examples of the use of Lie algebra to analyze physical phenomena completely.

Two systems are considered. One is a charged harmonic oscillator in a constant magnetic field, treated quantum mechanically, and the other is a solvable three-body problem, treated in Hamiltonian dynamics. In each case, the procedure is to address the problem as one of constructing representations of a Lie algebra. A maximal set of mutually commuting quantities (or quantities that have zero Lie bracket with one another) is determined. Then using undetermined coefficients, the remaining variables are combined into ladder operators, in the quantum mechanical case, or fundamental solutions in the classical case. The result is a pair of detailed examples of Lie algebra techniques applied to physical systems.
CHAPTER 1

INTRODUCTION

Symmetry plays a central role in the analysis of physical systems and the foundation of physical laws. Lie groups and their corresponding Lie algebras constitute the underlying mathematical framework. Lie algebraic techniques have contributed greatly to a wide range of problems in high energy and condensed matter physics. It is useful to have a clear understanding of Lie algebras and Lie algebraic techniques in order to apply them to solve physical systems effectively.

Much progress has been made in the development of Lie algebraic techniques applied to physics. For example, Lie transformation groups such as SU(2) and SO(3) are well known and used extensively. Consequently, a large amount of literature devoted to the subject is aimed particularly at physicists [1, 2, 3, 4, 5]. However, the connections between Lie algebra and physics are not accessible to typical physics graduate students without a thorough mathematical background. On the other hand, when the mathematics is clearly explained for the average student of physics, worked examples may be in short supply. This paper attempts to mitigate this situation by providing two explicitly-worked examples of simple physical systems treated by Lie algebraic techniques. These simple examples are solved exactly and can be extended to model more complicated systems.

Two systems are considered. One is a charged anisotropic, three-dimensional harmonic oscillator in a constant magnetic field, treated quantum mechanically, and the other is a
solvable three-body problem, treated in classical Hamiltonian dynamics. The problem of non-interacting, non-relativistic charged particles in an external magnetic field has been treated for example by Kennard in 1927 [6] and Johnson and Lippmann in 1949 [7]. These results have been extended more recently within a group theoretical framework by Beckers and Hussin [8]. Reference [8] is the primary motivation for the first part of the current work. The second portion of the present paper was intended to extend these results. After devoting nearly six months to the work reported here, I discovered many (not all) of the results had been published previously by N.F. Johnson and Payne [9] along with the 1993 follow-up paper by Hasse and N.F. Johnson [10]. The latter paper presents a solution for N harmonically interacting particles in a quantum dot, which is an interesting problem particularly to condensed matter physicists. The results in the third chapter of the present paper can be compared to the model in Ref.[10] for the case of three interacting particles.

This thesis is organized as follows. In chapter I, the basic concepts of vector spaces and a Lie algebra are reviewed. This chapter serves as a framework underlying the entire paper, introducing explicitly the mathematical concepts employed throughout this work. Chapter II addresses the well-known problem of calculating the energy eigenvalues of a particle under the influence of a constant magnetic field subject to a quantum dot-like confinement potential. A generating function for all of the possible excited states is obtained, which may not have been published previously. Also, selection rules for allowed transitions are computed. The problem is treated quantum-mechanically using Lie algebraic techniques, with a first-order correction for spin-orbit coupling. The application of Lie algebra is continued in chapter III, this time to classical Hamiltonian dynamics. Here, a completely solvable three-body problem is addressed, using the techniques developed in the first chapter. This problem is
solvable due to the nature of the Hamiltonian, which is a quadratic form. Finally, a set of elementary solutions corresponding to quantum mechanical raising and lowering operators is obtained. It is shown that the results correlate directly to familiar results in appropriate limiting cases.
CHAPTER 2

LIE ALGEBRA

In general, an algebra is a vector space with a multiplication defined between vectors. The multiplication takes two vectors into another vector. The reader is assumed to be familiar with vector spaces, however for completeness I include some basic concepts.

2.1 Basic Concepts

A vector space over the complex numbers \( \mathbb{C} \) is a set \( V \) of vectors and two operations, vector addition (+), under which \( V \) forms an abelian group, and multiplication of a vector \( v \) by a scalar \( f \). If \( v_1 \) and \( v_2 \) are in \( V \), then so is \( v_1 + v_2 \), also the following properties hold:

1. \( v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3 \) and \( v_1 + v_2 = v_2 + v_1 \)
2. There exists a zero vector 0, such that \( v + 0 = v \) for any \( v \in V \).
3. Each \( v \in V \) has an additive inverse, \( -v \) such that \( v + (-v) = 0 \) which defines subtraction \( v_1 - v_2 = v_1 + (-v_2) \).
4. For \( f_1, f_2 \in \mathbb{C} \) and \( v_1, v_2 \in V \), \( f(v_1 + v_2) = fv_1 + fv_2 \)
5. \( (f_1 + f_2)v_1 = f_1v_1 + f_2v_1 \)
6. \( (f_1f_2)v_1 = f_1(f_2v_1) \)
7. It is also necessary to state that \( 1v = v \).

In this section, \( V \) is used to represent the vector space itself, as well as the set of all vectors. A subspace \( S \) of \( V \) is a subset \( S \subseteq V \), such that \( S \) itself is a complex vector space with respect to the operations of \( V \). A set \( B = \{b_1, b_2, ..., b_n\} \) is linearly independent if whenever
\[ c_1 b_1 + c_2 b_2 + \ldots + c_n b_n = 0 \] for \( c_i \in \mathbb{C} \) every coefficient \( c_i \) is zero. \( \mathbf{B} \) is a basis in \( V \) if there is no set of \( n + 1 \) linearly independent vectors in \( V \). Then \( n \) is the dimension of \( V \). Thus any \( \mathbf{v} \in V \) has a unique representation \( \mathbf{v} = v_1 \mathbf{b}_1 + v_2 \mathbf{b}_2 + \ldots + v_n \mathbf{b}_n \).

An elementary example is the space of ordinary 3D Cartesian vectors, with basis \( \{ \hat{x}, \hat{y}, \hat{z} \} \). In this case, the vectors are written in familiar form \( \mathbf{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z} \), where the scalars \( v_x, v_y, v_z \) are components of \( \mathbf{v} \). Alternatively, one can write \( \mathbf{v} \) as a row of components, or as a column matrix

\[
\mathbf{v} = (v_1, v_2, v_3) \quad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}
\]

A Lie algebra is an algebra of vectors \( V \) over \( \mathbb{C} \) with a multiplication called the Lie bracket \( [x, y] \) which satisfies the following axioms:

1. \( [x, y] = -[y, x] \) \hspace{1cm} -antisymmetry

2. \( [x + \alpha y, z] = [x, z] + \alpha[y, z] \) \hspace{1cm} -linearity

3. \( [[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \) \hspace{1cm} - Jacobi identity

One sees immediately that the algebra generated by \( \{ \hat{x}, \hat{y}, \hat{z} \} \) with the usual cross product is a Lie algebra. Note that in the examples below, the Lie bracket will often be the commutator as in quantum mechanics \( [A, B] = AB - BA \). The algebra is abelian if \( [x, y] = 0 \) for all \( x \) and \( y \).

Let \( \mathbf{G} \) denote a Lie algebra. A subalgebra of \( \mathbf{G} \) is a subspace \( \mathbf{H} \) of \( \mathbf{G} \), closed under bracket multiplication, such that if \( h_1 \) and \( h_2 \) are in \( \mathbf{H} \), then \( [h_1, h_2] \in \mathbf{H} \). An ideal of \( \mathbf{G} \) is a subalgebra \( \mathbf{I} \) of \( \mathbf{G} \) such that if \( h \in \mathbf{I}, g \in \mathbf{G} \), then \( [h, g] \in \mathbf{I} \). Thus, bracketing with any element in the ideal always gives another element in the ideal. A trivial ideal is an ideal
consisting of the entire algebra $\mathbf{G}$ or the ideal containing only the zero element. A *simple* Lie algebra is one with no non-trivial ideals, and a *semi-simple* Lie algebra has no abelian ideals.

As an example of an algebra that contains an abelian ideal, consider $\mathcal{A}$ generated by translations and rotations in the $xy$ plane. Thus, $\mathcal{A} = \{A, B, C\}$ where $A = \frac{\partial}{\partial x}$, $B = \frac{\partial}{\partial y}$, and $C = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$. One finds that


Thus, $A$ and $B$ form an abelian ideal in the algebra $\mathcal{A}$.

A subalgebra $\mathcal{K}$ of $\mathbf{G}$ is called a *Cartan subalgebra* if $\mathcal{K}$ is nilpotent and of maximal dimension\[1\], meaning that there is no other nilpotent subalgebra with greater dimension. Thus for elements $g_1, g_2$ in a Cartan subalgebra $\mathcal{K}$, $[g_1, g_2] = 0$, or in this subalgebra all vectors commute. In physics, the dimension of the Cartan subalgebra corresponds to the maximal number of quantum numbers of the system. One recalls that when a set of Hermitian matrices commute, they can share a basis of mutual eigenvectors.

### 2.2 Structure of Lie Algebras

In order to understand further the structure of Lie algebras, it is useful to introduce the *adjoint* representation of $\mathbf{G}$. Each element $x \in \mathbf{G}$ can be thought of as a linear operator $\text{ad}_x$ acting on $\mathbf{G}$, defined as

$$[x, y] = \text{ad}_x(y)$$

for all $y \in \mathbf{G}$. Given a particular basis, the operators $\text{ad}_x$ can be represented as matrices of the *General Linear group*, $\text{GL}(n, \mathbb{C})$, consisting of all complex non-singular $n \times n$ matrices.
The eigenspace formed by the adjoint operator $\text{ad} x$ with eigenvalue $\lambda \in \mathbb{C}$ is

$$\{ y \in G | (\text{ad} x - \lambda 1) y = 0 \}.$$  

Consequently, the generalized eigenspace of $\text{ad} x$ with eigenvalue $\lambda$ is defined as

$$\{ y \in G | (\text{ad} x - \lambda 1)^n y = 0 \text{ for some } n > 1 \}.$$  

Since the generators $\{K^i\}$ of the Cartan subalgebra $K$ have zero brackets amongst themselves, the matrices $\text{ad} k$ for all elements $k \in K$ commute and hence are simultaneously diagonalizable. Thus $G$ is spanned by such elements $y$ that are simultaneous eigenvectors of all the maps $\text{ad} k$, which satisfy

$$[k, y] = \text{ad} k(y) = \alpha_y(k)y .$$

The purpose of the latter equation is to define $\alpha_y(k)$. For any fixed element $y \in G$ of this type, the eigenvalue $\alpha_y(k)$ of $y$ is some complex number which depends linearly on $k \in K$. In quantum theory, $\alpha_y(k)$ is the quantum of excitation in an observable quantity $k$ generated by ladder operator $y$. Thus $\alpha_y$ is a linear functional of $k$, because it maps $K \to \mathbb{C}$. (This is a linear functional since it is linear and takes an element of the vector space into a complex number.) The set of all eigenvalues (linear functionals) $\{\alpha_y\}$ forms a vector space $K^*$ dual to $K$. The eigenvalues, $\alpha_y(k)$, are the roots of the characteristic equation for $k$, thus $\alpha_y$ is a root of the algebra $G$.

$G$ is spanned by elements satisfying (for $y \in G$)

$$[k, y] = \alpha_y(k)y .$$

Thus, $G$ can be written as a direct sum of the Cartan subalgebra and the remaining 1-dimensional vector spaces

$$G = K \oplus \bigoplus_{\alpha} \mathbb{C} u_\alpha ,$$

(2.1)
which is called the Cartan decomposition of \( G \) relative to \( K \) [12]. Therefore, a semi-simple Lie algebra is the direct sum of simple Lie algebras including the cartan subalgebra [2]

\[
G = K \oplus G_{\alpha_1} \oplus G_{\alpha_2} \cdots \oplus G_{\alpha_n}.
\]

This decomposition means that there exists a basis \( \mathcal{B} \) of \( G \) which, apart from a basis of the Cartan subalgebra, consists entirely of elements \( u_{\alpha_i} \in K_{\alpha}^\perp \) which satisfy

\[
[k_i, u_{\alpha_i}] = \alpha_y u_{\alpha_i}.
\]

The elements \( u_{\alpha_i} \) that generate \( K_{\alpha}^\perp \) are the ladder operators associated to the roots \( \alpha \). Thus, these eigenvectors are called root vectors.

The set of all roots of the algebra, which spans the dual space \( K^* \), is denoted as \( \Phi(G) \) and called the root system of \( G \). For semisimple Lie algebras, the root system is non-degenerate. Thus, the eigenvectors are unique up to an overall constant. Therefore, the dimension of \( G \) is equal to the dimension of \( K \) plus the number of roots. Also if \( \alpha \in \Phi(G) \), then \( -\alpha \in \Phi(G) \) which I prove in the next section.

The basis of \( G \) is of the form

\[
\mathcal{B} = \{k_i | i = 1, ..., r\} \cup \{u_{\alpha_i} | \alpha \in \Phi\}.
\]

A basis of this form is called the Cartan-Weyl (canonical) basis of \( G \) [13]. From a physical point of view, the Cartan subalgebra \( \{k_i | i = 1, ..., r\} \) is a complete set of commuting observables, and the root vectors \( \{u_{\alpha_i} | \alpha \in \Phi\} \) form a complete set of ladder operators.

Finite-dimensional Lie algebras have only finitely many roots, thus it is possible to divide the roots into positive and negative roots, which correspond to, in quantum mechanics, raising and lowering operators. Thus, \( \Phi \) can be divided as follows

\[
\Phi_+ = \{\alpha \in \Phi | \alpha > 0\} \quad \Phi_- = \{\alpha \in \Phi | \alpha < 0\}.
\]
This allows one to decompose further the algebra $G$

$$G = K_+ \oplus K \oplus K_-$$

This is called the Gauss decomposition of $G$ [13].

### 2.2.1 The Killing Form

In order to understand better the relation between the roots, a geometrical picture of the algebra is established through the Killing form. The Killing form is a pairing between elements in the finite-dimensional Lie algebra (not a scalar product) $G$ defined as

$$\langle x, y \rangle = \text{trace}(adx \ ady).$$

Thus, the Killing form is the trace of any matrix representing ad$x$ ad$y$, independent of basis (since the trace is independent of the choice of basis). Also, trace$(AB) = \text{trace}(BA)$ for any two square matrices $A$ and $B$, so the Killing form is a symmetric bilinear form

$$\langle x, y \rangle = \langle y, x \rangle.$$

The Killing form is also an invariant form

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle$$

According to Cartan’s Criterion for semi-simplicity [14], if and only if $G$ is semi-simple, the Killing form is non-degenerate. Thus,

$$\langle x, y \rangle = 0 \ \forall y \in G \text{ implies } x = 0.$$

From non-degeneracy of the Killing form, it can be shown that if $\alpha$ and $\beta$ are any two roots and $\beta \neq -\alpha$, then from Eq. (2.1) the subalgebras $G_\alpha$ and $G_\beta$ are orthogonal relative to the Killing form [14]. This fact allows one to prove that in any semi-simple Lie algebra, if $\alpha$ is a
root, then so is \(-\alpha\). In other words, for every raising operator, there must be a corresponding lowering operator.

**Proof.** If \(z \in K\) and \(z \perp K\) (\(\langle\langle z, k \rangle\rangle = 0\)), then \(z \perp G\) since by the above assertion that \(G_{\alpha} \perp G_{\beta}\) for any two roots \(\alpha\) and \(\beta\), \(z \perp G_{\alpha}\ \forall \alpha \neq 0\). Then, \(z = 0\) by the non-degeneracy of \(\langle\langle x, y \rangle\rangle\). If \(\alpha\) is a root and \(-\alpha\) is not a root, then \(G_{\alpha} \perp G_{\beta}\) for all roots \(\beta\), so \(G_{\alpha} \perp G\). This is a contradiction to the non-degeneracy of the Killing form, since \(\langle\langle u_{\alpha}, \ell \rangle\rangle = 0 \ \forall \ell \in G\). \(\square\)

Also, one can show that the bracket product between a raising and lowering operator is proportional to an element of the Cartan subalgebra with respect to this operation \(\langle\langle [u_{-\alpha}, u_{\alpha}], k \rangle\rangle = \langle\langle u_{-\alpha}, u_{\alpha} \rangle\rangle k_{\alpha}\).

**Proof.** \(\langle\langle [u_{-\alpha}, u_{\alpha}], k \rangle\rangle = \langle\langle u_{-\alpha}, [u_{\alpha}, k] \rangle\rangle = \langle\langle u_{-\alpha}, \alpha(k) u_{\alpha} \rangle\rangle = \alpha(k) \langle\langle u_{-\alpha}, u_{\alpha} \rangle\rangle\) Now, take the quantity \([u_{-\alpha}, u_{\alpha}] = \langle\langle u_{-\alpha}, u_{\alpha} \rangle\rangle k_{\alpha}\) and show it gives the same result. \(\langle\langle [u_{-\alpha}, u_{\alpha}], k, h \rangle\rangle = \langle\langle u_{-\alpha}, u_{\alpha} \rangle\rangle \langle\langle k_{\alpha}, h \rangle\rangle = \langle\langle u_{-\alpha}, u_{\alpha} \rangle\rangle \alpha(k)\) thus, \([u_{-\alpha}, u_{\alpha}] = \langle\langle u_{-\alpha}, u_{\alpha} \rangle\rangle k_{\alpha}\). \(\square\)

This leads to a relation between different basis elements \(u_{\alpha}\) and \(u_{\beta}\). Let \(k \in K\), then using Jacobi non-associativity,

\[
[u_{\alpha}, u_{\beta}] = ([u_{\beta}, k], u_{\alpha}) + ([k, u_{\alpha}], u_{\beta})
\]

\[
= [u_{\alpha}, [k, u_{\beta}]] + [k, [u_{\alpha}, u_{\beta}]]
\]

\[
= \beta(k)[u_{\alpha}, u_{\beta}] + \alpha(k)[u_{\alpha}, u_{\beta}]
\]

\[
= (\alpha + \beta)(k)[u_{\alpha}, u_{\beta}].
\]

Thus, if \(\alpha + \beta\) is not in \(\Phi\), then \([u_{\alpha}, u_{\beta}] = 0\) and if \(\alpha + \beta\) is in \(\Phi\), then \([u_{\alpha}, u_{\beta}, u_{\alpha}] = n u_{\alpha + \beta}\) for some integer \(n\). A root \(\alpha\) is called *simple* if \(\alpha > 0\) and it cannot be written as a sum of two other positive roots. In general, the root system \(\Phi\) is not linearly independent since the
Cartan subalgebra contains all linear combinations of its elements \([\alpha k_1 + \beta k_2] = (\alpha + \beta)u_\alpha\).

Thus, it is natural to seek a subset of \(\Phi\) composed entirely of simple roots \(\alpha\), so that any root \(\beta\) can be expressed as a linear combination of the simple roots.

To illustrate the connection between Lie algebras and physics, two simple examples are presented in the next section.

### 2.3 Familiar Physical Examples

A clear example of applying Lie algebra to physics is the one-dimensional quantum Harmonic oscillator problem. The Hamiltonian for the problem is

\[
H = \frac{1}{2} m \omega^2 x^2 + \frac{1}{2m} p^2 .
\]  

(2.2)

The objective is to solve the eigenvalue equation

\[
H |\Psi\rangle = E |\Psi\rangle
\]

for the energy eigenvalue \(E\), and particular eigenfunction \(|\Psi\rangle\) given the potential \(V(x) = \frac{1}{2} m \omega^2 x^2\). The algebra is generated by the operators that correspond to physical quantities in the problem, in this case the set \(\{H, p, x, 1\}\), forms a basis. The constant 1 is included because \([x, p]\) is a constant. The Cartan subalgebra is composed of all operator combinations that leave the number of energy quanta constant, hence do not change the overall energy of a given state. It is clear that in this example, \(H\) and 1 form a basis for the Cartan subalgebra \([H, H] = 0\) and \([1, anything] = 0\), while \([H, x], [H, p] \neq 0\). To focus attention on the essence of using Lie algebra, scale to dimensionless variables, such that \(m = k = \hbar = 1\),

\[
H = \frac{1}{2} p^2 + \frac{1}{2} x^2 .
\]

Now I review the idea of constructing ladder operators [15], but in the context of a Lie
algebra. In particular, how does one find raising and lowering operators? Thus consider the following.

The subspace $K^\perp$ complimentary to $K$ is treated in the following way. First, form linear combinations of the basis vectors spanning $K^\perp$, so in the current example let $B = \alpha x + \beta p$, where $\alpha$ and $\beta$ are numbers to be determined. The task is to calculate the coefficients $\alpha$ and $\beta$ such that $B$ satisfies an operator eigenvalue equation of the form

$$[H, B] = \lambda B,$$

where operator $B$ is a sort of eigenvector, and $\lambda$ is its eigenvalue relative to $H$. After calculating the left side, one gets

$$i\beta x - i\alpha p = \lambda (\alpha x + \beta p).$$

Now recall that $x$ and $p$ are independent basis vectors in the Lie algebra. Thus, comparing the coefficients of $x$ and $p$,

$$i\beta = \lambda \alpha \quad \text{and} \quad -i\alpha = \lambda \beta.$$

Eliminating $\beta$, I reduce the two equations to the single equation

$$(1 - \lambda^2)\alpha = 0,$$

so, either $\alpha = 0$ or $\lambda = \pm 1$. The coefficient $\alpha$ cannot be zero, otherwise both $\alpha = 0$ and $\beta = 0$, so one gets the following possible solutions:

either $\lambda = +1$, $\beta = -i\alpha$, $B = \alpha(x - ip)$,

or $\lambda = -1$, $\beta = i\alpha$, $B = \alpha(x + ip)$. 

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Therefore, we can rename the two possible solutions of $B$ so as to correspond to the more familiar $a$ and $a^\dagger$ notation. In order to normalize $a$ and $a^\dagger$ so that $[a, a^\dagger] = 1$, let $\alpha = \frac{1}{\sqrt{2}}$.

One can easily see that $B = \frac{1}{\sqrt{2}}(x + ip)$, which prompts us to define $a = B$. Since both $x$ and $p$ are Hermitian, $a^\dagger = \frac{1}{\sqrt{2}}(x - ip)$, which is the Hermitian conjugate of $a$. Continuing, as usual [15]

$$ aa^\dagger = \left( \frac{1}{\sqrt{2}} \right)^2 (x - ip)(x + ip) $$

$$ = \frac{1}{2} (x^2 + p^2 + i[x, p]) . $$

Thus,

$$ a^\dagger a = H - \frac{1}{2} \quad \text{and} \quad aa^\dagger = H + \frac{1}{2} , $$

which allows two equivalent expressions for the Hamiltonian $H = a^\dagger a + \frac{1}{2}$ and $H = aa^\dagger - \frac{1}{2}$.

Now, one can solve the time-independent Schrödinger equation $H|\Psi\rangle = E|\Psi\rangle$ easily (as Schrödinger did [16]) by using the operator $a$ in the eigenvalue equation

$$ [H, a] = -a . $$

So, operating on an arbitrary eigenfunction $|\Psi\rangle$, with eigenvalue $E$, one gets

$$ [H, a]|\Psi\rangle = -a|\Psi\rangle , $$

and then, because in quantum mechanics the Lie product is the commutator,

$$ H(a|\Psi\rangle) - a(H|\Psi\rangle) = -(a|\Psi\rangle) $$

$$ H(a|\Psi\rangle) - E(a|\Psi\rangle) = -(a|\Psi\rangle) , $$

therefore

$$ H(a|\Psi\rangle) = (E - 1)(a|\Psi\rangle) . $$

There are two possibilities:
1. $a|\Psi\rangle$ is a new eigenfunction with energy $E - 1$, or

2. $a|\Psi\rangle = 0$, corresponding to a ground state.

One can continue the harmonic oscillator example to determine all of its physical properties. It is presented here rather as a simple template for the general idea of constructing ladder operators in the language of Lie algebras as described generally below.

Another familiar example is that of angular momentum. Consider the generalized angular momentum operators $J_x$, $J_y$, and $J_z$, which satisfy the commutation rules

$$[J_x, J_y] = iJ_z$$
$$[J_y, J_z] = iJ_x$$
$$[J_z, J_x] = iJ_y .$$

From these operators, one can construct the *Casimir operator* $J^2 = \vec{J} \cdot \vec{J}$, which commutes with all of the angular momentum operators

$$[J^2, J_x] = [J^2, J_y] = [J^2, J_z] = 0 .$$

Thus, the angular momentum algebra $\mathbf{J}$ is generated by $\{ J_x, J_y, J_z, J^2 \}$. $\mathbf{J}$ is an extension in $\text{SU}(2)$ of the algebra generated by $\{ J_x, J_y, J_z \}$. It is important to note that the casimir operator $J^2$ only has meaning for representations, and not as an element of the algebra since products like $J^2_x$ do not exist in $\mathbf{J}$.

Next, choose a basis for the Cartan subalgebra $\mathbf{K}$ of elements that mutually commute. By convention we usually choose $\{ J^2, J_z \}$. Again, form a vector $A = \alpha J_x + \beta J_y$, which is a linear combination of elements outside the Cartan sububalgebra. The rest of the algebra $\mathbf{K}^\perp$ should be spanned by vectors that are *mutual eigenvectors* of the Cartan subalgebra so,

$$[J^2, A] = 0 ,$$

$$[J_z, A] = \lambda A .$$
Notice that the fact that the operators in $\mathbf{K}$ commute guarantees they have mutual eigenvectors. We continue by using the linearity property of the bracket

$$[J_z, A] = [J_z, (\alpha J_x + \beta J_y)] = \alpha [J_z, J_x] + \beta [J_z, J_y].$$

Thus, from this and the commutation rules of the angular momentum operators, one gets the following relation:

$$\alpha i J_y - \beta i J_x = \lambda (\alpha J_x + \beta J_y).$$

Comparing the coefficients, gives

$$\alpha = -i\lambda \beta, \quad \beta = i\lambda \alpha.$$

These equations are homogeneous, so I am free to choose $\alpha = 1$. Thus,

- either $\lambda = +1, \quad \beta = +i$,
- or $\lambda = -1, \quad \beta = -i$.

Substituting the $\alpha$ and $\beta$ solutions into the expression for the vector $A$, I get

$$A = J_1 \pm iJ_2$$

As in the harmonic oscillator example, this is how we arrive at the familiar ladder operators used in elementary quantum mechanics

$$J_\pm = J_x \pm iJ_y. \quad \text{(2.3)}$$

If one chooses as a basis of $\mathbf{J}$ the set $\{J_+, J_-, J_z\}$, the elements can be represented in adjoint form:

$$adJ_z = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad adJ_+ = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad adJ_- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 0 \end{pmatrix}. \quad \text{(2.4)}$$
Notice that with this choice of basis, the matrix corresponding to the element in the Cartan subalgebra $\mathbf{K}$ is diagonal. In general, this is part of the reason why ladder operators are constructed. The raising and lowering operators along with the operators within the Cartan subalgebra provide a convenient basis for the entire algebra $\mathbf{G}$. Therefore, the basis that spans $\mathbf{K}^\perp$ is always arranged so that $\text{ad} k$ for all $k \in \mathbf{K}$ is diagonal, this is the Cartan-Weyl basis $\mathfrak{B}$.

And so, in this section two examples from elementary quantum mechanics have been presented to illustrate the process of finding appropriate ladder operators. In the following, this method of constructing root vectors will be applied to the Lie algebras of an anisotropic oscillator in a magnetic field (including electron spin via perturbation theory) and of a solvable 3-body problem.
CHAPTER 3
CONFINED PARTICLE IN A MAGNETIC FIELD

Begin by setting up the Hamiltonian for a charged particle in a magnetic field. For simplicity, consider a constant magnetic field in the positive $z$ direction. Thus, the magnetic field is $\vec{B} = B_o \hat{z}$, which suggests a possible gauge potential

$$A(\vec{r}) = \frac{1}{2} B_o (x \hat{y} - y \hat{x}) .$$

(3.1)

Also, ignoring spin, the Hamiltonian for a charged particle in a magnetic field is, in general

$$H = \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right) \cdot \left( \vec{p} - \frac{q}{c} \vec{A} \right) + V(\vec{r}) .$$

In this case, we add a containment force, an anisotropic harmonic restoring force, so that calculation may apply in some abstract way to a quantum dot. This force suggests the following choice of potential:

$$V(\vec{r}) = \frac{1}{2} m \alpha^2 (x^2 + y^2) + \frac{1}{2} m \beta^2 z^2 .$$

(3.2)

Therefore, the Hamiltonian for a charged particle in a constant magnetic field, under the influence of the containment force is

$$H = \frac{1}{2m} \left| \vec{p} - \frac{q}{c} \vec{A} \right|^2 + \frac{1}{2} m \alpha^2 (x^2 + y^2) + \frac{1}{2} m \beta^2 z^2 .$$

(3.3)

Expanding the terms of the Hamiltonian one gets

$$\vec{A} \cdot \vec{p} = \frac{1}{2} B_o (x p_y - y p_x) = \frac{1}{2} B_o L_z , \quad A^2 = \frac{1}{4} B_o^2 (x^2 + y^2) .$$
After simplifying, it is clear that the Hamiltonian decouples into separate terms, one for the $xy$ plane and another for the $z$ direction

$$H = H_{xy} + H_z.$$ 

The $z$ terms in the Hamiltonian make up a one-dimensional oscillator

$$H_z = \frac{1}{2m} p_z^2 + \frac{1}{2} m \beta z^2,$$

and the other term constitutes a Hamiltonian for the $xy$ motion,

$$H_{xy} = \frac{1}{2m} (p_x^2 + p_y^2) - \frac{1}{2m} \frac{q^2}{c^2} B_o L_z + \frac{1}{2} m \alpha^2 (x^2 + y^2) + \frac{1}{2} \frac{m q^2 B_o^2}{4 m^2 c^2} (x^2 + y^2).$$

This motivates the definitions

$$\omega_b = \frac{q B_o}{2 m c}, \quad \Omega = \sqrt{\alpha^2 + \omega_b^2},$$

where $\omega_b$ is the Larmor frequency. Thus, the $xy$ part of the Hamilton simplifies to

$$H_{xy} = \frac{1}{2m} (p_x^2 + p_y^2) - \omega_b L_z + \frac{1}{2} m \Omega^2 (x^2 + y^2). \quad (3.4)$$

We now focus specifically on $H_{xy}$, since the solution to the $z$-degree of freedom corresponding to a one-dimensional harmonic oscillator is straight-forward. Again, the objective is to solve the eigenvalue equation $H|\Psi\rangle = E|\Psi\rangle$ for all allowed energies. In this problem, \{ $H_{xy}, L_z, p_x, p_y, x, y, 1$ \} forms a basis of the algebra $\mathcal{A}$. As usual, one can construct orbital angular momentum out of these basis elements. In this problem, we only consider the $z$ component of the total angular momentum $L_z = x p_y - y p_x$, since it will be conserved due to the oblate spheroidal symmetry of the potential. It is clear that the Cartan subalgebra $\mathbf{K}$ for this problem is spanned by \{ $H_{xy}, L_z, 1$ \}, since $H_{xy}$ and $L_z$ commute. Before proceeding
further, we compute the Lie bracket product for elements of $\mathbf{K}$ with the basis elements.

Starting with $L_z$, I have

$$[L_z, x] = [-y p_x, x] = i \hbar y, \quad [L_z, y] = -i \hbar x,$$

$$[L_z, p_x] = [x p_y, p_x] = i \hbar p_y, \quad [L_z, p_y] = -i \hbar p_x.$$

One can see that commutators of the form

$$[L_z, f(x^2 + y^2)], \quad \text{and} \quad [L_z, g(p_x^2 + p_y^2)]$$

vanish, for any functions $f$ and $g$, and hence $[L_z, H_{x,y}] = 0$, so $L_z$ is indeed conserved (corresponding to a good quantum number). For completeness, I continue computing bracket products of $H$ with each of the other basis vectors of the algebra:

$$[H_{xy}, x] = -\frac{i \hbar}{m} p_x - i \hbar \omega_b y$$

$$[H_{xy}, y] = -\frac{i \hbar}{m} p_y + i \hbar \omega_b x$$

$$[H_{xy}, p_x] = -i \hbar \omega_b p_y + i \hbar m \Omega^2 x$$

$$[H_{xy}, p_y] = +i \hbar \omega_b p_x + i \hbar m \Omega^2 y$$

(3.5)

Once the general form of the bracket product $[k, y]$ ($k \in \mathbf{K}, y \in \mathbf{A}$) is determined, one can construct an orthonormal basis for the algebra more easily. In order to form the eigenvector, or root vector $u$ (a quantum mechanical operator), which is a linear combination of all the basis elements not in $\mathbf{K}$, one introduces undetermined coefficients

$$u = a_1 x + a_2 y + b_1 p_x + b_2 p_y.$$

We wish to choose these coefficients ($a_1, a_2, b_1, b_2$) such that $u$ satisfies simultaneously the eigenvalue equations

$$[L_z, u] = \lambda u.$$
\[ [H_{xy}, u] = \epsilon u. \]

These are in fact matrix eigenvalue equations for \( \text{ad}L_z \) and \( \text{ad}H_{xy} \) which operate on the algebra. It is customary to describe them simply as eigenvalue equations where operator \( u \) (ladder operator) is the eigenvector and \( \lambda \) and \( \epsilon \) are the respective eigenvalues. Starting with the \( L_z \) eigenvalue equation, one finds

\[
[L_z, u] = a_1 [L_z, x] + a_2 [L_z, y] + b_1 [L_z, p_x] + b_2 [L_z, p_y]
= i \hbar a_1 y - i \hbar a_2 x + i \hbar b_1 p_y - i \hbar b_2 p_x.
= \lambda u \quad (3.6)
\]

After comparing the coefficients from each side of Eq. (3.6) we have

\[
\lambda a_1 = -i \hbar a_2, \quad \lambda a_2 = i \hbar a_1, \quad \lambda b_1 = -i \hbar b_2, \quad \lambda b_2 = i \hbar b_1.
\]

There are two possibilities, either \( \lambda = +\hbar \) or \( \lambda = -\hbar \).

\[
\lambda = +\hbar \quad \Rightarrow \quad a_2 = i a_1, \quad b_2 = i b_1,
\]
\[
\lambda = -\hbar \quad \Rightarrow \quad a_2 = -i a_1, \quad b_2 = -i b_1.
\]

For the second eigenvalue equation involving \( H_{xy} \), we get

\[
[H_{xy}, u] = a_1 [H_{xy}, x] + a_2 [H_{xy}, y] + b_1 [H_{xy}, p_x] + b_2 [H_{xy}, p_y]
= a_1 \left( -\frac{i \hbar}{m} p_x - i \hbar \omega_b y \right) + a_2 \left( -\frac{i \hbar}{m} p_y + i \hbar \omega_b x \right)
+ b_1 \left( -i \hbar \omega_b p_y + i \hbar m \Omega^2 x \right) + b_2 \left( +i \hbar \omega_b p_x + i \hbar m \Omega^2 y \right).
\]

Comparing the coefficients as in the case for \( L_z \),

\[
\epsilon a_1 = +i \hbar \omega_b a_2 + i \hbar m \Omega^2 b_1,
\]
\[\epsilon a_2 = -i\hbar \omega_b a_1 + i\hbar m\Omega^2 b_2,\]
\[\epsilon b_1 = -i\frac{\hbar}{m} a_1 + i\hbar \omega_b b_2,\]
\[\epsilon b_2 = -i\frac{\hbar}{m} a_2 - i\hbar \omega_b b_1 .\]

Now, of course there are two choices for the relation between \(a_1\) and \(a_2\) also between \(b_1\) and \(b_2\), which correspond to the choice of \(\lambda = \pm\hbar\). When \(\lambda = +\hbar\), \(a_2 = i a_1\) and \(b_2 = i b_1\) so

\[\epsilon a_1 = -\hbar \omega_b a_1 + i\hbar m\Omega^2 b_1,\]
\[i \epsilon a_1 = -i\hbar \omega_b a_1 - \hbar m\Omega^2 b_1,\]
\[\epsilon b_1 = -i\frac{\hbar}{m} a_1 - \hbar \omega_b b_1,\]
\[i \epsilon b_1 = +\frac{\hbar}{m} a_1 - i\hbar \omega_b b_1,\]

So

\[(\epsilon + \hbar \omega_b) a_1 = i\hbar m\Omega^2 b_1,\]
\[(\epsilon + \hbar \omega_b) b_1, = -i\frac{\hbar}{m} a_1 .\]

Eliminating \(b_1\), I reduce two equations to one

\[(\epsilon + \hbar \omega_b)^2 a_1 = \hbar^2 \Omega^2 a_1, \quad \text{or} \quad \epsilon = \hbar (\pm \Omega - \omega_b).\]

Similarly, when \(\lambda = -\hbar\), one has \(a_2 = -i a_1\) and \(b_2 = -i b_1\), so

\[\epsilon a_1 = +\hbar \omega_b a_1 + i\hbar m\Omega^2 b_1,\]
\[-i \epsilon a_1 = -i\hbar \omega_b a_1 + \hbar m\Omega^2 b_1,\]
\[\epsilon b_1 = -i\frac{\hbar}{m} a_1 + \hbar \omega_b b_1,\]
\[-i \epsilon b_1 = -\frac{\hbar}{m} a_1 - i\hbar \omega_b b_1 .\]
Again, the four equations reduce to the following two:

\[
(\epsilon - \hbar \omega_b) a_1 = i \hbar m \Omega^2 b_1,
\]

\[
(\epsilon - \hbar \omega_b) b_1, = - \frac{i \hbar}{m} a_1.
\]

Finally, for the choice of \( \lambda = -\hbar \), one has

\[
(\epsilon - \hbar \omega_b)^2 a_1 = \hbar^2 \Omega^2 a_1, \quad \text{or} \quad \epsilon = \hbar (\pm \Omega + \omega_b).
\]

Thus we have four operators in the following table.

**Table 1: Complete Set of Ladder Operators**

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>( \epsilon )</th>
<th>operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>(+ \hbar) ( h(-\Omega - \omega_b))</td>
<td>( u ) = ( \frac{1}{2} \sqrt{\frac{m \Omega}{\hbar}} \left( x + i y + \frac{i}{m \Omega} P_x - \frac{1}{m \Omega} P_y \right) )</td>
<td></td>
</tr>
<tr>
<td>(- \hbar) ( +\Omega + \omega_b)</td>
<td>( u^\dagger ) = ( \frac{1}{2} \sqrt{\frac{m \Omega}{\hbar}} \left( x - i y - \frac{i}{m \Omega} P_x - \frac{1}{m \Omega} P_y \right) )</td>
<td></td>
</tr>
<tr>
<td>(- \hbar) ( -\Omega + \omega_b)</td>
<td>( v ) = ( \frac{1}{2} \sqrt{\frac{m \Omega}{\hbar}} \left( x - i y + \frac{i}{m \Omega} P_x + \frac{1}{m \Omega} P_y \right) )</td>
<td></td>
</tr>
<tr>
<td>(+ \hbar) ( +\Omega - \omega_b)</td>
<td>( v^\dagger ) = ( \frac{1}{2} \sqrt{\frac{m \Omega}{\hbar}} \left( x + i y - \frac{i}{m \Omega} P_x + \frac{1}{m \Omega} P_y \right) )</td>
<td></td>
</tr>
</tbody>
</table>

To make the operators dimensionless, notice that the quantity

\[
b = \sqrt{\frac{\hbar}{m \Omega}}
\]

has units of length. The operators in the table have been chosen with the following normalization:

\[
[u, u^\dagger] = 1, \quad [v, v^\dagger] = 1,
\]

\[
[u, v] = 0, \quad [u, v^\dagger] = 0. \quad (3.7)
\]

Since the eigenvalues \( \epsilon \) and \( \lambda \) are computed using the results of \( \text{ad}L_z(y) \) and \( \text{ad}H_{xy}(y) \) for all \( y \) in \( \mathcal{A} \), they constitute the roots of the algebra. Notice that the roots are non-degenerate and appear in \( \pm \) pairs, which is a consequence of semi-simplicity (section 2.2.1).
Once the operators are constructed, one considers the eigenvalue equations corresponding to the members of $K$, the Cartan subalgebra. These eigenvalue equations permit explicit calculation of the ground state, since successive application of the the lowering operators must eventually annihilate the state function $|\psi_{\ell,k}\rangle$. I denote the energy and angular momentum quantum numbers as $k$ and $\ell$ respectively. Starting with the angular momentum operator,

$$[L_z, u]|\psi_{\ell,k}\rangle = \hbar u|\psi_{\ell,k}\rangle$$

$$L_z u|\psi_{\ell,k}\rangle - u L_z|\psi_{\ell,k}\rangle = \hbar u|\psi_{\ell,k}\rangle$$

$$L_z(u|\psi_{\ell,k}\rangle) = (\ell + 1) \hbar |\psi'_{\ell,k}\rangle.$$ 

Thus, either $u|\psi_{\ell,k}\rangle = 0$ or $\psi'_{\ell,k}$ is a new eigenvector with eigenvalue (not normalized) $(\ell + 1)\hbar$ for $L_z$. Similarly,

$$[H_{xy}, u]|\psi_{\ell,k}\rangle = \hbar (-\omega_b \pm \Omega) u|\psi_{\ell,k}\rangle$$

$$H_{xy} u|\psi_{\ell,k}\rangle - u H_{xy}|\psi_{\ell,k}\rangle = \hbar (-\omega_b \pm \Omega) u|\psi_{\ell,k}\rangle$$

$$H_{xy}(u|\psi_{\ell,k}\rangle) = \left( k - \hbar(\Omega + \omega_b) \right) |\psi'_{\ell,k}\rangle$$

Again, either $|\psi'_{\ell,k}\rangle = 0$ or else $k - \hbar(\Omega + \omega_b)$ is an eigenvalue for $H_{xy}$. Therefore, either $|\psi'_{\ell,k}\rangle = 0$ or $|\psi'_{\ell,k}\rangle = (\text{const.})|k - \hbar(\Omega + \omega_b), \ell + 1\rangle$. Since the squared norm of a vector cannot be negative, it must be that

$$\langle \psi_{\ell,k} | u^\dagger u | \psi_{\ell,k} \rangle = \| u | \psi_{\ell,k} \|^2 \geq 0.$$ 

The next step is to solve for what $\ell$ and $k$ must be. The product of $u^\dagger$ and $u$ is

$$u^\dagger u = \frac{1}{2\hbar \Omega} (H_{xy} + \omega_b L_z) - \frac{L_z}{2\hbar} - \frac{1}{2}$$

so

$$(C_1)^2 = \langle \psi_{\ell,k} | u^\dagger u | \psi_{\ell,k} \rangle = \frac{k}{2\Omega\hbar} + \frac{\ell}{2} \left( \frac{\omega_b}{\Omega} - 1 \right) - \frac{1}{2}$$ \hspace{1cm} (3.8)
In a similar manner, either \( v|\psi_{\ell,k}\rangle = 0 \) or \( v|\psi_{\ell,k}\rangle = C_2|\psi'_{\ell,k}\rangle \), where the product of \( v^\dagger \) and \( v \) is

\[
v^\dagger v = \frac{1}{2\hbar\Omega}(H_{xy} + \omega_bL_z) + \frac{L_z}{2\hbar} - \frac{1}{2}
\]

Thus, the inner product is

\[
(C_2)^2 = \langle \psi_{\ell,k}|v^\dagger v|\psi_{\ell,k}\rangle = \frac{k}{2\Omega\hbar} + \frac{\ell}{2}(\omega_b + 1) - \frac{1}{2} \quad (3.9)
\]

Starting from any mutual eigenstate of \( H_{xy} \) and \( L_z \) and operating successively with \( u \), each time lowering the squared norm, one must eventually get to a state \( |\psi\rangle \) such that \( u|\psi\rangle = 0 \). However, since \([ u, v ] = 0 \), and \([ u, v^\dagger ] = 0 \), it follows that \( u \) also annihilates \( v|\psi\rangle \) and \( v^\dagger|\psi\rangle \) or any power \((v^\dagger)^p|\psi\rangle\). Thus, \( u \) annihilates an infinite set of \( |\psi\rangle \). Starting with \( |\psi\rangle \) annihilated by \( u \), and lowering successively with \( v \) one must arrive at a unique state \( |\psi_o\rangle \) such that \( u|\psi_o\rangle = 0 \) and \( v|\psi_o\rangle = 0 \). Setting the right hand sides of Eqs. (3.8) and (3.9) equal to zero, I get for the ground state

\[
\ell = 0 \quad \text{and} \quad k = \hbar\Omega
\]

So one is looking for the ground state \( |\psi_{\ell,k}\rangle \) which is the unique, normalized solution of \( u|\psi_o\rangle = 0 \) and \( v|\psi_o\rangle = 0 \).

Transferring the \( u \) operator in particular from Dirac to coordinate representation, one has the following differential equation

\[
u\psi_o(x, y) = \frac{1}{2}\sqrt{\frac{m\Omega}{\hbar}} \left( x + iy + \frac{\hbar}{m\Omega} \frac{\partial}{\partial x} + \frac{i\hbar}{m\Omega} \frac{\partial}{\partial y} \right) \psi_o(x, y) = 0. \quad (3.10)
\]

A linear partial differential equation of this form can be solved by standard means. The solution is of exponential form \( \psi_o(x, y) = e^\phi \). Substitution of this trial solution into the
differential equation Eq.(3.10) and invoking the definition of \( b \), yields the following result

\[
\frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} = \frac{-1}{b^2} (x + iy).
\] (3.11)

In order to solve this partial differential equation, invoke the method of characteristics [17]. The basic strategy is to convert the coordinates \((x, y)\) to a new system of coordinates in which the partial differential equation becomes a pair of ordinary differential equations. These new coordinates are the characteristic variables or canonical coordinates of the differential equation. Thus, Eq.(3.11) is converted into two ordinary differential equations

\[
\frac{dx}{1} = \frac{dy}{i} = \frac{-b^2}{(x + iy)} \, d\phi.
\] (3.12)

Solving the first ODE, one obtains the characteristic \( \xi = x + iy \). Using this coordinate (considered constant) in the second ODE from Eq.(3.12) and solving for \( \phi \), one finds that \( \phi = -\frac{\xi}{b^2} x + f(\xi) \). So, one has determined \( \phi \) up to a constant function \( f(\xi) \). The partial differential equation \( v\psi_o = 0 \) must also be satisfied, which by the same method outlined for \( u\psi_o = 0 \) leads to the second canonical coordinate for this system \( \eta = x - iy \).

Therefore, by the first PDE \((u\psi_o = 0)\) the ground state is of the form

\[
\psi_o = f(x + iy) e^{-\frac{\eta(x+iy)}{b^2}}.
\]

In order to know what the function \( f \) actually is, recall that \( \psi_o \) must be annihilated by \( v \). Thus, applying \( v \) to the current result gives the first-order ordinary differential equation

\[
bf'(x + iy) - (x + iy)f(x + iy) = 0
\] (3.13)

solving this, and normalizing the result gives the true ground state

\[
\psi_o(x, y) = \frac{1}{b\sqrt{\pi}} e^{\frac{-m\Omega^2}{2\hbar}(x^2 + y^2)}.
\] (3.14)
We can see from their commutation with the Cartan subalgebra, Table 1, that the raising operators $u^\dagger$ and $v^\dagger$ have the following effect on the energy and angular momentum quantum numbers of the wave function:

\begin{align*}
    u^\dagger : \ell &\to \ell - 1, \ k \to k + \hbar(\Omega + \omega_b) \\
    v^\dagger : \ell &\to \ell + 1, \ k \to k + \hbar(\Omega - \omega_b).
\end{align*}

Every state can be constructed from the ground state by raising $n_1$ times with $u^\dagger$, and raising $n_2$ times with $v^\dagger$. We can reorganize the quantum numbers according to the number of times $u^\dagger$ and $v^\dagger$ would have to operate. Thus the angular momentum quantum number $\ell$ is

$$\ell = n_2 - n_1$$

and the energy eigenvalue is

$$E = (n_1 - n_2)\hbar\omega_b + (n_1 + n_2 + 1)\hbar\Omega = k_o\hbar\omega_b + k\Omega\hbar\Omega.$$ 

Using these rules, I reorganize the quantum numbers and define a new basis $\{|n_1, n_2\rangle\}$ for $H_{xy}$. The next task is to calculate the normalization constant $C_1$ such that $u^\dagger |n_1, n_2\rangle = C_1 |n_1 + 1, n_2\rangle$. This facilitates calculating all subsequent states.

$$u^\dagger |n_1, n_2\rangle = C_1 |n_1 + 1, n_2\rangle$$

$$\langle n_1, n_2 | uu^\dagger | n_1, n_2 \rangle$$

$$C_1^2 = \frac{1}{2}(1 + n_1 + n_2) + \frac{\omega_b}{2\Omega}(n_1 - n_2) + \frac{\omega_b}{2\Omega}(n_2 - n_1) - \frac{1}{2}(n_2 - n_1) + \frac{1}{2}$$

Simplification yields

$$C_1^2 = n_1 + 1.$$
hence
\[ C_1 = \sqrt{n_1 + 1}. \]

Also
\[ v^\dagger |n_1, n_2\rangle = C_2 |n_1, n_2 + 1\rangle, \]
\[ C_2^2 = \frac{1}{2} (1 + n_1 + n_2) + \frac{\omega_b}{2\Omega} (n_1 - n_2) + \frac{\omega_b}{2\Omega} (n_2 - n_1) + \frac{1}{2} (n_2 - n_1) + \frac{1}{2}, \]
\[ C_2^2 = n_2 + 1. \]

Therefore, as expected
\[ C_2 = \sqrt{n_2 + 1}. \]

Now one gets all the allowed states by application of the operators \( u^\dagger \) and \( v^\dagger \), which raise the quantum numbers \( n_1 \) and \( n_2 \) associated with the \( xy \)-degrees of freedom of the Hamiltonian, and of \( a_z^\dagger \), which raises the \( z \) quantum number \( n_3 \). These operators can be applied \( n_1, n_2, \)
and \( n_3 \) times respectively. The general formula for the normalized state \( |n_1, n_2, n_3\rangle \) is
\[ (a_z^\dagger)^{n_3} (v^\dagger)^{n_2} (u^\dagger)^{n_1} |0, 0, 0\rangle = \sqrt{n_1!} \sqrt{n_2!} \sqrt{n_3!} |n_1, n_2, n_3\rangle. \] (3.15)

3.1 A Generating Function

Let us restrict attention to \( H_{xy} \). A more convenient method to manipulate all excited states at once is to use a generating function. To obtain the excited state \( \psi_{n_1,n_2}(x, y) \) recall first that
\[ u^\dagger v^\dagger \psi_{n_1,n_2}(x, y) = \sqrt{n_1 + 1} \sqrt{n_2 + 1} \psi_{n_1+1,n_2+1}(x, y), \]
and thus the \( n_1, n_2 \) excited state is
\[ \psi_{n_1,n_2}(x, y) = \frac{1}{\sqrt{n_1!} \sqrt{n_2!}} (u^\dagger)^{n_1}(v^\dagger)^{n_2} \psi_0(x, y). \] (3.16)
To illustrate the general method, consider first the form of a generating function for a set of single variable functions of $x$. The familiar definition [18] of a generating function for an infinite set of functions $\{f_n(x)\}$ is

$$G(x, t) = \sum_{n=0}^{\infty} C_n f_n(x) t^n \quad (3.17)$$

where the constant $C_n$ depends in some preassigned way on $n$, but is independent of $x$ and $t$. For the current problem, one is looking to obtain a generating function for the set of eigenfunctions $\{\psi_{n_1,n_2}(x, y)\}$, which depends on both $x$ and $y$. Therefore, in view of the way $v^\dagger$ and $u^\dagger$ act, a convenient generating function would appear to be of the following form:

$$F(x, y, s, t) = \sum_{n_1,n_2} C_{n_1} C_{n_2} s^{n_1} t^{n_2} \psi_{n_1,n_2}(x, y)$$

$$= \sum_{n_1,n_2} \frac{1}{\sqrt{n_1!} \sqrt{n_2!}} s^{n_1} t^{n_2} \psi_{n_1,n_2}(x, y).$$

Using Eq.(3.16), one can recover an exponential expansion,

$$F(x, y, s, t) = \sum_{n_1,n_2} \frac{1}{n_1! n_2!} s^{n_1} t^{n_2} (u^\dagger)^{n_1} (v^\dagger)^{n_2} \psi_o(x, y)$$

$$= \exp(su^\dagger + tv^\dagger) \psi_o(x, y). \quad (3.18)$$

In the latter equation, I have used the fact that $[u^\dagger, v^\dagger] = 0$. The reader may recognize this as a coherent state, an eigenstate of the lowering operator of the Hamiltonian, (the coherent state of the 1-D harmonic oscillator can be written as $|\alpha\rangle = e^{-\alpha^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$) [19]. The group-theoretic method employed in the following to obtain a generating function is the Weisner method [20]. This method relies on the fact that certain partial differential equations are invariant with respect to a nontrivial continuous group of transformations (a Lie group). The strategy is to find canonical variables such that $u^\dagger$ and $v^\dagger$ are essentially derivatives and then to evaluate Eq.(3.18) via the Taylor theorem. The canonical coordinates for the two
PDE’s \( u\psi_0 = 0 \) and \( v\psi_0 = 0 \), are \( \xi = x + iy \) and \( \eta = x - iy \), so to make these coordinates dimensionless we introduce the new coordinates \( \zeta \) and \( \zeta^* \).

\[
\zeta = \frac{x + iy}{b\sqrt{2}}
\]

\[
\zeta^* = \frac{x - iy}{b\sqrt{2}}
\]

Solving for \( x \) and \( y \)

\[
x = \frac{b}{\sqrt{2}} (\zeta + \zeta^*)
\]

\[
y = -\frac{ib}{\sqrt{2}} (\zeta - \zeta^*)
\]

Hence, from the chain rule

\[
\frac{\partial}{\partial \zeta} = \frac{\partial x}{\partial \zeta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \zeta} \frac{\partial}{\partial y}
\]

\[
\frac{\partial}{\partial \zeta^*} = \frac{\partial x}{\partial \zeta^*} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \zeta^*} \frac{\partial}{\partial y}.
\]

After taking the appropriate derivatives, one obtains the following expressions:

\[
\frac{\partial}{\partial \zeta} = \frac{b}{\sqrt{2}} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)
\]

\[
\frac{\partial}{\partial \zeta^*} = \frac{b}{\sqrt{2}} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right)
\]

Inserting the definitions of \( p_x \) and \( p_y \), I convert the two operators \( u^\dagger \) and \( v^\dagger \) into

\[
u^\dagger = \frac{1}{\sqrt{2}} \left( \frac{x + iy}{b\sqrt{2}} - \frac{b}{\sqrt{2}} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \right),
\]

thus using \( \zeta \) and \( \zeta^* \)

\[
u^\dagger = \frac{1}{\sqrt{2}} \left( \zeta^* - \frac{\partial}{\partial \zeta} \right),
\]

\[
u^\dagger = \frac{1}{\sqrt{2}} \left( \zeta - \frac{\partial}{\partial \zeta^*} \right).
\]
The next goal is to relate $u^\dagger$ and $v^\dagger$ each by a gauge transformation to a derivative operator, and then to evaluate the generating function by summing a Taylor series, thus treating $\exp(\alpha \frac{d}{dx})$ as a shifting operator. The Taylor series expansion of $F(x + \alpha)$ can be written in terms of a derivative operator on $F(x)$, an example of a 1-parameter Lie group [21].

$$F(x + \alpha) = F(x) + \alpha F'(x) + \frac{\alpha^2}{2!} F''(x) + \cdots$$

$$= \left(1 + \alpha \frac{d}{dx} + \frac{\alpha^2}{2!} \frac{d^2}{dx^2} + \cdots\right) F(x) = e^{(\alpha \frac{d}{dx})} F(x).$$

(3.19)

Therefore, the operator $\exp(\alpha \frac{d}{dx})$ translates the function $F(x)$ by an amount $\alpha$. Thus, starting with $u^\dagger$, suppose one has a function $\mu = \mu(\zeta, \zeta^*)$

$$u^\dagger = -\frac{1}{\sqrt{2}} \frac{1}{\mu} \frac{\partial}{\partial \zeta} \mu,$$

then by the Leibniz rule for differentiating a product

$$u^\dagger = -\frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \zeta} + \frac{1}{\mu} \frac{\partial \mu}{\partial \zeta} \right)$$

so, comparing to the expression above for $u^\dagger$

$$\frac{1}{\mu} \frac{\partial \mu}{\partial \zeta} = -\zeta^*$$

this is a separable differential equation, so holding $\zeta^*$ constant

$$\ln \mu = f(\zeta^*) - \zeta \zeta^*.$$

Solving for the integrating factor $\mu$, it is not necessary to retain the arbitrary constant of integration $f(\zeta^*)$ since only a particular solution is required, so

$$\mu = e^{-\zeta \zeta^*}.$$
hence the raising operator becomes

\[ u^\dagger = e^{\zeta \zeta^*} \left( -\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta} \right) e^{-\zeta \zeta^*}. \]  

(3.20)

Similarly, for \( v^\dagger \)

\[ v^\dagger = -\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta} \mu + \frac{1}{\mu} \frac{\partial \mu}{\partial \zeta^*}. \]

\[ = -\frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial \zeta^*} + \frac{1}{\mu} \frac{\partial \mu}{\partial \zeta^*} \right). \]

Again, comparing one gets the separable differential equation

\[ \frac{1}{\mu} \frac{\partial \mu}{\partial \zeta^*} = -\zeta. \]

After solving, one obtains the same integrating factor

\[ \mu = e^{-\zeta \zeta^*}. \]

Thus,

\[ v^\dagger = e^{\zeta \zeta^*} \left( -\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta^*} \right) e^{-\zeta \zeta^*}. \]  

(3.21)

Now it is easy to evaluate the exponential series that defines the generating function of Eq.(3.18). Consider first the \( t \) dependence. This leads one to consider the action of \((v^\dagger)^n\).

Operating twice with \( v^\dagger \), one obtains the following expression

\[ (v^\dagger)^2 = e^{\zeta \zeta^*} \left( -\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta^*} \right) e^{-\zeta \zeta^*} e^{\zeta \zeta^*} \left( -\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta^*} \right) e^{-\zeta \zeta^*} \]

\[ = e^{\zeta \zeta^*} \left( -\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta^*} \right)^2 e^{-\zeta \zeta^*}. \]

Hence, equations 3.20 and 3.21 can be generalized to

\[ (u^\dagger)^n = e^{\zeta \zeta^*} \left( -\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta} \right)^n e^{-\zeta \zeta^*} \]
\[(v^\dagger)^{n_2} = e^{\zeta\zeta^*} \left(-\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta^*}\right)^{n_2} e^{-\zeta\zeta^*}\]

(3.22)

This allows one to express \(F(x,y,s,t)\) as a function of \(\zeta\) and \(\zeta^*\). Let \(\tilde{f}(\zeta,\zeta^*,s,t) = f(x(\zeta,\zeta^*),y(\zeta,\zeta^*),s,t)\). Inserting the expressions for \((u^\dagger)^{n_1}\) and \((v^\dagger)^{n_2}\) from Eqs.(3.22), one obtains

\[
\tilde{f}(\zeta,\zeta^*,s,t) = \sum_{n_1,n_2} \frac{1}{n_1!n_2!} s^{n_1} e^{\zeta\zeta^*} \left(-\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta}\right)^{n_1} e^{-\zeta\zeta^*} t^{n_2} e^{\zeta\zeta^*} \left(-\frac{1}{\sqrt{2}} \frac{\partial}{\partial \zeta^*}\right)^{n_2} \tilde{\psi}_o(\zeta,\zeta^*)
\]

\[
= \frac{1}{\mu(\zeta,\zeta^*)} \sum_{n_1,n_2} \frac{1}{n_1!n_2!} \left(-\frac{s}{\sqrt{2}} \frac{\partial}{\partial \zeta}\right)^{n_1} \left(-\frac{t}{\sqrt{2}} \frac{\partial}{\partial \zeta^*}\right)^{n_2} \mu(\zeta,\zeta^*) \tilde{\psi}_o(\zeta,\zeta^*)
\]

\[
= \frac{1}{\mu(\zeta,\zeta^*)} \exp \left(-\frac{s}{\sqrt{2}} \frac{\partial}{\partial \zeta}\right) \exp \left(-\frac{t}{\sqrt{2}} \frac{\partial}{\partial \zeta^*}\right) \mu(\zeta,\zeta^*) \tilde{\psi}_o(\zeta,\zeta^*)
\]

Therefore, by Eq.(3.19) the function \(\tilde{f}(\zeta,\zeta^*,s,t)\) is translated in \(\zeta\) by an amount \(\frac{s}{\sqrt{2}}\) and in \(\zeta^*\) by an amount \(\frac{t}{\sqrt{2}}\), which leads to the following closed-form expression for the generating function:

\[
\tilde{f}(s,t,\zeta,\zeta^*) = \frac{1}{\mu(\zeta,\zeta^*)} \mu \left(\zeta - \frac{s}{\sqrt{2}},\zeta^* - \frac{t}{\sqrt{2}}\right) \tilde{\psi}_o \left(\zeta - \frac{s}{\sqrt{2}},\zeta^* - \frac{t}{\sqrt{2}}\right).
\]

Expressing the ground state in terms of \(\zeta\) and \(\zeta^*\)

\[
\tilde{\psi}_o(\zeta,\zeta^*) = \frac{1}{b\sqrt{\pi}} e^{-\zeta\zeta^*},
\]

(3.23)

one finally obtains the generating function

\[
\tilde{f}(s,t,\zeta,\zeta^*) = e^{-\zeta\zeta^*} e^{-\left(\zeta - s/\sqrt{2}\right) \left(\zeta^* - t/\sqrt{2}\right)} \frac{1}{b\sqrt{\pi}} e^{-\left(\zeta - s/\sqrt{2}\right) \left(\zeta^* - t/\sqrt{2}\right)}
\]

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= \frac{1}{b\sqrt{\pi}} e^{-\zeta^*} \exp \left(2\zeta^* \frac{s}{\sqrt{2}} + 2\zeta \frac{t}{\sqrt{2}} - st\right) \\
= \tilde{\psi}_o(\zeta, \zeta^*) \exp \left(2\zeta^* \frac{s}{\sqrt{2}} + 2\zeta \frac{t}{\sqrt{2}} - st\right) .

Transforming back to the original cartesian coordinates,

\[ f(s, t, x, y) = \psi_o(x, y) \exp \left[\frac{(x + iy)s}{b} + \frac{(x - iy)t}{b}\right] e^{-st} , \tag{3.24} \]

a convenient generating function has been constructed that allows one to obtain all of the possible excited states easily. Hence, expanding in powers of \( s \) and \( t \) and taking coefficients (multiplying the \( n^{th} \) coefficient by \( \sqrt{n!} \)), one can obtain the desired quantum states. For example, the first four are as follows:

\[
\psi_{0,0} = \frac{1}{b\sqrt{\pi}} e^{-\frac{(x^2 + y^2)}{2b^2}} \\
\psi_{0,1} = \frac{(x + iy)}{b^2\sqrt{\pi}} e^{-\frac{(x^2 + y^2)}{2b^2}} \\
\psi_{1,0} = \frac{(x - iy)}{b^2\sqrt{\pi}} e^{-\frac{(x^2 + y^2)}{2b^2}} \\
\psi_{1,1} = \frac{(x^2 + y^2 - b^2)}{b^3\sqrt{\pi}} e^{-\frac{(x^2 + y^2)}{2b^2}}
\]

Before going further, simplify the calculations by writing \( x, y, p_x \) and \( p_y \) in terms of the operators \( u, u^\dagger, v, v^\dagger \). Using the definitions of the ladder operators one has

\[
x = \frac{1}{2} b(u + u^\dagger + v + v^\dagger) , \\
y = \frac{i}{2} b(u^\dagger - u + v - v^\dagger) , \\
p_x = \frac{m\Omega}{2i} b(u - u^\dagger + v - v^\dagger) , \\
p_y = \frac{m\Omega}{2} b(v + v^\dagger - u - u^\dagger) . \tag{3.25}
\]

Implementing Eqns.(3.25), gives the general energy eigenvalue for the Hamiltonian \( H_{xy} \) in more transparent form. Start by rewriting the Hamiltonian, Eq.(3.4), in terms of the oper-
ators $u, u^\dagger, v, \text{ and } v^\dagger$. Thus, with the normalization expressed in Eq. (3.7)

$$H_{xy} = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2} m \Omega^2 (x^2 + y^2) - \omega_b L_z$$

$$= \frac{\hbar \Omega}{4} (uw^\dagger + u^\dagger u + vv^\dagger + v^\dagger v - 2uv - 2u^\dagger v^\dagger)$$

$$+ \frac{\hbar \Omega}{4} (uu^\dagger + u^\dagger u + vv^\dagger + v^\dagger v + 2uv + 2u^\dagger v^\dagger)$$

$$- \frac{\hbar \omega_b}{2} (vv^\dagger + v^\dagger v - uu^\dagger - u^\dagger u).$$

After simplification, the $xy$ portion of the Hamiltonian becomes

$$H_{xy} = \frac{\hbar \Omega}{4} (uu^\dagger + u^\dagger u + vv^\dagger + v^\dagger v) + \frac{\omega_b \hbar}{2} (uu^\dagger + u^\dagger u - vv^\dagger - v^\dagger v). \quad (3.26)$$

In order to calculate the energy, I insert this into the Schrodinger equation $H | \Psi_n \rangle = E | \Psi_n \rangle$, where $| \psi_n \rangle = |n_1, n_2, n_3 \rangle$ so

$$E_{n_1n_2n_3} = \langle n_1, n_2, n_3 | H_{xy} | n_1, n_2, n_3 \rangle.$$

Thus, simple calculation gives

$$E_{n_1n_2n_3} = \frac{\hbar \Omega}{2} (n_1 + n_2 + 1) - \omega_b \hbar (n_2 - n_1) + \hbar \beta (n_3 + \frac{1}{2}) \quad (3.27)$$

A plot of this result as a function of the Larmor frequency is shown in Figure 1.

### 3.2 Selection Rules

If one wishes to allow for transitions from one energy level to another, a time-dependent potential is introduced. The time-dependent portion of the Hamiltonian is assumed to be small compared to the time-independent part, thus the time-dependent Hamiltonian $H_{\text{rad}}$ is treated as a perturbation (time-dependent perturbation theory). Thus the full Hamiltonian of the problem becomes

$$H(t) = H_o + H_{\text{rad}}(t).$$
Fermi’s golden rule is an expression that gives the rate of transition from $|\psi_i\rangle$ to $|\psi_f\rangle$

$$\Gamma = \frac{2\pi}{\hbar} |\langle \psi_f | H_{\text{rad}} | \psi_i \rangle|^2 \rho(E_f) \delta(E_f - E_i \pm \hbar \omega) ,$$  

(3.28)

which is due to the absorption or emission of a photon. The $\rho(E_f)$ term in Eq.(3.28) denotes the density of final states and the $\delta(E_f - E_i \pm \hbar \omega)$ allows only for transitions between states of equal energies.

Specifically, the time dependence comes from an electric field $\vec{E}(\vec{r}, t)$ which is not assumed to be constant ($\nabla \phi = 0$ at $t = 0$) thus,

$$\vec{E}(\vec{r}, t) = \frac{\partial \vec{A}(\vec{r}, t)}{\partial t} .$$

As usual, the time and position dependence of this vector potential are assumed to be exponential. This is distinct from the vector potential $\vec{A}_b(\vec{r})$ due to the static magnetic field in the problem, thus it is denoted

$$\vec{A}_{\text{rad}}(\vec{r}, t) = A_o \hat{\epsilon} e^{i(\vec{k} \cdot \vec{r} - \omega t)} ,$$

which satisfies the Coulomb gauge condition $\nabla \cdot \vec{A}_{\text{rad}} = 0$. Due to the linearity of the electromagnetic field, $\vec{A}(\vec{r}, t) = \vec{A}_b(\vec{r}) + \vec{A}_{\text{rad}}(\vec{r}, t)$. Thus, the system is coupled to the radiation
through the time dependent Hamiltonian

\[ H_{\text{rad}}(t) = \frac{q}{c} \vec{A}_{\text{rad}} \cdot \vec{p}. \]

The time dependent portion of this Hamiltonian forms the energy-conserving delta functions. Also, the incident photon has a given polarization \( \hat{\epsilon} \) and wave vector \( \vec{k} = \frac{2\pi}{\lambda} \hat{k} \) which are assumed to always be mutually perpendicular (the angular frequency is \( \omega_k = ck \)). Thus, the time dependent Hamiltonian is

\[
H_{\text{rad}} = A_o e^{\pm i\vec{k} \cdot \vec{r}} \hat{\epsilon} \cdot \vec{p} \\
= A_o \left( 1 \pm i \frac{2\pi}{\lambda} \hat{k} \cdot \vec{r} \pm \cdots \right) (\hat{\epsilon} \cdot \vec{p}). \tag{3.29}
\]

In order to compute the selection rules, one need only consider matrix elements between the initial and final states. Clearly a non-zero result indicates an allowed transition. In most problems, the first term in the expansion Eq.(3.29) dominates, which is the electric dipole term \( (\hat{\epsilon} \cdot \vec{p}) \). The extent to which the other terms of the expansion contribute depends on a length ratio. The second term in the expansion, the magnetic dipole plus the electric quadrupole, has a strength on the order of \( \frac{a}{\lambda} \), where \( a \) is the diameter of the problem of interest. For an atomic or molecular problem, the diameter \( a \) is usually much smaller than \( \lambda \). However in this case a quantum dot is considered, which has a diameter much larger than that of the typical atomic problem. Therefore, presumably the higher multipole terms are more important. One can readily compute the matrix elements between final and initial states by writing \( \vec{p} \) and \( \vec{r} \) in terms of the ladder operators of the problem. The lowest-order electric dipole transitions were computed by considering

\[
\langle n_{1f} n_{2f} n_{3f} | \hat{\epsilon} \cdot \vec{p} | n_{1i} n_{2i} n_{3i} \rangle,
\]

while the lowest order magnetic dipole and electric quadrupole terms were calculated through
elements of the form
\[ \langle n_{1f} n_{2f} n_{3f} | (\hat{\epsilon} \cdot \vec{p}) (\hat{k} \cdot \vec{r}) | n_{1i} n_{2i} n_{3i} \rangle . \]

Selected results are displayed in Table 2.

### 3.3 A Charged Particle with Spin

The next step is to make the problem perhaps a bit more realistic by adding spin to the particle. In order to form the spin Hamiltonian, I take into account spin-orbit coupling, as well as Zeeman energy. Thus, assume the spin portion of the Hamiltonian is

\[ H_{\text{spin}} = -\vec{\mu} \cdot \vec{B} \tag{3.30} \]

where the magnetic field in the electron’s frame of reference [22] is

\[ \vec{B}_{\text{electron}} = \vec{B}_o - \frac{\vec{v}_c}{c} \times \vec{E}_{\text{lab}}, \tag{3.31} \]

where \( \vec{B}_o \) is the applied field in the laboratory frame (\( \vec{B}_o = \vec{B}_{\text{lab}} \)),

\[ \vec{E}_{\text{lab}} = -\nabla \phi \]

and

\[ \vec{\mu} = -\frac{e}{m} \vec{S} . \]

The magnetic scalar potential \( \phi \) comes from \( U \) in the model described above

\[ U = q\phi = \frac{m}{2} \alpha^2 (x^2 + y^2) + \frac{m}{2} \beta^2 z^2 . \]

Substituting this into Eq. (3.30),

\[ H_{\text{spin}} = \frac{e}{m} \vec{S} \cdot \vec{B}_o - \left[ \frac{\vec{v}}{c} \times \left\{ \frac{1}{e} (\alpha^2 (x\hat{x} + y\hat{y}) + \beta^2 z\hat{z}) \right\} \right] . \]

Then, expressing velocity in terms of canonical momentum

\[ \vec{v} = \frac{1}{m} \left( \vec{p} - \frac{q}{c} \vec{A} \right) , \]
Table 2: Selection Rules

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<th></th>
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and finally substituting this into Eq.(3.30) as well, gives

\[ H_{\text{spin}} = \frac{e}{m} \vec{S} \cdot \left( \vec{B}_0 - \left\{ \frac{1}{e mc} (\vec{p} - \frac{q}{c} \vec{A}) \right\} \times \left\{ \frac{1}{e} \alpha^2 (x \hat{x} + y \hat{y}) + \beta^2 z \hat{z} \right\} \right) . \]

Simplifying, one gets the final expression for the spin portion of the Hamiltonian

\[ H_{\text{spin}} = -\hbar^2 mc^2 \left[ \frac{\alpha^2 (x\sigma_y - y\sigma_x) p_z + \alpha^2 (yp_x - xp_y) \sigma_z}{2} \right. \]
\[ \left. - \beta^2 z(p_x\sigma_y - p_y\sigma_x) - m \omega_b \alpha^2 (x^2 + y^2) \sigma_z + m \omega_b \beta^2 z(x\sigma_x + y\sigma_y) \right]. \tag{3.32} \]

The total Hamiltonian for the system is then a sum of three terms,

\[ H = H_{xy} + H_z + H_{\text{spin}} . \tag{3.33} \]

This new overall Hamiltonian is not completely solvable using the same method. This is so since, taking commutations, one soon realizes that the set of generators is not closed with respect to the bracket operation. (The Lie algebra is not finite dimensional.) Hence, I use perturbation theory, treating \( H_{\text{spin}} \) as the perturbing Hamiltonian. The first-order corrections to the energies are calculated using

\[ E^{(1)}_n = \langle \Psi_n | \langle \uparrow | H_{\text{spin}} | \uparrow \rangle | \Psi_n \rangle \]

and

\[ E^{(1)}_{\downarrow} = \langle \Psi_n | \langle \downarrow | H_{\text{spin}} | \downarrow \rangle | \Psi_n \rangle \tag{3.34} \]

Similarly, the second order corrected energies could be calculated with

\[ E^{(2)}_n = \sum_{m \neq n} \left| \langle \Psi_m | \langle s_m | H_{\text{spin}} | s_n \rangle | \Psi_n \rangle \right|^2 \frac{E^{(0)}_n - E^{(0)}_m}{} , \tag{3.35} \]

although I have not done this.

Starting with the ground state, calculate the first order correction for each allowed state.

For the \(|\uparrow\rangle\) state,

\[ \langle \uparrow | H_{\text{spin}} | \uparrow \rangle = -\frac{\hbar}{2mc^2} [\alpha^2 (yp_x - xp_y) - m \omega_b \alpha^2 (x^2 + y^2)] \]
while for the $|\downarrow\rangle$ state the sign is reversed
\[ \langle \downarrow | H_{\text{spin}} | \downarrow \rangle = -\langle \uparrow | H_{\text{spin}} | \uparrow \rangle . \]
Matrix elements between states of different spin vanish. For a given set $\{n_1,n_2\}$, the Hamiltonian in matrix form is block diagonal
\[
H = \begin{pmatrix} H_{\uparrow} & 0 \\ 0 & H_{\downarrow} \end{pmatrix}
\]
The next step is to calculate the first order correction to the energies using this perturbing Hamiltonian:
\[
E_n^{(1)} = \langle \Psi_{n_1,n_2,n_3} | \langle \uparrow | H_{\text{spin}} | \uparrow \rangle | \Psi_{n_1,n_2,n_3} \rangle = -\frac{\hbar c^2}{2mc^2} \langle n_1,n_2,n_3 | (y p_x - x p_y - m \omega_b (x^2 + y^2)) | n_1,n_2,n_3 \rangle .
\]
Then, replacing $x$, $y$, $p_x$, and $p_y$ in terms of the four ladder operators $u$, $u^\dagger$, $v$, and $v^\dagger$ from above and breaking up the calculation for clarity, start with the first non-vanishing term of $H_{\text{spin}}$ from Eq.(3.32),
\[
\langle n_1,n_2,n_3 | y p_x - x p_y | n_1,n_2,n_3 \rangle = \frac{\hbar}{4} (n_1 + n_2 + 1 - n_2 - n_2 - 1 - n_2 + n_1 + n_1 + 1 + n_1)
= \hbar(n_1 - n_2).
\]
The second non-vanishing term is
\[
\langle n_1,n_2,n_3 | x^2 + y^2 | n_1,n_2,n_3 \rangle = \frac{\hbar}{4m \Omega} (n_1,n_2,n_3 | 2uu^\dagger + 2u^\dagger u + 2v^\dagger v + 2v v^\dagger | n_1,n_2,n_3)
= \frac{\hbar}{2m \Omega} (n_1 + 1 + n_2 + 1 + n_2 + n_1)
= \frac{\hbar}{m \Omega} (n_1 + n_2 + 1)
\]
Collecting these together I get the general first order correction for the \( |↑\rangle \) state

\[
E_{n↑} = \frac{\hbar^2 \alpha^2}{2mc^2} \left[ \frac{\omega_b}{\Omega} (n_1 + n_2 + 1) - (n_1 - n_2) \right].
\]

(3.36)

The first-order correction to the energy for the spin down state is \( E_{n↓} = -E_{n↑} \), thus one can write the corrected energy including spin as

\[
\begin{pmatrix}
\frac{\hbar^2 \alpha^2}{2mc^2} \left[ \frac{\omega_b}{\Omega} (n_1 + n_2 + 1) - (n_1 - n_2) \right] & 0 \\
0 & -\frac{\hbar^2 \alpha^2}{2mc^2} \left[ \frac{\omega_b}{\Omega} (n_1 + n_2 + 1) - (n_1 - n_2) \right]
\end{pmatrix}
\]

In general, the corrected energies for \( s = \pm \) are

\[
E_{n_1,n_2,n_3,\pm} = \frac{\hbar \Omega}{2} (n_1 + n_2 + 1) - \omega_b \hbar (n_2 - n_1) + \frac{\hbar \Omega}{2} (n_1 - n_2)
\]

\[
\pm \frac{\hbar^2 \alpha^2}{2mc^2} \left[ \frac{\omega_b}{\Omega} (n_1 + n_2 + 1) - (n_1 - n_2) \right].
\]

(3.37)

A plot of the corrected energies as a function of the Larmor frequency is shown in Figure 2.

\[\text{Figure 2: Plot of Energy Including Spin as a Function of the Larmor Frequency.}\]
CHAPTER 4

COMPLETELY SOLVABLE THREE-BODY PROBLEM

The most famous n-body problem is one where particles interact by an inverse square-law force. However, there is a class of exactly solvable n-body problems in which interactions are harmonic. In this section a Lie-algebraic method is illustrated by treating a solvable 3-body problem. In Lagrangian mechanics, the $2n$ coordinates of the tangent space comprising $x = \{x_1, ..., x_n\}$ and $\dot{x} = \{\dot{x}_1, ..., \dot{x}_n\}$ are independent. Lagrange’s equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \quad (4.1)$$

so, defining canonical momentum

$$p_i = \frac{\partial L}{\partial \dot{x}_i},$$

the Hamiltonian

$$H(x, p, t) = \sum_i p_i \dot{x}_i - L$$

is a function of *phase-space coordinates* $x$ and $p$. In general, one has $2n$ coordinates, $x = \{x_1, ..., x_n\}$ and $p = \{p_1, ..., p_n\}$. Treating these as independent allows one to obtain *Hamilton’s canonical equations*:

$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x_i}. \quad (4.2)$$

The time derivative of any function $A(x, p)$ of phase-space coordinates is

$$\frac{dA}{dt} = \sum_{i=1}^{n} \left( \frac{\partial A}{\partial x_i} \dot{x}_i + \frac{\partial A}{\partial p_i} \dot{p}_i \right).$$
Thus, following the sign convention of Perelomov [1], I define the Poisson bracket between any two functions of phase-space variables \(A\) and \(B\)

\[
[A, B] = \sum_i \left( \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial x_i} - \frac{\partial B}{\partial p_i} \frac{\partial A}{\partial x_i} \right).
\] (4.3)

Using Eqs. (4.2) and (4.3), \(\frac{dA}{dt}\) is the bracket of the Hamiltonian \(H\) with \(A\)

\[
\frac{dA}{dt} = [H, A].
\]

If \([H, A] = 0\), then \(A\) is a constant of the motion, or a conserved quantity. The Poisson bracket is another example of a Lie bracket \([x, y]\) which, as the reader will recall, has three main properties:

1. \([x, y] = -[y, x]\) - antisymmetry

2. \([x + \alpha y, z] = [x, z] + \alpha[y, z]\) - linearity

3. \([[x, y], z] + [[y, z], x] + [[z, x], y] = 0\) - Jacobi identity

As noted before, a Lie algebra is an algebra with a Lie bracket product.

Classical mechanics involves the study of Hamiltonian systems which are characterized by the Heisenberg algebra \(\mathcal{H}\) over \(\mathbb{C}\) generated by functions of the phase space variables including the Hamiltonian, with the Poisson bracket as the product. (So in general, it is infinite dimensional.) The Poisson bracket has the following additional property:

\[
[x, yz] = y[x, z] + [x, y]z \quad \text{-Leibniz rule.}
\]

In the problem below, I consider the finite dimensional extended Lie algebra

\[
\mathcal{A} = \{x_1, \ldots, z_3, p_{x_1}, \ldots, p_{z_3}, H(x, p), L_z, L_{\alpha\beta\gamma}, 1\}.
\]
The three-body problem involves three particles, interacting with each other via harmonic forces. I choose these particles to have equal masses in order not to obscure the important points of the following discussion with inessential details. Thus, \( m_1 = m_2 = m_3 = m \), which have respective positions \( \vec{r}_1, \vec{r}_2, \) and \( \vec{r}_3 \). (Here harmonic forces are forces associated with harmonic motion, where the Hamiltonian is of quadratic form.) There are three particles in this problem, each with three degrees of freedom, corresponding to nine configuration coordinates. Hence the full phase space is 18 dimensional. Also, there is an external force similar to the force caused by a magnetic field acting on moving charges. (The force considered is not actually magnetic, since if the origin of the force were actually through electric charge, there would also be mutual electric repulsion of inverse square type, rendering the problem unsolvable.) So, the potential energy is

\[
V(\vec{r}) = \frac{1}{2} \sum_{\alpha \neq \beta} k \left( |\vec{r}_\alpha - \vec{r}_\beta|^2 \right),
\]
where \( k \) is like a spring constant except that the springs have zero equilibrium length. This potential leads to a homogeneous quadratic Hamiltonian, so that the motion is governed by a set of linear homogenous differential equations with constant coefficients. This is what makes the problem exactly solvable. The Hamiltonian is

\[
H = \frac{1}{2m} \sum_{\alpha=1}^{3} |\vec{p}_\alpha - \frac{q}{c} \vec{A}|^2 + \frac{1}{2} \sum_{\alpha \neq \beta} k \left( |\vec{r}_\alpha - \vec{r}_\beta|^2 \right),
\]  

(4.4)

where again the symmetric vector potential \( \vec{A} = \frac{B_0}{2}(x\hat{y} - y\hat{x}) \) is chosen. The total angular momentum \( \vec{L} \) is

\[
\vec{L} = \left( \frac{1}{2}(\vec{r}_1 + \vec{r}_2 + \vec{r}_3) \times (\vec{p}_1 + \vec{p}_2 + \vec{p}_3) \right).
\]  

(4.5)
In order to solve the problem, it is helpful as usual to identify the conserved quantities. Due to the magnetic field, the vector $\vec{L}$ is not conserved. However, since $\vec{B}$ is parallel to the $z$-axis, the problem is invariant with respect to rotation about this axis. Recall Nöther’s theorem states that to each symmetry of the the Lagrangian corresponds a conserved quantity of the system. The axial symmetry guarantees that $L_z$ is conserved. Also, one finds that the $z$-components $L_{\alpha\beta z}$ of the relative angular momentum vector of two particles $\alpha$ and $\beta$ with respect to one another is conserved

$$\vec{L}_{\alpha\beta} = (\vec{r}_\beta - \vec{r}_\alpha) \times (\vec{p}_\beta - \vec{p}_\alpha).$$  \hspace{1cm} (**4.6**)

Finally, the $z$-component of the total linear momentum vector of the system is also conserved

$$p_z = (p_{1z} + p_{2z} + p_{3z}).$$  \hspace{1cm} (**4.7**)

Thus, in Poisson bracket notation

$$[H, L_z] = 0, \quad [H, L_{\alpha\beta z}] = 0, \quad [H, p_z] = 0.$$  \hspace{1cm} (**4.8**)

Again consider $\mathcal{A}$, the finite dimensional extended sub-algebra of $\mathcal{H}$. All of the elements such that the bracket between any two gives zero, form a Lie subalgebra of $\mathcal{A}$ which is the Cartan subalgebra (section 2.1). In other words, the Cartan subalgebra contains all constants of motion. In the three-body problem, it contains five elements $L_z$, $L_{12z}$, $p_z$, 1, and $H$. As usual, a linear combination of all the basis elements of the algebra $\mathcal{A}$ can be constructed such that it is an eigenvector of the Poisson bracket operation (a root vector of $\mathcal{A}$) with each generator of the Cartan subalgebra $\{K\}$, including the Hamiltonian. One defines for notational convenience the set of 18-dimensional phase-space coordinates as

$$\xi = \{\xi_1, \xi_2, \ldots, \xi_{18}\} = \{x_1, y_1, z_1, x_2, \ldots, p_{3y}, p_{3z}\}.$$  

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Thus, an eigenvector of $ad_k$ (where $k \in K$) constructed from combinations of $\xi$ is of the form

$$u = \sum_{k=1}^{18} a_k \xi_k ,$$

where $\{a_k\}$ are complex coefficients to be determined. Thus, as before, treat the set of generators $\xi = \{\xi_1, \xi_2, ..., \xi_{18}\}$ as a basis of independent vectors and compare coefficients. In this way, one determines what the coefficients must be in order to form, for example, an eigenvector of the eigenvalue equation

$$[H, u_\alpha] = \lambda_\alpha u_\alpha .$$

(4.9)

In other words, the motivation here is to obtain the decoupled differential equation

$$\dot{u} = \lambda u ,$$

with elementary solutions that have exponential time dependence $u(t) = u(0)e^{\lambda t}$ (since, again, this is a set of linear homogeneous DE’s with constant coefficients). This decoupling method is the same as in the theory of small oscillations where $u$ would represent a normal mode. This differential equation describes the motion of $u$ throughout time. Since $u$ is a linear combination of the phase-space coordinates of the system, the solutions of a complete set of 18 such decoupled equations will allow one to find the position and momentum of each particle at any given time. Using the oscillator frequency $\omega_o$ and cyclotron frequency $\omega_b$

$$\omega_o = \sqrt{\frac{k}{m}} , \quad \omega_b = \frac{q B_o}{2mc} ,$$

and making the substitution

$$\Omega = \sqrt{\omega_b^2 + 3\omega_o^2} ,$$

two roots appear in the solution to the eigenvalue equation, Eq.(4.9), namely

$$r_- = -\omega_b + \Omega \quad \text{and} \quad r_+ = \omega_b + \Omega .$$
Therefore, solving the set of DE’s for the Hamiltonian alone one obtains Table 3 of eigenvalues and eigenvectors.

Physically, the eigenvectors \( \{u_\alpha\} \) correspond to the normal mode coordinates of the three-mass system, providing those solutions to the motion that have exponential time dependence. Each combination of eigenvectors with a particular corresponding eigenvalue \( \lambda = i\omega_0 \) defines a mode of the system. From the table, it is clear that in the 3-body problem, the eigenvectors appear in complex conjugate pairs. These pairs each correspond to two possible motions, both of which have the same frequency.

Complete solutions to the equations of motion for this system involve combinations of the normal modes weighted with appropriate amplitude and phase factors \(^\text{23}\). A solution to the physical motion of the system is the real or imaginary part of the complex conjugate combination.

From Table 3 (of solutions to Eq.(4.9)), it is clear that the problem decouples into \( xy \) and \( z \) parts. Thus, the Hamiltonian can be rewritten as

\[
H = \frac{1}{2m} \sum_\alpha \left( p_{\alpha x}^2 + p_{\alpha y}^2 \right) - \frac{2q}{c} L_{\alpha z} + \frac{q^2}{c^2} \left( A_{\alpha x}^2 + A_{\alpha y}^2 \right)
\]

\[
+ \frac{1}{2} \sum_{\alpha \neq \beta} k \left( |x_\alpha \hat{x} y_\alpha \hat{y} - x_\beta \hat{x} y_\beta \hat{y}| \right)^2 + \frac{1}{2m} \sum_\alpha p_{\alpha z}^2 + \frac{1}{2} \sum_{\alpha \neq \beta} k \left( |z_\alpha \hat{z} - z_\beta \hat{z}| \right)^2.
\]

The motion along the \( z \)-axis is completely determined by eigenvectors \( u_1, u_4, u_7, u_8, u_9, \) and \( u_{10} \). The eigenvector \( u_1 \) corresponds to a translational mode along the \( z \)-axis, unaffected by the field (the particles move with constant velocity \( u_1 = \text{const.} + (p_{1z} + p_{2z} + p_{3z}) t \)). The eigenvectors \( u_7, u_8, u_9, \) and \( u_{10} \) describe an oscillatory motion of two particles with respect to a third, confined to the \( z \)-axis moving with oscillator frequency \( \pm \omega_0 \). The motion corresponding to the vector \( u_4 \) does not have an exponential solution, so is of the form \( u_4 = \text{const.} + (z_1 + z_2 + z_3) t \). Physically, this corresponds to the particles moving with a
Table 3: Eigenvalues and Eigenvectors of the 3-D three body problem

<table>
<thead>
<tr>
<th></th>
<th>( \lambda )</th>
<th>eigenvector</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>( u_1 = p_{z1} + p_{z2} + p_{z3} )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>( u_2 = p_{y1} + p_{y2} + p_{y3} + mx_1\omega_b + mx_2\omega_b + mx_3\omega_b )</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>( u_3 = p_{x1} + p_{x2} + p_{x3} - my_1\omega_b - my_2\omega_b - my_3\omega_b )</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>( u_4 = z_1 + z_2 + z_3 ) (generalized)</td>
</tr>
<tr>
<td>5</td>
<td>(-2i\omega_b)</td>
<td>( u_5 = p_{y1} + p_{y2} + p_{y3} - i(p_{x1} + p_{x2} + p_{x3}) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-m\omega_b [(x_1 + x_2 + x_3) + i(y_1 + y_2 + y_3)] )</td>
</tr>
<tr>
<td>6</td>
<td>(2i\omega_b)</td>
<td>( u_6 = p_{y1} + p_{y2} + p_{y3} + i(p_{x1} + p_{x2} + p_{x3}) )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-m\omega_b [(x_1 + x_2 + x_3) - i(y_1 + y_2 + y_3)] )</td>
</tr>
<tr>
<td>7</td>
<td>(-i\sqrt{3}\omega_o)</td>
<td>( u_7 = p_{z3} - p_{z1} + i\sqrt{3}m\omega_o(z_1 - z_3) )</td>
</tr>
<tr>
<td>8</td>
<td>(-i\sqrt{3}\omega_o)</td>
<td>( u_8 = p_{z2} - p_{z1} + i\sqrt{3}m\omega_o(z_1 - z_2) )</td>
</tr>
<tr>
<td>9</td>
<td>(i\sqrt{3}\omega_o)</td>
<td>( u_9 = p_{z3} - p_{z1} - i\sqrt{3}m\omega_o(z_1 - z_3) )</td>
</tr>
<tr>
<td>10</td>
<td>(i\sqrt{3}\omega_o)</td>
<td>( u_{10} = p_{z2} - p_{z1} - i\sqrt{3}m\omega_o(z_1 - z_2) )</td>
</tr>
<tr>
<td>11</td>
<td>(-ir_+)</td>
<td>( u_{11} = p_{y3} - p_{y1} - i(p_{x3} - p_{x1}) + m\Omega(x_1 - x_3) + im\Omega(y_1 - y_3) )</td>
</tr>
<tr>
<td>12</td>
<td>(-ir_+)</td>
<td>( u_{12} = p_{y2} - p_{y1} - i(p_{x2} - p_{x1}) + m\Omega(x_1 - x_2) + im\Omega(y_1 - y_2) )</td>
</tr>
<tr>
<td>13</td>
<td>(ir_+)</td>
<td>( u_{13} = p_{y3} - p_{y1} + i(p_{x3} - p_{x1}) + m\Omega(x_1 - x_3) - im\Omega(y_1 - y_3) )</td>
</tr>
<tr>
<td>14</td>
<td>(ir_+)</td>
<td>( u_{14} = p_{y2} - p_{y1} + i(p_{x2} - p_{x1}) + m\Omega(x_1 - x_2) - im\Omega(y_1 - y_2) )</td>
</tr>
<tr>
<td>15</td>
<td>(-ir_-)</td>
<td>( u_{15} = p_{y3} - p_{y1} + i(p_{x3} - p_{x1}) - m\Omega(x_1 - x_3) + im\Omega(y_1 - y_3) )</td>
</tr>
<tr>
<td>16</td>
<td>(-ir_-)</td>
<td>( u_{16} = p_{y2} - p_{y1} + i(p_{x2} - p_{x1}) - m\Omega(x_1 - x_2) + im\Omega(y_1 - y_2) )</td>
</tr>
<tr>
<td>17</td>
<td>(ir_-)</td>
<td>( u_{17} = p_{y3} - p_{y1} - i(p_{x3} - p_{x1}) - m\Omega(x_1 - x_3) - im\Omega(y_1 - y_3) )</td>
</tr>
<tr>
<td>18</td>
<td>(ir_-)</td>
<td>( u_{18} = p_{y2} - p_{y1} - i(p_{x2} - p_{x1}) - m\Omega(x_1 - x_2) - im\Omega(y_1 - y_2) )</td>
</tr>
</tbody>
</table>
constant velocity in the $z$ direction. This vector is a generalized eigenvector of order two.

For simplicity, I consider only the $xy$ portion in the following discussion.

Motivated by the desire to construct raising and lowering operators for this system in the corresponding quantum mechanical problem (see chapter I), one looks for a set of eigenvectors that satisfy simultaneously the eigenvalue equations corresponding to all elements in the Cartan subalgebra

$$[H, u_\alpha] = \lambda_\alpha u_\alpha, \quad [L_z, u_\alpha] = \gamma_\alpha u_\alpha, \quad [L_{12z}, u_\alpha] = \mu_\alpha u_\alpha. \quad (4.10)$$

In order to construct such a set of eigenvectors, start by obtaining solutions to the eigenvalue equation corresponding to $L_z$ by the same method as was employed for $H$. Then, collect the eigenvectors $\{u_\alpha\}$ degenerate in $\lambda, \mu$ and $\gamma$. Finally, combine these degenerate eigenvectors into linear combinations

$$v_\beta = \sum_{\alpha} a_{\beta \alpha} u_\alpha$$

to form the desired set $\{v_\beta\}$ of vectors that satisfy the entire system. (The coefficient $a$ may be zero.) In the process, one finds twelve ladder operators for the spectrum listed in Table 4. Once one has the list of eigenvectors which satisfy all of the bracket relations simultaneously, Eq.(4.10), the Hamiltonian can be rewritten in terms of these vectors. Technically, these eigenvectors are the elementary solutions of the system. The new eigenvectors of the total system appear in complex conjugate pairs

$$v_2 = v_1^*, \quad v_4 = v_3^*$$
$$v_7 = v_5^*, \quad v_8 = v_6^*$$
$$v_{11} = v_9^*, \quad v_{12} = v_{10}^* \quad (4.11)$$
Table 4: Eigenvalues and Eigenvectors of the entire Cartan subalgebra

<table>
<thead>
<tr>
<th></th>
<th>λ</th>
<th>γ</th>
<th>μ</th>
<th>operators</th>
</tr>
</thead>
</table>
| 1 | 0   | i  | 0  | \[ v_2 = p_{y_1} + p_{y_2} + p_{y_3} + m\omega_b(x_1 + x_2 + x_3) \]
|   |     |    |    | \[ -i(p_{x_1} + p_{x_2} + p_{x_3} - m\omega_b(y_1 + y_2 + y_3)) \]     |
| 2 | 0   | -i | 0  | \[ v_3 = p_{y_1} + p_{y_2} + p_{y_3} + m\omega_b(x_1 + x_2 + x_3) \]
|   |     |    |    | \[ +i(p_{x_1} + p_{x_2} + p_{x_3} - m\omega_b(y_1 + y_2 + y_3)) \]     |
| 3 | -2i \omega_b | i  | 0  | \[ v_5 = -i(p_{x_1} + p_{x_2} + p_{x_3}) + p_{y_1} + p_{y_2} + p_{y_3} \]
|   |     |    |    | \[ -m\omega_b((x_1 + x_2 + x_3) + i(y_1 + y_2 + y_3)) \]               |
| 4 | 2i \omega_b | -i | 0  | \[ v_6 = i(p_{x_1} + p_{x_2} + p_{x_3}) + p_{y_1} + p_{y_2} + p_{y_3} \]
|   |     |    |    | \[ -m\omega_b((x_1 + x_2 + x_3) - i(y_1 + y_2 + y_3)) \]               |
| 5 | -i(\omega_b + \Omega) | 0  | 0  | \[ v_{11} = -i(p_{x_1} + p_{x_2} - 2p_{x_3}) + p_{y_1} + p_{y_2} - 2p_{y_3} \]
|   |     |    |    | \[ -m\Omega((x_1 + x_2 - 2x_3) + i(y_1 + y_2 - 2y_3)) \]              |
| 6 | -i(\omega_b + \Omega) | 0  | 2i | \[ v_{12} = i(p_{x_1} - p_{x_2}) - p_{y_1} + p_{y_2} \]
|   |     |    |    | \[ +m\Omega((x_1 - x_2) + i(y_1 - y_2)) \]                            |
| 7 | i(\omega_b + \Omega)  | 0  | 0  | \[ v_{13} = i(p_{x_1} + p_{x_2} - 2p_{x_3}) + p_{y_1} + p_{y_2} - 2p_{y_3} \]
|   |     |    |    | \[ -m\Omega((x_1 + x_2 - 2x_3) - i(y_1 + y_2 - 2y_3)) \]              |
| 8 | i(\omega_b + \Omega)  | 0  | -2i| \[ v_{14} = -i(p_{x_1} - p_{x_2}) - p_{y_1} + p_{y_2} \]
|   |     |    |    | \[ +m\Omega((x_1 - x_2) - i(y_1 - y_2)) \]                            |
| 9 | -i(\Omega - \omega_b) | 0  | 0  | \[ v_{15} = i(p_{x_1} + p_{x_2} - 2p_{x_3}) + p_{y_1} + p_{y_2} - 2p_{y_3} \]
|   |     |    |    | \[ +m\Omega((x_1 + x_2 - 2x_3) - i(y_1 + y_2 - 2y_3)) \]              |
| 10| -i(\Omega - \omega_b) | 0  | -2i| \[ v_{16} = -i(p_{x_1} - p_{x_2}) - p_{y_1} + p_{y_2} \]
|   |     |    |    | \[ +m\Omega((x_2 - x_1) + i(y_1 - y_2)) \]                            |
| 11| i(\Omega - \omega_b)  | 0  | 0  | \[ v_{17} = -i(p_{x_1} + p_{x_2} - 2p_{x_3}) + p_{y_1} + p_{y_2} - 2p_{y_3} \]
|   |     |    |    | \[ +m\Omega((x_1 + x_2 - 2x_3) + i(y_1 + y_2 - 2y_3)) \]              |
| 12| i(\Omega - \omega_b)  | 0  | 2i | \[ v_{18} = i(p_{x_1} - p_{x_2}) - p_{y_1} + p_{y_2} \]
|   |     |    |    | \[ +m\Omega((x_2 - x_2) - i(y_1 - y_2)) \]                            |
With this in mind, the Hamiltonian of the system becomes:

\[
H_{xy} = \frac{1}{24m\Omega_2} \left[ \omega_b^2 v_9 v_9^* + 3\omega_b^2 v_{10} v_{10}^* + 4\omega_b^2 v_3 v_3^* - \omega_b \Omega v_9 v_9^* \\
-3\omega_b \Omega v_{10} v_{10}^* + \omega_b (\Omega + \omega_b) v_5 v_5^* + 3\omega_b (\Omega + \omega_b) v_6 v_6^* \\
+3\omega_b^2 (v_5 v_5^* + 3v_6 v_6^* + v_9 v_9^* + 3v_{10} v_{10}^* + 4v_3 v_3^*) \right]
\] (4.12)

The latter could be viewed as a quantum mechanical Hamiltonian for which we have now constructed ladder operators giving the complete excitation spectrum of a quantum 3-body problem. Converting each of the phase-space coordinates into combinations of the eigenvectors \( \{v_\beta\} \), one obtains Table 5. As in the case of the raising and lowering operators in chapter I, these elementary solutions with exponential time dependence can be used to determine the motion of the system corresponding to each mode. (To make this a quantum mechanical problem, the Hamiltonian would be rewritten in terms of symmetrized combinations of the eigenvectors \( \frac{1}{2} \sum_\alpha v_\alpha v_\alpha^\dagger + v_\alpha^\dagger v_\alpha \).)

In order to trace out this motion, one begins by collecting the appropriate elementary solutions. For example when analyzing modes with frequency \( \lambda = (\omega_b + \Omega) \), one must form linear combinations of the elementary solutions \( v_{11}, v_{12}, v_{13}, \) and \( v_{14} \). When combined properly, these elementary solutions lead to real functions, which constitute actual motion. So, including the proper time dependence, \( v_\alpha = c_\alpha e^{\lambda_\alpha t} \) (\( \alpha = 1, 2, ..., 18 \)), one can graph the motion of each particle as a function of time. Therefore, the coefficients \( c_\alpha \) are the initial conditions of the system \( v(t) = v(0) e^{\lambda_\alpha t} \). In this way, the entire orbit of each particle is determined over all time.

A 2-D parametric plot of the motion corresponding to each mode of the system is shown in Figure 3. From this figure, it is clear that the solutions \( v_5 \) and \( v_6 \) represent a mode of the system in which all three particles move in a circle about the center of mass. Analysis
of initial conditions for this mode reveals that the three particles begin at the same position on the $x$ or $y$ axis, and move in unison around the circle. This mode allows only a clockwise direction, since no spring force exists between the particles.

The modes corresponding to solutions $v_{11}$, $v_{12}$, $v_{13}$, or $v_{14}$ is slightly more complicated. There are a number of possible situations for motion in this mode, depending on the initial conditions given. In each case, the particles move around the center of mass in both the clockwise and counter-clockwise directions. The final oscillatory mode in the $xy$ plane is determined by elementary solutions $v_{15}$, $v_{16}$, $v_{17}$, and $v_{18}$. From the figure, it is clear that this mode is characterized by the same motion as was displayed for the previous mode. The only difference is in the angular frequency of rotation and the direction of motion.

The final mode corresponding to $xy$ motion of the particles (not included in the figure) is the zero frequency mode given by solutions $v_2$ and $v_3$. In this case, the particles are stationary relative to each other throughout time, only translating in the $xy$ plane deflected by the magnetic field.

All possible motion of this system can be described by appropriate linear combinations of these normal mode solutions. Thus, by analyzing the elementary solutions that satisfy the entire Cartan subalgebra (Eq.(4.10)) simultaneously, one has solved this harmonic 3-body problem completely.
Table 5: Phase Space Coordinates in terms of Elementary Solutions

<p>| | |</p>
<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$\frac{1}{24m\omega_i}(\omega_b(v_{11} + 3v_{12} + v_{13} + 3v_{14} - v_{15} - 3v_{16} - v_{17} - 3v_{18})$ $+ 2\Omega(v_2 + v_3 - v_5 - v_6))$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$\frac{1}{24m\omega_i}(\omega_b(v_{11} - 3v_{12} + v_{13} - 3v_{14} - v_{15} + 3v_{16} - v_{17} + 3v_{18})$ $+ 2\Omega(v_2 + v_3 - v_5 - v_6))$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$\frac{1}{12m\omega_i}(\omega_b(-v_{11} - v_{13} + v_{15} + v_{17})$ $+ \Omega(v_2 + v_3 - v_5 - v_6))$</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$\frac{-i}{24m\omega_i}(\omega_b(v_{11} + 3v_{12} - v_{13} - 3v_{14} + v_{15} + 3v_{16} - v_{17} - 3v_{18})$ $+ 2\Omega(v_2 - v_3 - v_5 + v_6))$</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$\frac{-i}{24m\omega_i}(\omega_b(v_{11} - 3v_{12} - v_{13} + 3v_{14} + v_{15} - 3v_{16} - v_{17} + 3v_{18})$ $+ 2\Omega(v_2 - v_3 - v_5 + v_6))$</td>
</tr>
<tr>
<td>$y_3$</td>
<td>$\frac{-i}{12m\omega_i}(\omega_b(-v_{11} + v_{13} - v_{15} + v_{17})$ $+ \Omega(v_2 - v_3 - v_5 + v_6))$</td>
</tr>
<tr>
<td>$p_{x_1}$</td>
<td>$\frac{-i}{24}(v_{11} + 3v_{12} - v_{13} - 3v_{14} - v_{15} - 3v_{16} + v_{17} + 3v_{18} - 2v_2 + 2v_3 - 2v_5 + 2v_6)$</td>
</tr>
<tr>
<td>$p_{x_2}$</td>
<td>$\frac{-i}{24}(v_{11} - 3v_{12} - v_{13} + 3v_{14} - v_{15} + 3v_{16} + v_{17} - 3v_{18} - 2v_2 + 2v_3 - 2v_5 + 2v_6)$</td>
</tr>
<tr>
<td>$p_{x_3}$</td>
<td>$\frac{i}{12}(v_{11} - v_{13} - v_{15} + v_{17} + v_2 - v_3 + v_5 - v_6)$</td>
</tr>
<tr>
<td>$p_{y_1}$</td>
<td>$\frac{i}{24}(-v_{11} - 3v_{12} - v_{13} - 3v_{14} - v_{15} - 3v_{16} - v_{17} - 3v_{18} + 2v_2 + 2v_3 + 2v_5 + 2v_6)$</td>
</tr>
<tr>
<td>$p_{y_2}$</td>
<td>$\frac{i}{24}(-v_{11} + 3v_{12} - v_{13} + 3v_{14} - v_{15} + 3v_{16} - v_{17} + 3v_{18} + 2v_2 + 2v_3 + 2v_5 + 2v_6)$</td>
</tr>
<tr>
<td>$p_{y_3}$</td>
<td>$\frac{i}{12}(v_{11} + v_{13} + v_{15} + v_{17} + v_2 + v_3 + v_5 + v_6)$</td>
</tr>
</tbody>
</table>
Figure 3: 2D parametric plots of possible $xy$ motion generated by combinations of oscillatory elementary solutions, which determine three separate modes of the system. Paths (a) - (c) correspond to orbits generated by solutions $v_{11}$ - $v_{18}$ (double arrows indicate direction of motion), while path (d) represents the mode generated by $v_5$ and $v_6$. 
CHAPTER 5

CONCLUSION

The Lie-algebraic techniques have been applied to both the quantum-mechanical problem of a particle in a magnetic field constrained by an additional quadratic potential, and a classical three body problem. In the second problem, the elementary solutions were computed using the poisson bracket, which are interpreted classically as ladder operators of the system. In both cases, this approach provided not only a straight-forward solution method, but also a glimpse of the connection between Lie algebra and physics, for typical physics graduate students. In each of these examples, the method of undetermined coefficients was used to construct ladder operators based on a complete set of invariants. The invariant operators, or functions (which formed the Cartan subalgebra) were first identified, then used to represent the entire algebra in terms of the Cartan-Weyl basis $\mathfrak{B}$ (consisting of the invariants and corresponding ladder operators). This general method can be applied to quite a few physical problems. This paper provides two explicitly-worked examples of simple physical systems treated by Lie algebraic techniques.

For the first example problem, in addition to calculating the energy eigenvalues, a generating function was constructed for the eigenstates of a harmonically constructed particle in a magnetic field. I am unaware that the full generating function has been published previously (we were not able to find any mention of it in the literature). Therefore, this may constitute a new result.
REFERENCES


