Our goal as we close out the semester is to give several “Fundamental Theorem of Calculus”-type theorems which relate volume integrals of derivatives on a given domain to line and surface integrals about the boundary of the domain.

The general form of these theorems, which we collectively call the fundamental theorems of vector calculus, is the following:

The integral of a “derivative-type object” on a given domain $D$ may be computed using only the function values along the boundary of $D$. 
Motivation: Fundamental Theorems of Vector Calculus

As an overview, we will roughly and informally summarize the content of the fundamental theorems of vector calculus. First let’s start with the ones we have already seen:

1. **FTC**: An area integral of the form

   \[ \int_{a}^{b} f'(x) \, dx \]

   may be computed by evaluating \( f \) at the boundary points \( a \) and \( b \) of the 1-dimensional domain interval \([a, b]\).
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2. **Fundamental Theorem for Conservative Vector Fields**: A line integral of the form

   \[ \int_{C} \nabla V \, ds \]

   may be computed by evaluating \( V \) at the boundary points of the 1-dimensional parametrized curve domain \( C \).
Fundamental Theorems of Vector Calculus, contd.

Now here are the new ones.

1. **Green’s Theorem**: A volume integral of the form

   \[
   \int \int_D \left( \frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} \right) d(x, y)
   \]

   may be computed by line integrating \( \vec{F} = (F_1, F_2) \) along the boundary of the 2-dimensional domain \( D \).
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2. **Stokes’ Theorem**: A surface integral of the form

   \[ \int \int_S \text{curl}(\vec{F})dS \]

   may be computed by line integrating \( \vec{F} \) along the boundary of the 2-dimensional domain \( S \).

   *Note:* The “curl” of \( \vec{F} \) is a derivative-type object we will define later.
Fundamental Theorems of Vector Calculus, contd.

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2. **Stokes’ Theorem**: A surface integral of the form

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   may be computed by line integrating $\vec{F}$ along the boundary of the 2-dimensional domain $S$.

   *Note*: The “curl” of $\vec{F}$ is a derivative-type object we will define later.

3. **Divergence Theorem**: A volume integral of the form

   $\int \int \int_W \text{div}(\vec{F}) d(x, y, z)$

   may be computed by surface integrating $\vec{F}$ along the boundary of the 3-dimensional domain $W$.

   *Note*: The "divergence" of $\vec{F}$ is another soon-to-be-introduced derivative-type object.
Definitions and Terminology

Definition
Let $D$ be a region in $\mathbb{R}^2$. Recall that a point $(x, y)$ is called a **boundary point** of $D$ if every open disk about $(x, y)$ intersects both $D$ and the exterior of $D$. Denote the set of all boundary points of $D$ by $\delta D$. We call $\delta D$ the **boundary** of $D$.
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Suppose $C$ may be parametrized by a continuous one-to-one $\mathbb{R}^2$-valued function $\vec{c}$ with domain $[a, b]$, where $\vec{c}(a) = \vec{c}(b)$. Then we call $C$ a **simple closed curve**.
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If $\delta D$ is a simple closed curve, then we choose to orient $\delta D$ in the counterclockwise direction. This is called the boundary orientation.
Green’s Theorem

**Theorem (Green’s Theorem)**

Let $D$ be a domain whose boundary $\delta D$ is a simple closed curve. Let $F = (F_1, F_2)$ be a vector field over $\mathbb{R}^2$. Then

$$\oint_{\delta D} \vec{F} \cdot ds = \int \int_D \left( \frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} \right) d(x, y).$$
Example

Verify Green’s theorem for the line integral \( \oint_C (xy^2, x) \cdot ds \) about the unit circle \( C \).
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Verify Green’s theorem for the line integral $\int_C (xy^2, x) \cdot ds$ about the unit circle $C$.

If $(F_1, F_2) = (xy^2, x)$, then

$$\frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} = 1 - 2xy^2,$$

and hence we are being asked to show that

$$\oint C (xy^2, x) \cdot ds = \int \int_R (1 - 2xy) d(x, y),$$

where $R$ is the interior of the unit circle.
Solution: The Line Integral

Goal: $\oint_C (xy^2, x) \cdot ds = \int \int_R (1 - 2xy) d(x, y)$
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We start with the line integral on the left. Parametrize $C$ by

$\vec{c}(t) = (\cos t, \sin t)$ for $0 \leq t \leq 2\pi$.

We have $\vec{c}'(t) = (-\sin t, \cos t)$.
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We have \( \vec{c}'(t) = (-\sin t, \cos t) \).

Now compute:

\[
\oint_C (xy^2, x) \cdot ds = \int_0^{2\pi} (\cos t \sin^2 t, \sin t) \cdot (-\sin t, \cos t) dt \\
= \int_0^{2\pi} (-\cos t \sin^3 t + \cos^2 t) dt \\
= \left[ -\frac{1}{4} \sin^4 t + \frac{1}{2} t + \frac{1}{4} \sin(2t) \right]_0^{2\pi} \\
= (0 + \pi + 0) - (0 + 0 + 0) \\
= \pi.
\]
Solution: The Volume Integral

Goal: \( \oint_C (xy^2, x) \cdot ds = \iint_R (1 - 2xy)d(x, y) \)
Known: \( \oint_C (xy^2, x) \cdot ds = \pi \)
Solution: The Volume Integral

Goal: $\oint_{C} (xy^2, x) \cdot ds = \int \int_{R} (1 - 2xy) d(x, y)$

Known: $\oint_{C} (xy^2, x) \cdot ds = \pi$

Now we compute the integral on the right and hope we get $\pi$: 
Solution: The Volume Integral

Goal: \( \oint_C (xy^2, x) \cdot ds = \int \int_R (1 - 2xy) \, d(x, y) \)

Known: \( \oint_C (xy^2, x) \cdot ds = \pi \)

Now we compute the integral on the right and hope we get \( \pi \):

\[
\int \int_R (1 - 2xy) \, d(x, y) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - 2xy) \, dy \, dx
\]

\[
= \int_{-1}^{1} \left[ y - xy^2 \right]_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \, dx
\]

\[
= \int_{-1}^{1} 2\sqrt{1-x^2} \, dx
\]

\[
= [2 \arcsin x]_{-1}^{1}
\]

\[= \pi. \]

So Green’s theorem is true in this case!
Example

Compute the circulation of \( \vec{F} = (\sin x, x^2 y^3) \) about the path \( C = \delta D \), where \( D \) is the triangle with vertices (0, 0), (2, 0), and (2, 2).
Example
Compute the circulation of $\vec{F} = (\sin x, x^2 y^3)$ about the path $C = \delta D$, where $D$ is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 2)$.

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Computing the circulation might be tedious in this case, as we would need to parametrize the three sides of the triangle separately. So we appeal to Green’s theorem instead.
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The interior of the triangle is bounded below by $y = 0$ and above by $y = x$, and on the left and right by $y = 0$ and $y = 2$. Check that

$$\frac{\delta F_2}{\delta x} = 2xy^3 \quad \text{and} \quad \frac{\delta F_1}{\delta y} = 0.$$
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\]

So by Green's theorem we get:

\[
\oint_{\delta D} (\sin x, x^2 y^3) \cdot ds = \iint_D (2xy^3 - 0)d(x, y)
\]

\[
= \int_0^2 \int_0^x 2xy^3 \, dy \, dx
\]

\[
= \int_0^2 \left[ \frac{1}{2} xy^4 \right]_y=0^x \, dx
\]

\[
= \frac{1}{2} \int_0^2 x^5 \, dx
\]

\[
= \frac{1}{2} \cdot \frac{1}{6} \cdot 2^6 = \frac{16}{3}.
\]
Corollary (Area Formula)

Let $D$ be a region in $\mathbb{R}^2$ and assume $C = \delta D$ is a simple closed curve. Then the area of $D$ is equal to

$$\frac{1}{2} \oint_C (-y, x) \cdot ds.$$
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Proof.

Set $\vec{F} = (F_1, F_2) = \frac{1}{2}(-y, x)$ and check that $\frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} = \frac{1}{2} - (-\frac{1}{2}) = 1$. Now apply Green's theorem:

$$ \text{Area of } D = \iint_D 1 \, d(x, y) $$

$$ = \iint_D \left( \frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} \right) d(x, y) $$

$$ = \oint_C \vec{F} \cdot ds $$

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Some Quick Terminology

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Definition

Let $\vec{F} = (F_1, F_2)$ be a vector field over $\mathbb{R}^2$ and let $\vec{c}(t) = (x(t), y(t))$ be a smooth parametrization of a curve in $\mathbb{R}^2$ with domain $[a, b]$. Define the $x$- and $y$-components of the vector field line integral of $\vec{F}$ over $C$ to be, respectively,

$$\int_C F_1 dx = \int_a^b F_1(\vec{c}(t))x'(t)dt$$

$$\int_C F_2 dy = \int_a^b F_2(\vec{c}(t))y'(t)dt.$$
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It follows from the definitions that

$$\int_C \vec{F} \cdot ds = \int_a^b \left( F_1(\vec{c}(t)), F_2(\vec{c}(t)) \right) \cdot (x'(t), y'(t)) \, dt$$

$$= \int_C F_1 \, dx + \int_C F_2 \, dy.$$
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$$

Lemma

The values of the $x$- and $y$-components of a vector field line integral are independent of the choice of parametrization $\vec{c}$.
Proof of a Special Case of Green’s Theorem

Goal: Show $\oint_{\delta D} \vec{F} \cdot ds = \int \int_{D} \left( \frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} \right) d(x, y)$.

We are unable to give a proof of Green’s theorem in its full generality, but we can prove it if we make the following very special simplifying assumptions:

▶ the boundary $\delta D$ may be described as the union of two graphs of the form $y = T(x)$ with $B(x) \leq T(x)$ for $a \leq x \leq b$; and

▶ $\delta D$ may also be described as the union of two graphs of the form $x = L(y)$ and $x = R(y)$ with $L(y) \leq R(y)$ for $c \leq y \leq d$.

(Picture on whiteboard!)
Proof of a Special Case, contd.

Goal: Show \( \oint_{\delta D} \vec{F} \cdot ds = \int \int_D \left( \frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} \right) d(x, y) \).

In order to observe Green’s theorem, we will break it up into two parts. Since \( \oint_{\delta D} \vec{F} \cdot ds = \oint_{\delta D} F_1 dx + \oint_{\delta D} F_2 dy \), it suffices for us to show the following two equalities:

\[
\oint_{\delta D} F_1 dx = -\int \int_D \frac{\delta F_1}{\delta y} d(x, y)
\]
\[
\oint_{\delta D} F_2 dy = \int \int_D \frac{\delta F_2}{\delta x} d(x, y).
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\oint_{\delta D} F_2 dy = \int \int_D \frac{\delta F_2}{\delta x} d(x, y).
\]

We begin by parametrizing the bottom \( C_1 \) and top \( C_2 \) of \( \delta D \).

\[
C_1: \tilde{c}_1(t) = (x_1(t), y_1(t)) = (t, B(t)) \\
C_2: \tilde{c}_2(t) = (x_2(t), y_2(t)) = (t, T(t))
\]

Note \( \tilde{c}_1 \) traverses \( C_1 \) along the boundary orientation but \( \tilde{c}_2 \) goes clockwise, i.e. \( \tilde{c}_2 \) parametrizes \(-C_2\).
Proof of a Special Case, contd.

Mid-Goal: Show that \( \oint \delta_D F_1 \, dx = - \int \int_D \frac{\delta F_1}{\delta y} \, d(x, y) \)
Proof of a Special Case, contd.

Mid-Goal: Show that $\oint_{\delta D} F_1 dx = - \int \int_D \frac{\delta F_1}{\delta y} d(x, y)$

Now we compute $\int \int_D \frac{\delta F_1}{\delta y} d(x, y)$.

$$\int \int_D \frac{\delta F_1}{\delta y} d(x, y) = \int_a^b \int_{B(x)}^{T(x)} \frac{\delta F_1}{\delta y} dy dx$$

$$= \int_a^b [F_1(x, T(x)) - F_1(x, B(x))] dx$$

$$= \int_a^b F_1(c_2(t))x_2'(t) dt - \int_a^b F_1(c_1(t))x_1'(t) dt$$

$$= \int_{-C_2}^{C_1} F_1 dx - \int_{C_1}^{C_2} F_1 dx$$

$$= - \oint_{\delta D} F_1 dx.$$

This completes the proof of the mid-goal.
Proof of a Special Case, contd.

Mid-Goal: Show that \( \oint_{\delta D} F_2 \, dy = \int \int_D \frac{\delta F_2}{\delta x} \, d(x, y) \)
Proof of a Special Case, contd.

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Now let \( C_3 \) and \( C_4 \) denote the left and right boundaries, and parametrize again:

\[
\begin{align*}
C_3: \; \vec{c}_3(t) &= (x_3(t), y_3(t)) = (L(t), t) \\
C_4: \; \vec{c}_4(t) &= (x_4(t), y_4(t)) = (R(t), t).
\end{align*}
\]

This time \( \vec{c}_3 \) parametrizes \(-C_3\) but \( \vec{c}_4 \) parametrizes \( C_4 \).
Proof of a Special Case, contd.

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\[
\int \int_D \frac{\delta F_2}{\delta x} \, d(x, y) = \int_c^{d} \int_{R(y)}^{L(y)} \frac{\delta F_2}{\delta x} \, dx \, dy
\]

\[
= \int_c^{d} F_2(R(y), y) \, dy - \int_c^{d} F_2(L(y), y) \, dy
\]

\[
= \int_c^{d} F_2(\vec{c}_4(t)) y_4'(t) \, dt - \int_c^{d} F_2(\vec{c}_3(t)) y_3'(t) \, dt
\]

\[
= \int_{C_4} F_2 \, dy - \int_{-C_3} F_2 \, dy
\]

\[
= \oint_{\delta D} F_2 \, dy.
\]
Proof of a Special Case, contd.

We conclude by observing that, as promised:

\[ \oint_{\delta D} \vec{F} \cdot ds = \oint_{\delta D} F_1 \, dx + \oint_{\delta D} F_2 \, dy = -\int \int_D \frac{\delta F_1}{\delta y} \, d(x, y) + \int \int_D \frac{\delta F_2}{\delta x} \, d(x, y) = \int \int_D \left( \frac{\delta F_2}{\delta x} - \frac{\delta F_1}{\delta y} \right) \, d(x, y). \]