Parametrized Surfaces

Recall from our unit on vector-valued functions at the beginning of the semester that an $\mathbb{R}^3$-valued function $\vec{c}(t)$ in one parameter is a mapping of the form

$$\vec{c} : I \rightarrow \mathbb{R}^3$$

where $I$ is some subinterval of the real line. If $\vec{c}$ is differentiable, we can think of $\vec{c}$ as smoothly embedding the interval $I$ as a closed curve $C$ in $\mathbb{R}^3$, a parametrized curve.
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Our goal now is to study parametrized surfaces, which are a natural next step from parametrized curves. Now instead of embedding a 1-dimensional domain $I$ into $\mathbb{R}^3$, we will embed 2-dimensional domain $D$ into $\mathbb{R}^3$ and do calculus on the embedded surface.
Definition
We consider a continuously differentiable $\mathbb{R}^3$-valued function of two variables $G: \mathbb{R}^2 \rightarrow \mathbb{R}^3$:

$$G(u, v) = (x(u, v), y(u, v), z(u, v))$$
Definition
We consider a continuously differentiable \( \mathbb{R}^3 \)-valued function of two variables \( G : \mathbb{R}^2 \rightarrow \mathbb{R}^3 : \)

\[
G(u, v) = (x(u, v), y(u, v), z(u, v))
\]

The **graph** of \( G \) is the set

\[
\{(x, y, z) \in \mathbb{R}^3 : \text{there exists } (u, v) \text{ with } G(u, v) = (x, y, z)\}.
\]

(This definition is analogous to that of the graph of a vector-valued function.)

The graph of \( G \) is also called a **parametrized surface**.

The domain \( D \) of \( G \) is called the **parameter domain**.
Example (Parametrization of a Sphere)

Fix any $R > 0$. Let $D = [0, 2\pi] \times [0, \pi]$, a rectangle in $\mathbb{R}^2$. Define a mapping $G$ on $D$ by

$$G(\theta, \phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi).$$

Then $G$ maps $D$ onto the sphere of radius $R$ in $\mathbb{R}^3$. 
Example (Parametrization of the Graph of a Real-Valued Function of Two Variables)

Let \( f(x, y) \) be any real-valued function of two variables with domain \( D \). Define mapping \( G \) on \( D \) by

\[
G(x, y) = (x, y, f(x, y)).
\]

Then \( G \) maps \( D \) onto the graph of \( f \). (In particular the graph of \( G \) equals the graph of \( f \).)
Definition
Let $G(u, v) = (x, y, z)$ be a continuously differentiable $\mathbb{R}^3$-valued function of two variables. Define the $u$-tangent vector and the $v$-tangent vector functions of $G$, respectively, to be

$$\vec{T}_u = (x_u, y_u, z_u) \text{ and } \vec{T}_v = (x_v, y_v, z_v).$$

$\vec{T}_u$ and $\vec{T}_v$ are both functions from $\mathbb{R}^2$ into $\mathbb{R}^3$, and they output non-parallel tangent vectors to the graph of $G$ at input $(u, v)$. 
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Assume both $\vec{T}_u$ and $\vec{T}_v$ are non-zero. Define the \textbf{normal vector} associated to $G$ to be the function

$$\vec{n} = \vec{T}_u \times \vec{T}_v.$$

Then $\vec{n}$ is always orthogonal to the tangent plane of the graph of $G$ at a point $G(u, v)$. 
Example

Let $G(\theta, z) = (2 \cos \theta, 2 \sin \theta, z)$ be a parametrization of the cylinder described by $x^2 + y^2 = 4$.

1. Compute $\vec{T}_\theta$, $\vec{T}_z$, and $\vec{n}$.

2. Find an equation for the tangent plane to the graph of $G$ at $G(\frac{\pi}{4}, 5)$. 
Solution to (1)

Known: \( G(\theta, z) = (2 \cos \theta, 2 \sin \theta, z) \)
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Taking derivatives, we have

\[
\vec{T}_\theta(\theta, z) = (-2 \sin \theta, 2 \cos \theta, 0) \text{ and } \vec{T}_z = (0, 0, 1),
\]

and hence

\[
\vec{n}(\theta, z) = \vec{T}_\theta \times \vec{T}_z = \det \begin{bmatrix}
\vec{i} & \vec{j} & \vec{k} \\
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To find the tangent plane, compute the normal at input \((\pi/4, 5)\):

\[
\vec{n}(\pi/4, 5) = (2 \cos \pi/4, 2 \sin \pi/4, 0) = (\sqrt{2}, \sqrt{2}, 0).
\]

The tangent plane passes through the point \( G(\pi/4, 5) = (\sqrt{2}, \sqrt{2}, 5) \), and so its equation [using \( 0 = \vec{n} \cdot ((x, y, z) - (x_0, y_0, z_0)) \)] is given by

\[
0 = \sqrt{2}(x - \sqrt{2}) + \sqrt{2}(y - \sqrt{2}).
\]
Surface Integrals

Definition
Let $G(u, v) = (x, y, z)$ be a one-to-one continuously differentiable parametrization of a surface $S$ in $\mathbb{R}^3$ with domain $D$, with non-zero tangent vectors $\vec{T}_u$ and $\vec{T}_v$. Let $f(x, y, z)$ be a real-valued function of three variables. Define the surface integral of $f$ over $S$ to be

$$\int \int_S f(x, y, z) dS = \int \int_D f(G(u, v)) \|\vec{n}(u, v)\| d(u, v).$$
Surface Integrals

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\[
\int \int_S f(x, y, z) \, dS = \int \int_D f(G(u, v)) ||\vec{n}(u, v)|| \, d(u, v).
\]

Fact
If \( G \) is as above, then the surface integral \( \int \int_S 1 \, dS \) is exactly the surface area of the graph of \( G \).
Example

1. Calculate the surface area of the portion $S$ of the cone $x^2 + y^2 = z^2$ within the cylinder $x^2 + y^2 = 4$.

2. For the same $S$, calculate $\int \int_S x^2 zdS$. 
Surface Area of $S$

First we parametrize the cone by $G$ using a variant of spherical coordinates:

$$G(\rho, \theta) = (x, y, z)$$

$$= \left( \rho \cos \theta \sin \frac{\pi}{4}, \rho \sin \theta \sin \frac{\pi}{4}, \rho \cos \frac{\pi}{4} \right)$$

$$= \left( \frac{\sqrt{2}}{2} \rho \cos \theta, \frac{\sqrt{2}}{2} \sin \theta, \frac{\sqrt{2}}{2} \rho \right)$$

on the domain $D = [0, 2\sqrt{2}] \times [0, 2\pi]$ in the $(\rho, \theta)$-plane.
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Now compute tangent and normal vectors:

\[ \vec{T}_\rho = \left( \frac{\sqrt{2}}{2} \cos \theta, \frac{\sqrt{2}}{2} \sin \theta, \frac{\sqrt{2}}{2} \right) \]
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$$\vec{n} = \vec{T}_\rho \times \vec{T}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\sqrt{2}}{2} \cos \theta & \frac{\sqrt{2}}{2} \sin \theta & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \rho \sin \theta & \frac{\sqrt{2}}{2} \rho \cos \theta & 0 \end{vmatrix}$$

$$= \left( -\frac{1}{2} \rho \cos \theta, -\frac{1}{2} \rho \sin \theta, \frac{1}{2} \rho \right).$$
Surface Area of $S$, contd.

Known: $\vec{n} = (-\frac{1}{2}\rho \cos \theta, -\frac{1}{2}\rho \sin \theta, \frac{1}{2}\rho)$

Now the length of $\vec{n}$ is

$$||\vec{n}|| = \sqrt{\frac{1}{4} \rho^2 \cos^2 \theta + \frac{1}{4} \rho^2 \sin^2 \theta + \frac{1}{4} \rho^2} = \frac{\sqrt{2}}{2} \rho,$$
Surface Area of $S$, contd.

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and so we are able to compute the area surface integral:

$$\int \int_S 1dS = \int_D \int ||\vec{n}(u, v)|| d(u, v)$$

$$= \int_0^{2\pi} \int_0^{2\sqrt{2}} \frac{\sqrt{2}}{2} \rho d\rho d\theta$$

$$= \frac{\sqrt{2}}{2} \left[\theta\right]_0^{2\pi} \cdot \left[\frac{1}{2} \rho^2\right]_0^{2\sqrt{2}}$$

$$= 4\sqrt{2}\pi.$$
Surface Integral of $x^2z$

Known: $\vec{n} = \frac{\sqrt{2}}{2} \rho$

$f(x, y, z) = x^2z$

$G(\rho, \theta) = \left( \frac{\sqrt{2}}{2} \rho \cos \theta, \frac{\sqrt{2}}{2} \sin \theta, \frac{\sqrt{2}}{2} \rho \right)$
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Compute the composition:

$$f(G(\rho, \theta)) = \frac{\sqrt{2}}{4} \rho^3 \cos^2 \theta,$$
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$f(G(\rho, \theta)) = \frac{\sqrt{2}}{4} \rho^3 \cos^2 \theta,$

and hence

$$\int \int_S f dS = \int \int_D f(G(u, v)) ||\vec{n}(u, v)|| d(u, v)$$

$$= \int_0^{2\pi} \int_0^{2\sqrt{2}} \frac{\sqrt{2}}{4} \rho^3 \cos^2 \theta \cdot \frac{\sqrt{2}}{2} \rho d\rho d\theta$$

$$= \frac{1}{4} \int_0^{2\pi} \int_0^{2\sqrt{2}} \rho^4 \cos^2 \theta d\rho d\theta$$

$$= \frac{1}{4} \left[ \frac{1}{5} \rho^5 \right]_0^{2\sqrt{2}} \left[ \frac{1}{2} \theta + \frac{1}{4} \sin 2\theta \right]_0^{2\pi}$$

$$= \frac{32\sqrt{2}\pi}{5}.$$
Fact

Suppose $g(x, y)$ is a function of two variables let $S$ be the graph of $g$. Suppose $G(x, y)$ is a parametrization of $S$. Then

$$\vec{T}_x = (1, 0, g_x) \text{ and } \vec{T}_y = (0, 1, g_y);$$

$$\vec{n} = \sqrt{1 + g_x^2 + g_y^2};$$
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and therefore for any function $f$ with domain $S$ we have

\[ \int \int_S f(x, y, z) dS = \int \int_D f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} d(x, y). \]
Proof.
Let $g(x, y)$ be a function. Then the graph of $g$ is parametrized by

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Let $g(x, y)$ be a function. Then the graph of $g$ is parametrized by

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Computing tangent vectors, we have

$$\vec{T}_u = (1, 0, g_x(u, v)) \text{ and } \vec{T}_v = (0, 1, g_y(u, v))$$
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$$||\vec{n}|| = (\sqrt{1 + g_x^2 + g_y^2}).$$

Therefore if we want to compute the surface integral of $f$ over the graph of $g$, we get

$$\int \int_S f(x, y, z) dS = \int \int_D f(x, y, g(x, y)) \sqrt{1 + g_x^2 + g_y^2} d(x, y).$$
Example
Compute \( \int \int_S (z - x) dS \), where \( S \) is the graph of \( z = x + y^2 \), \( 0 \leq x \leq y \), \( 0 \leq y \leq 1 \).
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Compute \( \int \int_S (z - x) \, dS \), where \( S \) is the graph of \( z = x + y^2 \), \( 0 \leq x \leq y \), \( 0 \leq y \leq 1 \).

Solution.
Take \( g(x, y) = x + y^2 \), so \( ||\vec{n}|| = \sqrt{1 + 1 + (2y)^2} = \sqrt{2 + 4y^2} \).
Example

Compute \( \int \int_{S} (z - x) \, dS \), where \( S \) is the graph of \( z = x + y^2 \), \( 0 \leq x \leq y \), \( 0 \leq y \leq 1 \).

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and \( f(x, y, g(x, y)) = g(x, y) - x = x + y^2 - x = y^2 \).
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Now compute (using substitution rule with $u = 2 + 4y^2$):

$$\int \int_S (z - x) dS = \int_0^1 \int_0^y y^2 \sqrt{2 + 4y^2} dx \, dy$$

$$= \int_0^1 y^3 \sqrt{2 + 4y^2} \, dy$$

$$= \frac{1}{30} (6\sqrt{6} + \sqrt{2}).$$
Our goal now is to develop an integral which computes the amount of flow of a given vector field in $\mathbb{R}^3$ through a given oriented surface $S$, which we call a \textbf{flux integral}.

**Definition**

Let $G(u, v)$ be a parametrization of the surface $S$ in $\mathbb{R}^3$, and let $\vec{n}$ be its associated normal vector. Let $\vec{F}$ be a vector field over $\mathbb{R}^3$. The normal component of $\vec{F}$ with respect to $G$ is $\vec{F} \cdot \vec{n}$. Informally speaking, the normal component of $\vec{F}$ is "how much" of the vector field $\vec{F}$ is orthogonal to the surface $S$.

The flux integral or vector surface integral $\int\int_S \vec{F} \cdot dS$ of $\vec{F}$ across $S$ is the surface integral of the normal component of $\vec{F}$, i.e.

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Flux Integrals

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Fact
Let $G$, $S$, $\vec{n}$, $\vec{F}$ be as in the previous definition, and let $D$ be the domain of $G$. Then

$$\int \int_S \vec{F} \cdot dS = \int \int_D \vec{F}(G(u, v)) \cdot \vec{n}(u, v) d(u, v).$$

To help remember the formula, note the analogy here: computationally, flux integrals are to surface integrals as vector field line integrals are to scalar line integrals.
Example

Compute $\int \int_S \vec{F} \cdot dS$, where $\vec{F} = (0, 0, x)$ and $S$ is the graph of $G(u, v) = (u^2, v, u^3 - v^2)$ with domain $D = [0, 1] \times [0, 1]$. 
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Solution.

Compute the tangents and normal of $G$:

$$\vec{T}_u = (2u, 0, 3u^2) \text{ and } \vec{T}_v = (0, 1, -2v),$$

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\end{bmatrix} = (-3u^2, 4uv, 2u).
\]

Now compute the normal component of \( \vec{F} \):

\[
\vec{F}(G(u, v)) \cdot \vec{n} = (0, 0, u^2) \cdot (-3u^2, 4uv, 2u) = 2u^3.
\]
Example

Compute \( \int \int_S \vec{F} \cdot dS \), where \( \vec{F} = (0, 0, x) \) and \( S \) is the graph of \( G(u, v) = (u^2, v, u^3 - v^2) \) with domain \( D = [0, 1] \times [0, 1] \).

Solution.

Compute the tangents and normal of \( G \):

\[
\vec{T}_u = (2u, 0, 3u^2) \quad \text{and} \quad \vec{T}_v = (0, 1, -2v), \quad \text{and} \quad \vec{n} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2u & 0 & 3u^2 \\ 0 & 1 & -2v \end{bmatrix} = (-3u^2, 4uv, 2u).
\]

Now compute the normal component of \( \vec{F} \):

\[
\vec{F}(G(u, v)) \cdot \vec{n} = (0, 0, u^2) \cdot (-3u^2, 4uv, 2u) = 2u^3.
\]

Finally compute the integral:

\[
\int \int_S \vec{F} \cdot dS = \int_0^1 \int_0^1 2u^3 \, du \, dv = \frac{1}{2} u^4 \Big|_0^1 \cdot [v]_0^1 = \frac{1}{2}.
\]
Example

Let \( \vec{v} = (x^2 + y^2, 0, z^2) \) model the velocity (in cm/s) of a three-dimensional body of water. Compute the volume of water passing each second through the upper hemisphere \( S \) of the unit sphere centered at the origin. (Hint: Compute the corresponding flux integral.)

Solution.

First parametrize the upper hemisphere using spherical coordinates:

\[
G(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi)
\]

on \([0, 2\pi] \times [0, \pi/2]\) in the \((\theta, \phi)\)-plane.

Now compute tangents and normals.

\[
\vec{T}_\theta = (-\sin \theta \sin \phi, \cos \theta \sin \phi, 0)
\] and

\[
\vec{T}_\phi = (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi);
\]

\[
\vec{n} = \vec{T}_\theta \times \vec{T}_\phi = (\cos \theta \sin^2 \phi, \sin \theta \sin^2 \phi, \cos \phi).
\]

Now \( \vec{v}(G(\theta, \phi)) = (\sin^2 \phi, 0, \cos^2 \phi) \), and so

\[
\vec{v}(G(\theta, \phi)) \cdot \vec{n}(\theta, \phi) = \sin^4 \phi \cos \theta + \sin \phi \cos^3 \phi.
\]

Then we compute the surface integral (using symmetry on the first half):

\[
\int \int_S \vec{v} \cdot dS = \int_{\pi/2}^0 \int_{2\pi}^0 (\sin^4 \phi \cos \theta + \sin \phi \cos^3 \phi) \, d\theta \, d\phi = 0 + \frac{\pi}{2}.
\]

So the volume of water flow is \( \frac{\pi}{2} \) cubic centimeters per second.
Example
Let \( \vec{v} = (x^2 + y^2, 0, z^2) \) model the velocity (in cm/s) of a three-dimensional body of water. Compute the volume of water passing each second through the upper hemisphere \( S \) of the unit sphere centered at the origin. (Hint: Compute the corresponding flux integral.)

Solution.
First parametrize the upper hemisphere using spherical coordinates:

\[
G(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi) \text{ on } [0, 2\pi] \times [0, \frac{\pi}{2}] \text{ in the } (\theta, \phi)-\text{plane.}
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Example

Let $\vec{v} = (x^2 + y^2, 0, z^2)$ model the velocity (in cm/s) of a three-dimensional body of water. Compute the volume of water passing each second through the upper hemisphere $S$ of the unit sphere centered at the origin. (Hint: Compute the corresponding flux integral.)

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$$\vec{T}_\theta = (-\sin \theta \sin \phi, \cos \theta \sin \phi, 0) \text{ and } \vec{T}_\phi = (\cos \theta \cos \phi, \sin \theta \cos \phi, -\sin \phi);$$

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Now $\vec{v}(G(\theta, \phi)) = (\sin^2 \phi, 0, \cos^2 \phi)$, and so

$$\int \int_S \vec{v} \cdot dS = \int_{\pi/2}^0 \int_0^{2\pi} (\sin^2 \phi \cos \theta + \sin \phi \cos^3 \phi) \, d\theta \, d\phi = \frac{\pi}{2}.$$
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$$\int \int_S \vec{v} \cdot dS = \int_0^{\pi/2} \int_0^{2\pi} (\sin^4 \phi \cos \theta + \sin \phi \cos^3 \phi) d\theta d\phi$$
$$= 0 + \frac{\pi}{2}.$$