CALCULUS MATH 165 FALL 2016 (COHEN) LECTURE NOTES

Remark 0.1. Much of this set of lecture notes is adapted from a combination of Rogawski's *Calculus Early Transcendentals* and Briggs' and Cochran's *Calculus*, and many of the examples used appear previously in these texts.

1 Vague History, and Motivation From Physics

Remark 1.1 (History of the Calculus)**.** Most students tend to define calculus as "derivatives and integrals," and credit Isaac Newton and Gottfriend Wilhelm Leibniz with its "discovery" in the late 17th century. This viewpoint is not wholly incorrect, but reductive—the calculus actually has a long and complicated history which in many ways predates Newton and Leibniz, even as far back as the philosophical questions about "infinitely small distances" posed by ancient Greeks, and which developed far beyond the 17th century techniques. Broadly, calculus can be regarded as a collection of techniques for solving a variety of very different but related types of problems, such as: What is the slope of a tangent line to a curve? (differential calculus) What is the area under a curve? (integral calculus) How long is a curve? What is the volume of a body in space? What is the sum of infinitely many real numbers? and many others.

The standard solutions to these diverse types of problems are tied together by the notion of a *limit*, which we will spend the first few weeks of the course defining and studying. A limit in calculus is a fairly modern notion, which replaced previous concepts like *infinitesimals*, Newton's *fluxions* and Leibniz's *differentials*. These older notions proved problematic in establishing a rigorous theory for the calculus, although they still rear their head sometimes in the notation. (More on this later—see Remark 8.9 for instance.)

Newton and Leibniz deserve credit for giving a systematic treatment to the class of problems listed above, thus "inventing" the calculus. But our modern treatment owes a huge amount to Fermat, Euler, Lagrange, Gauss, Cauchy, Weierstrass, and too many others to list here.

To see how the fundamental notion of a limit might arise naturally, we consider the simple physics problem posed below.

Example 1.2. A ball is launched into the air at 96 ft/s. Gravity accelerates the ball downward at a rate of 32 ft/s. The height of the ball in feet after t seconds is given by the function:

$$h(t) = -16t^2 + 96t$$

Question 1: How long until the ball lands?

By setting $h(t) = 0$ and solving for t, we see that the height of the ball is 0 feet after exactly 6 seconds.

Question 2: How high does the ball go?

The graph of $h(t)$ is a parabola and hence symmetrical about some vertex. Since the ball has height 0 at $t = 0$ and $t = 6$, its vertex must lie directly between, at $t = 3$. Then its maximum height is given by $h(3) = 144$ feet.

Question 3: What is the average velocity of the ball in the first 3 seconds?

The ball goes up 144 feet in 3 seconds, so its average velocity is $\frac{144}{3} = 48$ ft/s.

Question 4: What is the average velocity of the ball from $t = 1$ to $t = 3$ seconds?

Velocity should be given by distance/time. The ball's height changes from $h(1)$ feet to $h(3)$ feet between $t = 1$ and $t = 3$. So its average velocity over this interval is given by:

$$\frac{h(3) - h(1)}{3 - 1} = \frac{144 - 80}{3 - 1} = 32 \text{ ft/s}.$$

The formula above should recall the *slope formula* $m = \frac{y_2 - y_1}{x_2 - x_1}$ learned in College Algebra. Indeed, the average velocity from any $t = 1$ to $t = 3$ is exactly the slope of the line passing through the points $(1, h(1))$ and $(3, h(3))$.

Question 5: Exactly how fast is the ball going at $t = 1$ second? (Our first calculus question!)

You can get good estimates of this *instantaneous* velocity by computing the *average* velocity over very small intervals around $t = 1$. For example, we can compute the average velocities for .1, then .01, then .00001 seconds after $t = 1$, as below:

$$\frac{h(1.1) - h(1)}{1.1 - 1} \approx 62.4 \text{ ft/s}$$
$$\frac{h(1.01) - h(1)}{1.01 - 1} \approx 63.8 \text{ ft/s}$$
$$\frac{h(1.00001) - h(1)}{1.00001 - 1} \approx 63.9998 \text{ ft/s}$$

The problem is that each of the above is just an *approximation* of the actual velocity at $t = 1$... but as mathematicians we are truly interested in the exact value! We can get better and better approximations by examining smaller and smaller intervals, but for our purposes, any positive interval at all will be "too large." To solve this problem, we wish to look at "arbitrarily small distances" on the real number line.

Newton/Leibniz approached this by relying on a notion which we now regard as non-rigorous: an *infinitesimal*, that is, a distance which is non-zero, yet "infinitely small," i.e. smaller than any positive real number. Many mathematicians in the 18th century and earlier freely used the idea of an infinitesimal in their work. Modern mathematicians, however, regard the notion of an "infinitely small" value as nebulous in meaning, and we abhor the violation of the Archimedean principle! So, in order to solve calculus problems rigorously, we need to develop some new technology, in particular the notion of *limits*, whose formal definition is due in an early form to Bernard Bolzano in 1817, and in its present form to Karl Weierstrass in the second half of the 19th century.

2 Limits

Definition 2.1 (Informal Definition)**.** Let $f(x)$ be a function and a be a real number. If there exists some number L such that $f(x)$ is arbitrarily close to L whenever x is sufficiently close (but not equal) to a, then we write:

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$ as x approaches a is L."

If no such number L exists, then we say the limit does not exist.

Remark 2.2. The student may note that the terms "arbitrarily close" and "sufficiently close" are somewhat vague. Rest assured, the notion of a limit has a precise and clearly stated mathematical definition! However, this definition is more easily understood after some intuition about limits is already developed, so we will omit this formal definition here and present it later after we have done a few exercises.

Example 2.3. Let $f(x) = x + 2$. Find $\lim_{x\to 1} f(x)$, if the limit exists.

Example 2.4. Let $g(x) = x^2$. Find $\lim_{x\to 4} g(x)$, if the limit exists.

Example 2.5. Graphical example (possibly 2.2 #12).

Example 2.6. Let $f(x) = \frac{\sqrt{x}-1}{x-1}$. Find $\lim_{x\to 1} f(x)$, if the limit exists.

Solution. We can try to solve the above problem by making a table of function values where x is very close to 1.

$x =$	.9	.09	.009	1.001	1.01	1.1
$f(x) =$	.5131670	.5012563	.5001251	.4998750	.4987562	.4880885

It appears that as x gets very close to 1, $f(x)$ gets close to .5, and we guess that $\lim_{x\to 1} f(x) = .5$. This will turn out to be correct, but note that at this point in our course we are merely making a *conjecture*, or educated guess! (How do you know the limit is .5 and not .500000001 or .499999999?)

□

Example 2.7 (A Limit That Does Not Exist)**.** Find $\lim_{x\to 0} \sin \frac{\pi}{x}$, if the limit exists.

Example 2.8. Let $f(x) = \frac{x^2-5x+6}{x-2}$. Find $\lim_{x\to 2} f(x)$, if the limit exists.

Solution. To solve this, we first note that $x^2 - 5x + 6 = (x - 2)(x - 3)$ in the numerator of $f(x)$. Our gut says to divide out the $(x - 2)$ from top and bottom and be done with it! However, it is important to note that the following equation is **NOT** true:

$$\frac{(x-2)(x-3)}{x-2} = x - 3$$

The reason the above equation is false is because x could take on the value of any real number. In particular, if $x = 2$, then the left side of the equation is undefined, while the right side is equal to -1; so we don't have true equality here.

However, when we are computing the limit of $f(x)$ as x approaches 2, we restrict our attention to all possible values of x which are near, but *not equal to* the value 2. This means that, since the above equation holds for all $x \neq 2$, the following equation makes perfect sense:

$$\lim_{x\to 2} \frac{(x-2)(x-3)}{x-2} = \lim_{x\to 2} x - 3$$

Since the latter limit is -1, we have $\lim_{x\to 2} f(x) = -1$. □

Now we will go back and cover our tracks by including the formal definition of a limit.

Definition 2.9 (Formal Definition)**.** Let $f(x)$ be a function and a be a real number. Suppose that $f(x)$ exists for all x in some open interval containing a. We say that the **limit of** $f(x)$ **as** x **approaches** a **is** L, written $\lim_{x\to a} f(x) = L$, if the following statement holds: For **any** positive number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ (depending on ϵ) such that

$$|f(x) - L| < \epsilon \text{ whenever } |x - a| < \delta.$$

Why does the above definition make sense? The definition above should give the student confidence that the intuitive notion of a limit may be made mathematically precise. Proceeding with such assurances, all future limit definitions will be presented in the informal style.

Definition 2.10. The **left-hand limit**: If $f(x)$ is arbitrarily close to L for all x sufficiently close to a, **with** $x < a$, then we say "the limit of $f(x)$ as x approaches a from the left is L," and write: $\lim_{x\to a^-} f(x) = L$.

Definition 2.11. The **right-hand limit**: If $f(x)$ is arbitrarily close to L for all x sufficiently close to a, **with** $x > a$, then we say "the limit of $f(x)$ as x approaches a from the right is L," and write: $\lim_{x\to a^+} f(x) = L$.

Example 2.12. Graphical example where left- and right-hand limits are not equal.

Fact 2.13. $\lim_{x\to} f(x) = L$ *if and only if* $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} = L$.

3 Continuity

Definition 3.1. We say that a function f is **continuous** at a point a if the following all hold:

(1) $f(a)$ is defined.

(2) $\lim_{x\to a} f(x)$ exists.

(3) $\lim_{x\to a} f(x) = f(a)$.

If f is continuous at every real number, then we say that f is continuous.

Example 3.2. (1) *Linear functions* $(y = mx + b)$ are continuous.

(2) *Polynomials* are continuous.

(3) *Rational functions*, or functions of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, are continuous at every point except where $q(x) = 0$.

(4) *Sine* and *cosine* are continuous everywhere. The *tangent* function is continuous everywhere except points of the form $\frac{\pi}{2} + k\pi$, where k is any integer.

(5) *Exponential* functions $(y = b^x$ for some positive real number $b)$ are continuous everywhere. *Logarithms* are continuous everywhere they are defined.

Remark 3.3 (The Significance of Continuity For Evaluating Limits)**.** If a function is continuous at a point a, then by definition $\lim_{x\to a} f(x) = f(a)$. This means to evaluate a continuous function's limit at a, you can just plug a into the function!

Example 3.4. Find the limits, if they exist:

(1) $\lim_{x\to 3} f(x)$, where $f(x) = \frac{1}{2}x - 7$

(2) $\lim_{x\to -1} (5x^4 - \pi x^3 - 1)$

(3) $\lim_{x\to 5} g(x)$, where $g(x) = 6$

(4) $\lim_{x\to 1} \dfrac{x^2 - 6x + 8}{x^2 - 9}$

Solution. For part (1), simply observe that $f(x)$ is a linear function and hence continuous. In that case, $\lim_{x\to 3} f(x) = f(3) = \dfrac{1}{2} \cdot 3 - 7 = -\dfrac{11}{2}$. The remaining problems are solved similarly. □

Theorem 3.5 (Intermediate Value Theorem)**.** *If f is a continuous function on a closed interval* $[a, b]$, *then for every value M between $f(a)$ and $f(b)$, there exists at least one $c \in (a, b)$ such that $f(c) = M$.*

Example 3.6. Prove that the equation $\sin x = 0.3$ has at least one solution.

Fact 3.7. *Suppose* $\lim_{x\to a} f(x)$ *and* $\lim_{x\to a} g(x)$ *exist. Suppose* c *is a real number, and* $m, n > 0$ *are integers. The following all hold:*

(1) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(2) $\lim_{x\to a}[cf(x)] = c[\lim_{x\to a} f(x)]$

(3) $\lim_{x\to a}[f(x)g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$

(4) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, *provided* $\lim_{x\to a} g(x) \neq 0$.

(5) $\lim_{x\to a}[f(x)]^{\frac{n}{m}} = [\lim_{x\to a} f(x)]^{\frac{n}{m}}$, *provided* $f(x) \geq 0$ *for* x *near* a *if* m *is even and* $\frac{n}{m}$ *is in reduced form.*

Example 3.8. Compute $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$, if it exists.

Solution. Recall: when we tackled this problem earlier using a table, we conjectured that the limit is $\frac{1}{2} = .5$. We will now show that our conjecture was correct by observing the following:

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1}$$

(Note that we may cancel to obtain the second equality, since we are taking a limit.) Now since $\dfrac{1}{\sqrt{x}+1}$ is continuous at $x = 1$, we may substitute to finish the problem.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

□

4 Infinite Limits

Example 4.1. Find $\lim_{x\to 0} \dfrac{1}{x^2}$, if it exists.

Solution. Note that if we take x to be a very small number (i.e. close to 0), then x^2 will be an even smaller POSITIVE number. Then its reciprocal $\frac{1}{x^2}$ will be a very large positive number. In fact, the smaller we take x to be, the larger the value of $\frac{1}{x^2}$ will be, i.e. the function $\frac{1}{x^2}$ becomes *arbitrarily large* as x approaches 0.

By our previous definition of limits, there is no number L which $\frac{1}{x^2}$ approaches when $x \to 0$, so we say that the limit does not exist. However, we wish to describe this kind of phenomenon in functions, so we will now expand our definition suitably. □

Definition 4.2. Let $f(x)$ be a function and a a real number. If $f(x)$ grows arbitrarily large for x sufficiently close to a, then we write $\lim_{x\to a} f(x) = \infty$.

Similarly, if $f(x)$ grows arbitrarily large in magnitude but in the *negative* direction, then we write $\lim_{x\to a} f(x) = -\infty$.

We can also define the one-sided limits $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ to be ∞ or $-\infty$, in the analogous way.

Example 4.3. Graphical example.

Example 4.4. Find (a) $\lim_{x \to \frac{\pi}{2}^-} \tan x$ and (b) $\lim_{x \to \frac{\pi}{2}^+} \tan x$, if they exist.

Solution. A quick sketch of the graph of the tangent function shows that $\lim_{x \to \frac{\pi}{2}^-} \tan x = \infty$ and $\lim_{x \to \frac{\pi}{2}^+} \tan x = -\infty$. □

Example 4.5. Find $\lim_{x \to 0} \frac{5+x}{x}$, if it exists.

Solution. Taking x to be a very small *positive* number, we observe that $5+x$ will be very close to 5, and hence we can approximate $\frac{5+x}{x} \approx \frac{5}{x}$. The latter fraction will grow arbitrarily large as x approaches 0 from the right, so we have $\lim_{x \to 0^+} \frac{5+x}{x} = \infty$.

On the other hand, if we take x to be a very small *negative* number, then we still have $5 + x \approx 5$ and hence $\frac{5+x}{x} \approx \frac{5}{x}$. This time, however, the latter fraction will grow arbitrarily large in magnitude in the *negative* direction, since a positive number divided by a negative number is negative. This implies $\lim_{x \to 0^-} \frac{5+x}{x} = -\infty$.

Since this function has two *different* one-sided limits, we say that the limit *does not exist.* □

Remark 4.6. The lesson we take from the above example is that for functions of the form $\frac{p(x)}{q(x)}$, if $p(x)$ stays relatively constant but $q(x)$ goes to 0 as $x \to a$, then $\frac{p(x)}{q(x)}$ will blow up in either the positive or negative direction. Whether the limit is ∞, $-\infty$, or does not exist will depend on the signs of $p(x)$ and $q(x)$ when x is near a.

Example 4.7. Find (a) $\lim_{x \to 3^+} \frac{2-5x}{x-3}$ and (b) $\lim_{x \to 3^-} \frac{2-5x}{x-3}$, if they exist.

5 Limits at Infinity

Definition 5.1. If $f(x)$ becomes arbitrarily close to L for all sufficiently large and *positive* x, we write $\lim_{x \to \infty} f(x) = L$.

If $f(x)$ becomes arbitrarily close to L for all sufficiently large and *negative* X, then we write $\lim_{x \to -\infty} f(x) = L$.

Example 5.2. Let $f(x) = \frac{x}{\sqrt{x^2+1}}$. Find $\lim_{x \to \infty} f(x)$, if it exists.

Example 5.3. Find $\lim_{x \to -\infty} (2 + \frac{10}{x^2})$, if it exists.

Example 5.4. Find $\lim_{x \to \infty} (2x + 8)$, if it exists.

Remark 5.5 (End Behavior of Polynomials)**.** The previous example shows that it makes sense to combine *infinite limits* and *limits at infinity*, when appropriate.

We can also use limits at infinity to characterize the *end behavior* of functions. The student may recall from a previous algebra course that for polynomial functions $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$, the end behavior of $f(x)$ is determined entirely by the first term $a_n x^n$, according to the following rules:

If n is even and a_n is positive, then $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = \infty$.

If n is even and a_n is negative, then $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = -\infty$.

If n is odd and a_n is positive, then $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$.

If n is odd and a_n is negative, then $\lim_{x\to-\infty} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -\infty$.

Now let us consider the limits at infinity of *rational* functions, i.e. fractions of polynomials.

6 Indeterminate Forms, and End Behavior of Rational Functions

Definition 6.1 (Informal Definition)**.** We say that the limit $\lim_{x\to a} f(x)$ is of **indeterminate form** if the initial attempt to evaluate $f(c)$ yields an undefined expression of the type $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, or $\infty - \infty$.

Example 6.2. Observe that the following limits are of indeterminate form, and try to evaluate them.

(1) $\lim_{x\to 3} \dfrac{x^2-4x+3}{x^2+x-12}$
(2) $\lim_{x\to\pi/2^-} \dfrac{\tan x}{\sec x}$
(3) $\lim_{x\to 1^+} \left(\dfrac{1}{x-1} - \dfrac{2}{x^2-1}\right)$
(4) $\lim_{x\to 0} x \ln x$

Example 6.3. Find the following limits if they exist.

(1) $\lim_{x\to\infty} \dfrac{12x-7}{x^3+1}$
(2) $\lim_{x\to\infty} \dfrac{5x^5-4x^2+2}{2x^4-\pi x^2+6x-1}$
(3) $\lim_{x\to\infty} \dfrac{3x^7-22x^5+19x}{8x^7-100x^6+22}$

Fact 6.4 (Limits at Infinity for Rational Functions)**.** *Suppose $f(x)$ is a rational function, i.e.*

$$f(x) = \frac{ax^n + ...}{bx^m + ...}$$

for some nonnegative integers n and m and some non-zero leading coefficients a and b. (Here the ellipsis on the numerator represents terms of degree less than n, and the ellipsis on the denominator represents terms of degree less than m.) Then the following hold:

(1) If $n < m$, then $\lim_{x\to\infty} f(x) = 0$.

(2) If $n > m$, then $\lim_{x\to\infty} f(x) = \infty$.

(3) If $n = m$, then $\lim_{x\to\infty} f(x) = \dfrac{a}{b}$.

Remark 6.5. The previous rule takes care of limits at *positive* infinity for all rational functions, what about limits at *negative* infinity? A very similar rule should apply for determining $\lim_{x\to-\infty} f(x)$, where f is a rational function, but it needs to depend on whether n and m are even or odd (as an even exponent will flip the sign on any large negative input x). We leave it to the student to develop this analogous rule for limits at minus infinity.

Remark 6.6. Note that the limits at infinity of any rational function are always of indeterminate form, and Fact 6.4 shows that limits of indeterminate form may take *any* possible value.

7 The Squeeze Theorem, and Some Important Trig Limits

Example 7.1. Find $\lim_{x\to\infty} \cos x$, if the limit exists.

Solution. Since $\cos x$ oscillates back and forth between -1 and 1 for arbitrarily large x, this limit does not exist. □

Example 7.2. Find $\lim_{x\to\infty} \dfrac{\cos x}{x}$, if the limit exists.

Solution. First note that $-1 \leq \cos x \leq 1$ for all values of x. It follows that $-\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$ for all values. Thus the values of $\frac{\cos x}{x}$ oscillate up and down, as in the previous example, but they never exceed $\frac{1}{x}$ nor go below $-\frac{1}{x}$. Since $\lim_{x\to\infty} -\dfrac{1}{x} = \lim_{x\to\infty} \dfrac{1}{x} = 0$ and our function simply wiggles in between, we must have $\lim_{x\to\infty} \dfrac{\cos x}{x} = 0$ as well.

□

This example suggests the following theorem, which we give without proof but which should seem intuitively obvious to the reader:

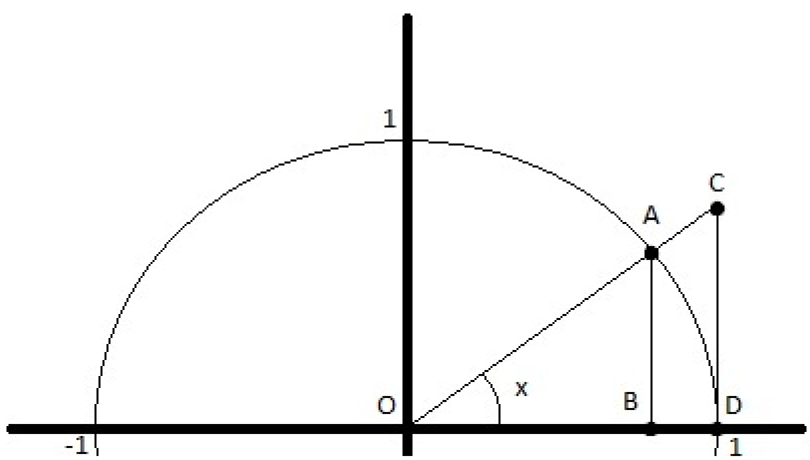
Theorem 7.3 (Squeeze Theorem)**.** *Assume f, g, and h are functions which satisfy $f(x) \leq g(x) \leq h(x)$ for values of x near a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.*

Remark 7.4. The analogous result also holds for limits at infinity.

Example 7.5. Evaluate $\lim_{x\to\infty} (5 + \dfrac{\sin x}{\sqrt{x}})$, if the limit exists.

Theorem 7.6. $\lim_{x\to 0} \dfrac{\sin x}{x} = 1.$

Proof. First let us compute the limit only from the right, i.e. $\lim_{x\to 0^+} \dfrac{\sin x}{x}$. So consider a positive angle x near 0, and label the following points: let A be the point on the unit circle determined by angle x; let B be the foot of the perpendicular dropped from A to the x-axis; let D be the point $(1, 0)$; and let C be the point directly above or below D which lies on the ray $\overrightarrow{OA}$. (See diagram below.)



Let A_1 denote the area of triangle $\triangle OAB$, and let A_2 denote the area of triangle $\triangle OCD$. Note that the lengths of the legs of $\triangle OAB$ are exactly $\cos x$ and $\sin x$, and therefore

$$A_1 = \tfrac{1}{2} \sin x \cos x$$

by the familiar area formula for triangles. On the other hand $\triangle OCD$ has a leg $\overline{OD}$ of length 1, and a leg $\overline{CD}$ of length $\tan x$. Therefore

$$A_2 = \tfrac{1}{2}\tan x = \tfrac{\sin x}{2\cos x}.$$

Lastly, consider the *sector* (i.e. pie slice) of the unit disk determined by the points O, A, and B. Let X denote the area of this sector. Since the area of the unit circle is π, we have

$$X = \tfrac{x}{2\pi}\cdot\pi = \tfrac{x}{2}.$$

Now observe that regardless of the choice of x, the inequality $A_1 \leq X \leq A_2$ holds. Written otherwise,

$$\tfrac{1}{2}\sin x\cos x \leq \tfrac{x}{2} \leq \tfrac{\sin x}{2\cos x}.$$

Dividing by $\sin x$ (for values of x near 0 but not equal to 0) and multiplying by 2, we get

$$\cos x \leq \tfrac{x}{\sin x} \leq \tfrac{1}{\cos x}$$

and therefore

$$\cos x \leq \tfrac{\sin x}{x} \leq \tfrac{1}{\cos x}.$$

Now taking the limit of the three expressions above as $x \to 0$, we see $\displaystyle\lim_{x\to 0^+}\cos x = \lim_{x\to 0^+}\frac{1}{\cos x} = 1$, and therefore $\displaystyle\lim_{x\to 0^+}\frac{\sin x}{x} = 1$ by the Squeeze Theorem.

To see that $\displaystyle\lim_{x\to 0^-}\frac{\sin x}{x} = 1$ as well, simply note that both $\sin x$ and x are *odd* functions of x, i.e. their graphs have a 180° rotation symmetry about the origin in the plane. Written out symbolically, we are simply observing that $\displaystyle\frac{\sin(-x)}{-x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}$, and therefore the limits at 0 from the left and the right must be the same. □

Corollary 7.7. $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x} = 0.$

Proof. For x near 0, observe that

$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\
&= \lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{\sin x}{1+\cos x}\right)\\
&= 1\cdot\frac{0}{2}\\
&= 0.
\end{aligned}$$

□

Example 7.8. Find $\lim_{x \to 0} \frac{\tan x}{x}$, if it exists.

Example 7.9. Find $\lim_{x \to 0} \frac{\sin 6x}{x}$, if it exists.

Example 7.10. Find $\lim_{x \to 0} \frac{\sin 7x}{\sin 4x}$, if it exists.

Example 7.11. Find $\lim_{x \to 0} \frac{\sec x - 1}{x}$, if it exists.

8 Differentiation

Definition 8.1. Let $f(x)$ be any function, and a be any real number. The **average rate of change** of f from $x = a$ to $x = b$ is given by:

$$\frac{f(b) - f(a)}{b - a}.$$

This is also the **slope of the secant line** connecting the points $(a, f(a))$ and $(b, f(b))$.

Example 8.2. Let $f(x) = x^2 + 4x$. Find the average rate of change of f from $x = 1$ to $x = 5$.

Example 8.3. A ball is launched into the air, and its height in feet after t seconds is given by $f(t) = -16t^2 + 96t$.

(1) Find the average velocity of the ball from $t = 1$ second to $t = 4$ seconds.
(2) Find the average velocity of the ball from $t = 1$ second to $t = 1 + h$ seconds, where h is any small positive interval of time.

Definition 8.4. Let f be a function and a a real number. The **difference quotient** of f at a is the expression

$$\frac{f(a + h) - f(a)}{h},$$

which computes the average rate of change of f from $x = a$ to $x = a + h$.

The **instantaneous rate of change** of f at $x = a$, or the **slope of the tangent line** to the graph of f at $x = a$, is given by:

$$\lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Example 8.5. Let $f(x) = x^2 + 4x$.

(1) Find the instantaneous rate of change (IROC) of f at $x = 1$.

(2) Find the equation of the line tangent to the graph of f at $x = 1$.

Solution. We compute the IROC of f at $x = 1$ below:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} &= \lim_{h \to 0} \frac{[(1 + h)^2 + 4(1 + h)] - [1^2 + 4(1)]}{h} \\
&= \lim_{h \to 0} \frac{1 + 2h + h^2 + 4 + 4h - 1 - 4}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 6h}{h} \\
&= \lim_{h \to 0} (h + 6) \\
&= 6.
\end{aligned}$$

So the slope of the tangent line to the graph of f at $x = 1$ is 6. To find the equation of this line, we need only note that the line must meet the graph at the point $(1, f(1)) = (1, -5)$, and apply the point-slope formula:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-5) &= 6(x - 1) \\ y &= 6x - 11 \end{aligned}$$

□

Definition 8.6. The **derivative** of f is the function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

wherever this limit exists. If $f'(x)$ exists, we say that f is **differentiable** at the point x.

Remark 8.7. The main idea of the derivative is this: given a function f, we define a new function f', the derivative of f, which describes the behavior of the function f at each point x. Specifically, f' takes a real number x for input, and for output returns the *slope of the tangent line* to the graph of f at the point x, or equivalently, the *instantaneous rate of change* of f at x.

Example 8.8. Let $f(x) = -16x^2 + 96x$. Find its derivative $f'(x)$. Compute $f'(1)$, $f'(0)$, $f'(3)$, $f'(5)$. Do the values make sense given the context of our physics example?

Solution. We compute the derivative by the definition below:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{[-16(x+h)^2 + 96(x+h)] - [-16x^2 + 96x]}{h} \\ &= \lim_{h\to 0} \frac{-16x^2 - 32xh - 16h^2 + 96x + 96h + 16x^2 - 96x}{h} \\ &= \lim_{h\to 0} \frac{-32xh - 16h^2 + 96h}{h} \\ &= \lim_{h\to 0} -32x - 16h + 96 \\ &= -32x - 16(0) + 96 \\ &= -32x + 96. \end{aligned}$$

So the derivative is $f'(x) = -32x + 96$.

Now we observe that $f'(1) = -32(1) + 96 = 64$, which is the instantaneous velocity at $x = 1$ we computed earlier.

$f'(0) = -32(0) + 96 = 96$, which is the velocity at which the ball is initially launched according to our original problem. (In other words, the instantaneous velocity at $x = 0$.

$f'(3) = -32(3) + 96 = 0$. So the velocity at $t = 3$, the moment the ball tops out its arc and begins to fall back to the earth, is exactly 0, as expected.

$f'(5) = -32(5) + 96 = -64$. So the velocity at $t = 5$, i.e. 2 seconds after the ball tops out its arc, is -64, i.e. the ball is falling toward the earth at 64 ft/s. Considering at $t = 1$ the ball was flying UP at 64 ft/s, this should not be surprising! □

Remark 8.9 (Remark on Old-Fashioned Notations)**.** The Greek letter Δ is often used to represent change. So instead of h one may write Δx, the "change in x." Moreover instead of writing $f(x+h)-f(x)$, one may regard this as the "change in y," and write Δy. So we have

$$\frac{f(x+h)-f(x)}{h} = \frac{\Delta y}{\Delta x}.$$

When computing the derivative, we wish to take take the limit as $\Delta x \to 0$. Now recall our earlier remarks about *infinitesimals* or "infinitely small distances": this notion was used freely before and during the development of the calculus, before it was discarded for perceived lack of rigor, and replaced with the modern notion of a limit. Leibniz's work, however, significantly predates limits, and he thought in terms of infinitesimals. So in his notation, the "infinitely small" version of Δx was denoted dx, an infinitesimal distance, and he used dy to represent the corresponding infinitesimal change in y. Thus the **Leibniz notation** for the derivative is as follows:

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

The Leibniz notation for the derivative has some advantages, but many drawbacks: in particular it confusingly portrays the derivative function as some type of fraction (see Remark 12.5 for more on this). However, the Leibniz notation is extremely common and the student is sure to encounter it in future coursework. So we will use it in this course interchangeably with $f'(x)$.

Some other notations which also stand for the derivative of a function $y = f(x)$ are the following: $\frac{df}{dx}$, $\frac{d}{dx}f(x)$, y', $y'(x)$.

Example 8.10. Let $y = f(x) = \sqrt{x}$.

(1) Find $\frac{dy}{dx}$.
(2) Find the equation of the line tangent to the graph of f at $(4, 2)$.

Example 8.11. Let $g(t) = \frac{1}{t^2}$. Find $g'(t)$.

Example 8.12. Graphical example: Sec 3.1 Example 5.

Example 8.13. Show that $f(x) = |x|$ is not differentiable at $x = 0$.

Example 8.14. Show that $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$.

Fact 8.15. *If a function is differentiable at a point, then it must be continuous at that point. The converse, however, is not true: A function may be continuous at a point but not differentiable there.*

A function is not differentiable at a point a if at least one of the following holds:

(1) f is not continuous at a.
(2) f has a corner at a. (Example: $f(x) = |x|$.)
(3) f has a vertical tangent at a. (Example: $f(x) = x^{\frac{1}{3}}$.)

9 Rules of Differentiation

Remark 9.1. Many of our derivative calculations in the previous section were long, tedious, and/or computationally difficult. We wish to develop shortcuts by which one may rapidly compute the derivatives of familiar functions. We'll begin by computing a few the old-fashioned way.

Example 9.2. Let c be any real number, and let $f(x) = c$. Find $f'(x)$.

Solution. Since the graph of f is a horizontal line, every tangent line to the graph should have slope 0. Thus we conjecture that $f'(x) = 0$. Computing below, we see that our chosen definition of the derivative confirms our intuition:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0.$$ □

Theorem 9.3 (Constant Rule)**.** *If c is a real number, then*

$$\boxed{\tfrac{d}{dx}(c) = 0.}$$

Example 9.4. Find the following:

(1) $\frac{d}{dx}(x)$
(2) $\frac{d}{dx}(x^2)$
(3) $\frac{d}{dx}(x^3)$

Solution. Some basic computations will show that the three derivatives are 1, $2x$, and $3x^2$, respectively. It is also easy to show that $\frac{d}{dx}(x^4) = 4x^3$, $\frac{d}{dx}(x^5) = 5x^4$, etc., which leads us to the following rule. □

Theorem 9.5 (Power Rule)**.** *For any exponent n, we have*

$$\boxed{\tfrac{d}{dx}(x^n) = nx^{n-1}.}$$

Example 9.6. Evaluate the following derivatives:

(1) $\frac{d}{dx}(x^9)$
(2) $\frac{d}{dx}(x^{275})$
(3) $\frac{d}{dx}(2^8)$
(4) $\frac{d}{dx}\sqrt[3]{x}$
(5) $\frac{d}{dx}\frac{1}{x^5}$

Theorem 9.7 (Constant Multiple Rule)**.** *If f is differentiable at x and c is a constant, then*

$$\boxed{\tfrac{d}{dx}[cf(x)] = cf'(x).}$$

Theorem 9.8 (Sum Rule)**.** *If f and g are differentiable at x, then*

$$\boxed{\tfrac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).}$$

Remark 9.9. The two rules above say that taking derivatives "distributes over" sums, differences, and multiples of functions.

Example 9.10. Find $\frac{d}{dx}(2x^3 + 9x^2 - 6x + 4)$.

Example 9.11. Let $f(x) = 2x^3 - 15x^2 + 24x$.

(1) Find an equation of the line tangent to the graph of f at the point $(2, 4)$.
(2) At what points on the graph of f is the tangent line horizontal?
(3) For what values of x does the tangent line have a slope of 6?

10 The Product Rule and Quotient Rule

Remark 10.1 (A Naive Product Rule?)**.** The "sum rule" above tells us that derivatives "distribute over addition", i.e. $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$ for any two functions f and g. We may be tempted to derive a similar conclusion for multiplication, i.e. that the derivative of a product of two functions is the product of the two derivatives. Unfortunately this is not the case, as the following example illustrates.

Example 10.2. Let $f(x) = x^2$ and $g(x) = x^3$.

(1) Find $f'(x)$, $g'(x)$, and $f'(x)\cdot g'(x)$.

(2) Find $\frac{d}{dx}(f(x)g(x))$.
(3) Does $\frac{d}{dx}(f(x)g(x)) = f'(x)g'(x)$?

Solution. By the power rule we have $f'(x) = 2x$ and $g'(x) = 3x^2$, so $f'(x)\cdot g'(x) = 6x^3$. On the other hand, $(fg)(x) = x^2\cdot x^3 = x^5$, and hence $(fg)'(x) = 5x^4$. So clearly we have $\frac{d}{dx}(f(x)g(x)) \neq f'(x)g'(x)$.

This example shows that the very simple "obvious" product rule is false. However, we can obtain a rule which is almost as simple and nice, which is stated in the next theorem. □

Theorem 10.3 (Product Rule for Derivatives)**.** *If f and g are differentiable at x, then*

$$\boxed{\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).}$$

Proof. To show the product rule, we simply compute the derivative $\frac{d}{dx}[f(x)g(x)]$ below. Note in our second line, we use the trick of adding and subtracting $\frac{f(x)g(x+h)}{h}$ to the equation (which is the same as adding 0).

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-f(x)g(x+h)}{h}+\frac{f(x)g(x+h)-f(x)g(x)}{h}\right]\\
&= \lim_{x\to 0}\left[\frac{f(x+h)-f(x)}{h}\cdot g(x+h)+f(x)\cdot\frac{g(x+h)-g(x)}{h}\right]\\
&= \left[\lim_{x\to 0}\frac{f(x+h)-f(x)}{h}\right](\lim_{h\to 0}g(x+h))+(\lim_{h\to 0}f(x))\left[\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\right]\\
&= f'(x)g(x)+f(x)g'(x).
\end{aligned}$$

□

Example 10.4. Use the product rule to find the following derivatives:
(1) $\frac{d}{dx}[x^2\cdot x^3]$

(2) $\frac{d}{dx}[(x^3-8)(x^2+4)]$

(3) $\frac{d}{dx}[(7x^5-4x^2+8x)(14x^3-19)]$

Remark 10.5. Our next objective is to provide a "quotient rule" for derivatives as well. To emphasize the importance of the order of the terms in this rule, we will change our function names from f and g to N and D, for "numerator" and "denominator" respectively.

Theorem 10.6 (Quotient Rule for Derivatives)**.** *If N and D are differentiable at x and $D(x)\neq 0$, then*

$$\boxed{\frac{d}{dx}\left[\frac{N(x)}{D(x)}\right] = \frac{D(x)N'(x) - N(x)D'(x)}{[D(x)]^2}.}$$

Proof. This rule should follow easily from the product rule, if we set things up the right way. Set $q(x) = \frac{N(x)}{D(x)}$. We wish to compute $q'(x)$.

First note that $N(x) = q(x)D(x)$, so the product rule tells us that

$$N'(x) = q'(x)D(x) + q(x)D'(x)$$

In that case, solving for $q'(x)$, we have $q'(x) = \frac{N'(x)-q(x)D'(x)}{D(x)}$. In order to simplify this fraction, we multiply by $\frac{D(x)}{D(x)}$, and observe that $q(x)D(x) = \frac{N(x)}{D(x)}D(x) = N(x)$ below:

$$\begin{aligned} q'(x) &= \frac{N'(x) - q(x)D'(x)}{D(x)} \cdot \frac{D(x)}{D(x)} \\ &= \frac{D(x)N'(x) - q(x)D(x)D'(x)}{[D(x)]^2} \\ &= \frac{D(x)N'(x) - N(x)D'(x)}{[D(x)]^2} \end{aligned}$$

So our quotient rule holds. ☐

Example 10.7. Find and simplify the following derivatives.

(1) $\frac{d}{dx}[\frac{x^2+3x+4}{x^2-1}]$

(2) $\frac{d}{dx}(2x^{-3})$

Example 10.8. Find an equation of the line tangent to the graph of $f(x) = \frac{x^2+1}{x^2-4}$ at the point $(3, 2)$.

11 Derivatives of Trigonometric Functions

Example 11.1. Sketch a rough graph of the derivative of $f(x) = \sin x$.

Solution. The graph of $f(x) = \sin x$ has horizontal tangent lines at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...,$ etc., so we know that $f'(x) = 0$ at each of these points.

At $x = 0$, the graph of $f(x) = \sin x$ has some positive slope m, which will recur every 2π units up and down the real line. At $x = \pi$ the graph should have slope $-m$, which will also recur every 2π units.

In general the graph of $f'(x)$ should look like a continuous periodic curve which oscillates back and forth between some values $-m$ and m. In particular, if $m = 1$, then the graph of $f'(x)$ should call to mind the graph of $\cos x$! ☐

Theorem 11.2 (Derivatives of Sine and Cosine)**.** *The following derivative formulas hold:*

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

and

$$\boxed{\frac{d}{dx}\cos x = -\sin x.}$$

Proof. First compute the derivative of $\sin x$, using the limit formulas we proved in Theorem 7.6 and Corollary 7.7. We use a common trig identity in line 2.

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\left(\frac{\sin h}{h}\right)\right] \\
&= \sin x\cdot 0+\cos x\cdot 1 \\
&= \cos x.
\end{aligned}$$

Next using a similar method, compute the derivative of $\cos x$.

$$\begin{aligned}
\frac{d}{dx}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h} \\
&= \lim_{h\to 0}\left[\cos x\left(\frac{\cos h-1}{h}\right)-\sin x\left(\frac{\sin h}{h}\right)\right] \\
&= \cos x\cdot 0-\sin x\cdot 1 \\
&= -\sin x.
\end{aligned}$$

□

Example 11.3. Calculate $\frac{dy}{dx}$ for the following functions.

(1) $y = x^2\cos x$

(2) $y = \sin x - x\cos x$

(3) $y = \frac{1+\sin x}{1-\sin x}$

Example 11.4. Calculate $\frac{d}{dx}\tan x$ using the quotient rule.

Theorem 11.5 (Derivative of Tangent and Cotangent)**.** *We have*

$$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$

and

$$\boxed{\frac{d}{dx}\cot x = -\csc^2 x.}$$

Theorem 11.6 (Derivatives of Secant and Cosecant)**.** *We have*

$$\boxed{\frac{d}{dx}\sec x = \sec x\tan x}$$

and

$$\boxed{\frac{d}{dx}\csc x = -\csc x\cot x.}$$

Proof. Do this using the quotient rule on $\sec = \frac{1}{\cos x}$ and $\csc = \frac{1}{\sin x}$. □

Example 11.7. Let $y = \sec x\csc x$, and compute y'.

12 The Chain Rule

Remark 12.1. Recall: If $f(x)$ and $g(x)$ are two functions, then the *composition* of f and g, denoted by $(f \circ g)$, is defined by the rule $(f \circ g)(x) = f(g(x))$. For example, if $f(x) = x^{90}$ and $g(x) = 5x + 4$, then $(f \circ g)(x) = (5x + 4)^{90}$ and $(g \circ f)(x) = 5x^{90} + 4$. We wish to be able to take derivatives of such compositions of functions, so we need a new rule, which we give below. This rule is often easier understood through practice than by staring at the formula.

Theorem 12.2 (Chain Rule for Derivatives)**.** *Let f and g be differentiable functions. Then $f \circ g$ is differentiable and*

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x).}$$

Example 12.3. Let $f(x) = x^{90}$ and $g(x) = 5x + 4$. Find the derivative of $(f \circ g)(x) = f(g(x)) = (5x + 4)^{90}$.

Solution. We already know $f'(x) = 90x^{89}$ and $g'(x) = 5$. The chain rule says to multiply the two derivatives, but evaluate $f'(x)$ at the point $g(x)$ as follows:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) = 90(5x + 4)^{89} \cdot 5 = 450(5x + 4)^{89}.$$

□

Example 12.4. Find derivatives of the following functions.

(1) $y = (7x^3 - 5x)^8$

(2) $y = (5x^2 + 15)^{-2}$

(3) $y = \sin^3 x$

(4) $y = \sin x^3$

Remark 12.5. Now we wish to note that there is another way to view the chain rule, as a "substitution" rule. For example, consider the function $y = (5x + 4)^{90}$, and suppose we wish to find the derivative y'. Set $u = 5x+4$, so $y = u^{90}$. Then $\frac{dy}{du} = 90u^{89}$ and $\frac{du}{dx} = 5$. In this case the chain rule says the following:

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

So we have $y' = \frac{dy}{dx} = [90u^{89}] \cdot [5] = 450u^{89} = 450(5x + 4)^{89}$.

We should caution the student. Writing the chain rule in the above manner is simple and makes it very easy to remember, as it jibes with our intuition of "cancelling out" terms in products of fractions. However, this intuition is misleading—derivatives are *not* fractions, and the fact that the functions $\frac{dy}{dx}$, $\frac{dy}{du}$, $\frac{du}{dx}$ are often written in this manner is an artifact of the obsolete notion of infinitesimals. (See Remark 8.9.)

Example 12.6. Find $\frac{d}{dx}[\sqrt{5x^2 + 1}]$.

Example 12.7. Find $\frac{d}{dt}[\left(\frac{5t^2}{3t^2+2}\right)^3]$.

Example 12.8. Let $y = \sin(\cos x^2)$. Find y'.

Proof of the Chain Rule. The naive idea of the proof is as follows: We want to compute the derivative of $f \circ g$ by rewriting the difference quotient as below.

$$
\begin{aligned}
(f\circ g)'(x) &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h} \\
&= \lim_{h\to 0}\left(\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\right).
\end{aligned}
$$

Naively, the expression on the left should go to $f'(g(x))$ and the expression on the right should go to $g'(x)$, as $h\to 0$. However the problem with the above is that $g(x+h)$ could be equal to $g(x)$ for small values of h (for instance this is true if $g(x)$ is a constant function). So the expression above is not necessarily well-defined. We need to introduce a new function to fix this gap.

To repair the argument, let us treat x as a constant, and piecewise define a new function Q in a variable u:

$$
Q(u)=\begin{cases}\dfrac{f(u)-f(g(x))}{u-g(x)} & : u\neq g(x); \\ f'(g(x)) & : u=g(x).\end{cases}
$$

Thus $Q(u)$ makes sense for any value of u. Observe also that $\lim\limits_{u\to g(x)} Q(u)=Q(g(x))$, since f is a differentiable function at $g(x)$. Therefore Q is continuous. Moreover, by our definition of Q, the following equality holds for all $h\neq 0$:

$$
\frac{f(g(x+h))-f(g(x))}{h}=Q(g(x+h))\cdot\frac{g(x+h)-g(x)}{h}.
$$

The above equality is clear when $g(x+h)\neq g(x)$, just by plugging $u=g(x+h)$ into Q. On the other hand if $g(x+h)=g(x)$, then the equality holds because both sides are 0. So we have shown that

$$
(f\circ g)'(x)=\lim_{h\to 0}\left(Q(g(x+h))\cdot\frac{g(x+h)-g(x)}{h}\right).
$$

So computing $(f\circ g)'(x)$ amounts to computing the limit of the product above. Both limits in the product converge separately—we have $\lim\limits_{h\to 0} Q(g(x+h))=Q(g(x))=f'(g(x))$ by the continuity of Q, and $\lim\limits_{h\to 0}\dfrac{g(x+h)-g(x)}{h}=g'(x)$ by the differentiability of g. Therefore $(f\circ g)'(x)=f'(g(x))g'(x)$ as claimed. □

13 Exponentials and Logarithms

Definition 13.1 (Definition of e)**.** Consider the function $f(x)=(1+x)^{1/x}$, defined for $x>-1$, $x\neq 0$. Examination of this function reveals a few properties: we see that $f(0)$ is undefined, $\lim\limits_{x\to -1} f(x)=\infty$, and $\lim\limits_{x\to\infty} f(x)=1$. Via numerical approximation, we can see that $f(x)$ appears to have a limit as $x\to 0$, and this limit is somewhere between 2.5 and 3. For instance:

$$\begin{aligned}
f(1) &= 2^1 && = 2 \\
f\left(\frac{1}{2}\right) &= \left(\frac{3}{2}\right)^2 && = 2.25 \\
f\left(\frac{1}{3}\right) &= \left(\frac{4}{3}\right)^3 && = 2.\overline{370} \\
f\left(\frac{1}{4}\right) &= \left(\frac{5}{4}\right)^4 && = 2.44140625 \\
f\left(\frac{1}{5}\right) &= \left(\frac{6}{5}\right)^5 && = 2.44832 \\
&\ldots \\
f\left(\frac{1}{100}\right) &= \left(\frac{101}{100}\right)^{100} && = 2.7048... \\
&\ldots \\
f\left(\frac{1}{1000}\right) &= \left(\frac{1001}{1000}\right)^{1000} && = 2.7169...
\end{aligned}$$

It turns out this limit really does exist, and we define the number e to be the limit of this function at 0:

$$\boxed{e = \lim_{x \to 0} (1+x)^{1/x}}$$

Recall that given real numbers $a, x > 0$, the logarithm $y = \log_a x$ is the unique number y for which $a^y = x$. We define

$$\ln x = \log_e x$$

for all $x > 0$. We call this function the **natural logarithm**.

Theorem 13.2 (Derivative of the Natural Logarithm)**.** *The following derivative formula holds:*

$$\boxed{\tfrac{d}{dx} \ln x = \tfrac{1}{x}.}$$

Proof. Applying the definition and using the properties of logarithms, we get that

$$\begin{aligned}
\frac{d}{dx} \ln x &= \lim_{h \to 0} \frac{\ln(x+h) - \ln x}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(\frac{x+h}{x}\right) \\
&= \lim_{h \to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right).
\end{aligned}$$

Now we introduce a change of variables: set $u = \frac{h}{x}$. Then as $h \to 0$, we also have $u \to 0$. Therefore, substituting u into the above and using our logarithm properties again, we get

$$\begin{aligned}\frac{d}{dx}\ln x &= \lim_{u\to 0}\frac{1}{xu}\ln(1+u)\\ &= \lim_{u\to 0}\frac{1}{x}\ln(1+u)^{1/u}\\ &= \frac{1}{x}\cdot\ln\left(\lim_{u\to 0}(1+u)^{1/u}\right)\\ &= \frac{1}{x}\cdot\ln e\\ &= \frac{1}{x}.\end{aligned}$$

Note that in line 3 of the equality above, we used the fact that $\ln x$ is a continuous function to pull the limit inside. This completes the proof. □

Example 13.3. Compute $\frac{d}{dx}\ln(x^3+10x)$.

Corollary 13.4 (Derivative of the Exponential Function)**.** *The following derivative formula holds:*

$$\boxed{\frac{d}{dx}e^x = e^x.}$$

Proof. Let $f(x) = e^x$ and let $g(x) = \ln x$. Note that

$$g(f(x)) = x.$$

Therefore, taking the derivative of both sides and using the chain rule, we get

$$g'(f(x))\cdot f'(x) = 1.$$

But $g'(f(x)) = \frac{1}{e^x}$. So the line above says $\frac{1}{e^x}\cdot f'(x) = 1$. Therefore $f'(x) = e^x$. □

Example 13.5. Find $\frac{d}{dx}e^{7x}$.

Example 13.6. Find $\frac{d}{dx}5^x$.

Theorem 13.7 (Derivatives of General Exponential Functions)**.** *Let* $b > 0$*. Then*

$$\boxed{\frac{d}{dx}b^x = (\ln b)b^x.}$$

Proof. Compute using the chain rule that

$$\begin{aligned}\frac{d}{dx}b^x &= \frac{d}{dx}(e^{\ln b})^x\\ &= \frac{d}{dx}e^{(\ln b)x}\\ &= (\ln b)e^{(\ln b)x}\\ &= (\ln b)b^x.\end{aligned}$$

□

Example 13.8. Find $f'(x)$ for the following.

(1) $f(x) = 4^{3x}$
(2) $f(x) = 5^{x^2}$

(3) $f(x) = 10^x \ln x$

Theorem 13.9 (Derivatives of General Logarithms)**.** *Let $b > 0$. Then*

$$\boxed{\frac{d}{dx} \log_b x = \frac{1}{(\ln b)x}.}$$

Proof. This is immediate from the fact that $\log_b x = \frac{\ln x}{\ln b}$. □

Example 13.10. Compute y' for the following.

(1) $y = x^x$
(2) $y = x^{\sin x}$

Solution. There are two good methods for solving these two (similar) derivative problems. We will do one method for each.

For $y = x^x$, simply rewrite as $y = (e^{\ln x})^x = e^{x \ln x}$. Then the derivative using the chain rule and the product rule is

$$y' = (\ln x + 1)e^{x \ln x} = (\ln x + 1)x^x.$$

For the second problem $y = x^{\sin x}$, let us introduce the logarithm on both sides and write

$$\ln y = \ln x^{\sin x} = \sin x \ln x.$$

Next take the derivative of both sides with respect to x. Since y is a function of x, the chain rule implies that $\frac{d}{dx} \ln y = \frac{1}{y} \cdot y'$. Therefore

$$\frac{1}{y} \cdot y' = \cos x \ln x + \sin x \cdot \frac{1}{x}.$$

Multiplying both sides above by $y = x^{\sin x}$ yields

$$y' = \left(\cos x \ln x + \frac{\sin x}{x}\right) x^{\sin x}.$$

□

14 Maxima and Minima

Definition 14.1. Let f be defined on an interval containing c. Then f has an **absolute maximum** value on I at c is $f(c) \geq f(x)$ for every $x \in I$. Similarly, f has an **absolute minimum** value on I at c is $f(c) \leq f(x)$ for every x in I.

Example 14.2. Graphical examples: $f(x) = x^2$ on: $(-\infty, \infty)$, $[0, 2]$, $(0, 2]$, and $(0, 2)$.

Remark 14.3. From the above examples we see that determining absolute maxima and minima depends on not only the function f but also the choice of interval I. However, we can guarantee their existence by requiring two things: first, that f be continuous, and second, that I contain its endpoints.

Theorem 14.4 (Extreme Value Theorem)**.** *A function that is continuous on a closed interval $[a, b]$ has an absolute maximum value and an absolute minimum value on that interval.*

Definition 14.5. Let f be defined at c. If there exists some open interval I containing c, such that f is defined on I, and $f(c) \geq f(x)$ for all x in I, then $f(c)$ is a **local maximum** of f. If $f(c) \leq f(x)$ for all x in I, then $f(c)$ is a **local minimum** of f.

We also refer to local maxima (and minima) as **relative** maxima (and relative minima). If $f(c)$ is a maximum (or minimum) then we say f has a maximum at $x = c$ (or a minimum at $x = c$.

Remark 14.6. Now, suppose that f is some polynomial function. Polynomials are continuous and differentiable at every point, so its graph should look like some smooth curve, perhaps winding up and down, and eventually going to either ∞ or $-\infty$ in either the left or right direction. How can we compute at which points f has local maxima or local minima?

It should be clear by now that for a smooth curve (like that of a polynomial) the only places x where f can "top out" or "bottom out" (i.e. have a local maximum or minimum) are points where the tangent line to the graph is horizontal, i.e. where $f'(x) = 0$. In fact, this applies to any differentiable function: If c is some point where $f'(c)$ exists, and such that f has a local maximum or local minimum at $x = c$, then $f'(c) = 0$.

On the other hand, we know that it is possible to have a local maximum or minimum at points where no derivative exists. For example, if $f(x) = |x|$, then f has a corner point at $x = 0$, so $f'(0)$ does not exist, but f certainly has a minimum at $x = 0$. It is also possible to have local maxima/minima at points of discontinuity, and if f is discontinuous at a point then it has no derivative there. These cases, however, account for all the possibilities: if f has a local maximum or minimum at $x = c$, then either $f'(c) = 0$ or f is not differentiable at c. This leads us to the following definition.

Definition 14.7. Suppose f is defined at the point c. We say c is a **critical point** of f if either $f'(c) = 0$ or if $f'(c)$ does not exist.

Fact 14.8. *If f has a local maximum/minimum at $x = c$, then c must be a critical point of f.*

Remark 14.9. On the other hand, if c is a critical point, then f does not necessarily have a local maximum or minimum at c. For example, consider $f(x) = x^3$ or $f(x) = \sqrt[3]{x}$ at $x = 0$.

Example 14.10. Find the critical points of $f(x) = \frac{x}{x^2+1}$.

Example 14.11. Find the absolute maximum and minimum values of the following:

(1) $f(x) = x^4 - 2x^3$ on the interval $[-2, 2]$

(2) $g(x) = x^{\frac{2}{3}}(2 - x)$ on the interval $[-1, 2]$

Solution. We will solve part (a) and leave part (b) to the student. By the Extreme Value Theorem we know that an absolute maximum and minimum value for f on $[-2, 2]$ are guaranteed to exist. Such values can only occur at either the critical points of f, or at the endpoints of the interval $[-2, 2]$. So all we need to do is find the critical values of f, and determine at which of the points f takes its highest and lowest values.

We start by finding the derivative $f'(x) = 4x^3 - 6x^2$. Since f' is a polynomial, it is defined everywhere, so there will be no critical values x for which $f'(x)$ does not exist. Then we need only find all points x where $f'(x) = 0$:

$$\begin{aligned} 4x^3 - 6x^2 &= 0 \\ 2x^2(2x - 3) &= 0 \end{aligned}$$

So either $2x^2 = 0$ or $2x - 3 = 0$; this gives us two solutions, $x = 0$ and $x = \frac{3}{2}$. To finish the problem we test f at the two critical values $x = 0$ and $x = \frac{3}{2}$, and also at the endpoints $x = -2$ and $x = 2$.

$$\begin{aligned} f(0) &= 0 \\ f(\tfrac{3}{2}) &= -\tfrac{27}{16} \\ f(-2) &= 32 \\ f(2) &= 0 \end{aligned}$$

So f obtains an absolute maximum of 32 at $x = -2$ and an absolute minimum of $-\frac{27}{16}$ at $x = \frac{3}{2}$. □

15 Increasing and Decreasing Functions

Definition 15.1. Suppose f is a function defined on an interval I. We say f is **increasing** if, whenever x_1 and x_2 are in I and $x_1 < x_2$, then $f(x_1) < f(x_2)$. We say f is **decreasing** if whenever x_1 and x_2 are in I and $x_1 < x_2$, then $f(x_1) > f(x_2)$.

Fact 15.2. *If f is differentiable on I and $f'(x) > 0$ for all x in I, then f's tangent lines all have a positive slope, and hence f is increasing on I. Likewise if $f'(x) < 0$ for all x in I, then f is decreasing on I.*

Example 15.3. Sketch a function f which is continuous on $(-\infty, \infty)$ and satisfies the following:

(1) $f' > 0$ on $(-\infty, 0)$, $(4, 6)$, and $(6, \infty)$

(2) $f' < 0$ on $(0, 4)$

(3) $f'(0)$ is undefined

(4) $f'(4) = f'(6) = 0$

Example 15.4. Find the intervals on which $f(x) = 2x^3 + 3x^2 + 1$ is increasing and decreasing.

Example 15.5. Find all local maxima and minima of $f(x) = 2x^3 + 3x^2 + 1$.

Theorem 15.6 (First Derivative Test)**.** *Suppose f is continuous on an interval I that contains a point x, and f is differentiable on I (except possibly at the point c).*

(1) If f' changes sign from positive to negative as x increases through c, then f has a local maximum at c.

(2) If f' changes sign from negative to positive as x increases through c, then f has a local minimum at c.

(3) If f' does not change sign at c then f has no local extreme value at c.

Example 15.7. Let $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Find all local extrema of f.

Example 15.8. Find all local extrema of $f(x) = x^{\frac{2}{3}}(2 - x)$.

Example 15.9. Find all absolute extrema of $f(x) = \frac{1}{4}x^4 - x^3 + \frac{3}{2}x^2 - 9x + 2$.

16 Concavity

Definition 16.1. Let f be a differentiable function, and assume its derivative f' is also differentiable. We denote by f'' the **second derivative** of f, defined by

$$f''(x) = \frac{d}{dx} f'(x).$$

Similarly we may define the **third derivative** f''', **fourth derivative** f'''', etc., when they exist. For any positive integer n (especially $n > 2$) we denote by $f^{(n)}$ the **n-th derivative** of f, if it exists.

Remark 16.2. We wish to apply the notion of a second derivative to our optimization problems. As a motivating example, consider the graph of $f(x) = x^3$. For values of $x > 0$, the curve of the graph bends upward. In other words, as we move from left to right, the tangent lines get steeper. In other words, the first derivative f' is increasing. Since f' is differentiable, we know that the second derivative f'' must satisfy $f''(x) > 0$ for all $x > 0$. We refer to this portion of the graph as **concave up**.

On the other hand, for all values of $x < 0$, the curve of the graph bends downward, i.e. the graph is **concave down**. As we move left to right on the graph, the tangent lines become less steep, i.e. the *first derivative* f' is *decreasing*, i.e. the *second derivative* f'' satisfies $f''(x) < 0$ for all $x < 0$.

Definition 16.3. Let f be differentiable on an open interval I. If f' is increasing on I then f is **concave up** on I. If f' is decreasing on I that f is **concave down** on I. If f is continuous at c and f changes concavity at c (from up to down, or vice versa), then f has an **inflection point** at c.

Fact 16.4. *If a graph is concave up at a given point x, then the graph near the point lies above the tangent line at x. Conversely if the graph is concave down at x, then the graph near the point lies below the tangent line at x.*

Example 16.5. Graphical example which shows no explicit relationship between the concavity of a function and whether it is increasing or decreasing. (A function can be increasing and concave down, or decreasing and concave up, or any other combination of these properties.)

Example 16.6. Let $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$. Identify the intervals on which f is concave up or concave down, and find any inflection points.

Theorem 16.7 (Second Derivative Test)**.** *Suppose that f'' is continuous on an open interval containing c with $f'(c) = 0$.*

(1) If $f''(c) > 0$ then f has a local maximum at c.

(2) If $f''(c) < 0$ then f has a local minimum at c.

(3) If $f''(c) = 0$, then the test is inconclusive.

Example 16.8. Use the Second Derivative Test to find all local extrema of $f(x) = 2x^3 - 3x^2 + 12$.

17 Some Optimization Problems

Example 17.1. Consider all pairs of positive numbers whose product is 10000. Is there a pair with minimal sum? Is there a pair with maximal sum? If so, identify the pairs.

Solution. Notice that there can be no maximal such pair, for if x is ANY positive real number, then $x \cdot \dfrac{10000}{x} = 10000$, and $x + \dfrac{10000}{x}$ is bigger than x. Since we can choose x as large as we want, we can get sums as large as we want.

On the other hand, we can use calculus to prove that there is a minimal such pair. The first step is to define an *sum function*, which, intuitively speaking, should take a pair of numbers which multiply to 10000 as input and spit out the sum of the pair as output. Then all we need to do is find the absolute minimum of this function using the techniques we have learned.

This function is easy to define but we need to make one change in order to use our techniques. Let x and y be any two numbers whose product is 10000; then their sum is given by:

$$S(x, y) = x + y$$

Our only problem with the above sum function is that it takes two variables for input, and we would like it to be a single-variable function. Note, now, that x and y are not independent pieces of information-that is, if x is any particular positive number, then there is exactly one y for which $xy = 10000$! In fact, we have $y = \dfrac{10000}{x}$ for any chosen x. We can now substitute this equality into our sum function to get a function in one variable:

$$S(x) = x + \frac{10000}{x} = x + 10000x^{-1}$$

Now we maximize $S(x)$. We have $S'(x) = 1 - 10000x^{-2}$. This derivative is 0 if $x = 100$ or $x = -100$, and is undefined at $x = 0$, so we have three critical points. But the problem asks us about *positive* numbers, so we need only consider the single critical point $x = 100$.

Since $S''(x) = 20000x^{-3}$, applying the second derivative test at $x = 100$ yields $S''(100) = \frac{1}{50} > 0$. So S is concave up at $x = 100$ and hence S indeed has a minimum value at $x = 100$.

If $x = 100$ then $y = 100$ and $S = 100 + 100 = 200$ is the minimum possible sum. □

Remark 17.2 (Strategy for Optimization Word Problems)**.** In the previous example, we (1) built a general "sum" function $S = x + y$; (2) reduced S to to a function of one variable by observing that $xy = 10000$ and subsituting for y; and (3) minimized S with our calculus techniques. This process is a typical example of the following general strategy:

(1) Identify what you are being asked to maximize or minimize, and build a function which expresses this value in terms of any possible inputs.
(2) Use any constraints given in the problem to reduce your function to just one variable input.
(3) Optimize your function using calculus.

Example 17.3. Let P be any fixed positive number, and consider all rectangles that have perimeter P. Is there such a rectangle with maximal area? Is there one with minimal area? If so, identify the rectangles.

Example 17.4. A rancher wishes to build a rectangular corral using 400 ft of fencing. One wall of the corral will lie alongside a barn, so the rancher doesn't have to use any fencing on one side. The corral will be split into three congruent rectangular sections, each with one wall alongside the barn. Which dimensions should the rancher choose for his corral in order to maximize its area?

18 Implicit Differentiation

Remark 18.1. Up to this point we have restricted our attention to functions which are defined *explicitly*, e.g. we say $y = f(x)$ where f is some function, and we compute the derivative y'. Now we wish to consider the concept of differentiation where the relationship between variables is defined *implicitly*. For example, the set of all solutions to the equation $x^2 + y^2 = 1$ is all the points on the unit circle. In this case, y is not a function of x (as the vertical line test fails) and x is not a function of y, but it still makes sense to consider tangent lines to the graph of the unit circle. The chain rule now gives us the tool we need to find a reasonable derivative function y'.

Example 18.2. Consider the unit circle $x^2 + y^2 = 1$. Find the slope of the tangent line to the unit circle at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$.

Solution. We will take the derivative of both sides of the equation $x^2 + y^2 = 1$. The crucial idea we will use is that although y is not *globally* a function of x, if we restrict our attention to a sufficiently small region around a point where the tangent line to the circle is non-vertical, then *locally* y can be regarded as a function of x.

Treating y as a function of x, by the chain rule, the derivative of y^2 (with respect to x is given by $\frac{d}{dx}y^2 = (\frac{d}{dy}y^2) \cdot \frac{dy}{dx} = 2y \cdot \frac{dy}{dx}$. We now compute y' below:

$$\begin{aligned}\frac{d}{dx}[x^2+y^2] &= \frac{d}{dx}(1)\\ \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) &= 0\\ 2x+\left(\frac{d}{dy}y^2\right)\left(\frac{dy}{dx}\right) &= 0\\ 2x+2y\cdot\frac{dy}{dx} &= 0\\ y'=\frac{dy}{dx} &= -\frac{x}{y}\end{aligned}$$

Notice that y' depends on both y and x! This should not be surprising, as the x-coordinate alone is not sufficient to pick out a point on the unit circle. We can now compute the appropriate slopes: $y'(\frac{1}{2},\frac{\sqrt{3}}{2}) = -\frac{1}{\sqrt{3}}$ and $y'(\frac{1}{2},-\frac{\sqrt{3}}{2}) = \frac{1}{\sqrt{3}}$. □

Example 18.3. Find an equation of the line tangent to the curve $x^2+xy-y^3=7$ at $(3,2)$.

Example 18.4. Find the slope of the tangent line at the point $(1,1)$ on the graph of $e^{x-y}=2x^2-y^2$.

Example 18.5. Find y'' if y is defined implicitly by the rule $x^2+4y^2=7$.

Theorem 18.6 (Derivatives of Arcsine and Arccosine)**.** *Let* $\arcsin x$ *denote the inverse function of* $\sin x$ *with domain* $[-1,1]$ *and range* $[-\frac{\pi}{2},\frac{\pi}{2}]$ *(i.e. the function for which* $\sin\arcsin x = x$ *for all* x *in* $[-1,1]$ *and* $\arcsin\sin x = x$ *for all* x *in* $[-\frac{\pi}{2},\frac{\pi}{2}]$*. Similarly, let* $\arccos x$ *denote the inverse function of* $\cos x$ *with domain* $[-1,1]$ *and range* $[0,\pi]$*. For all* x *in the domain of either function, we have*

$$\boxed{\tfrac{d}{dx}\arcsin x = \tfrac{1}{\sqrt{1-x^2}}}$$

and

$$\boxed{\tfrac{d}{dx}\arccos x = -\tfrac{1}{\sqrt{1-x^2}}.}$$

Proof. For the function $y=\arcsin x$, write $x=\sin y$ and use implicit differentiation to take the derivative of both sides of the equation with respect to x.

$$\begin{aligned}\sin y &= x\\ \cos y\cdot\left(\frac{dy}{dx}\right) &= 1\\ \frac{dy}{dx} &= \frac{1}{\cos y}.\end{aligned}$$

Written otherwise, we have computed that $y' = \frac{1}{\cos\arcsin x}$. To finish the computation, consider a right triangle with angle $y=\arcsin x$, where x is a member of the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$ as specified in the definition of $\arcsin x$. The ratio of opposite/hypotenuse in this triangle is $x = x/1$. If we label the opposite x and the hypotenuse 1, then by the Pythagorean theorem, the adjacent has length $\sqrt{1-x^2}$. Therefore $\cos y = \frac{1}{\sqrt{1-x^2}}$ and we get

$$\tfrac{dy}{dx} = \tfrac{1}{\sqrt{1-x^2}}$$

as claimed in the theorem. The computation for $\arccos x$ is very similar and we omit it. □

Example 18.7. Calculate $f'(\frac{1}{2})$, where $f(x)=\arcsin(x^2)$.

Theorem 18.8 (Derivatives of Other Inverse Trigonometric Functions)**.** *Labeling the inverse trigonometric functions in the usual way, we have*

$$\boxed{\frac{d}{dx}\arctan x = \frac{1}{1+x^2}}$$

$$\boxed{\frac{d}{dx}\text{arccot}x = -\frac{1}{1+x^2}}$$

$$\boxed{\frac{d}{dx}\text{arcsec}x = \frac{1}{|x|\sqrt{x^2-1}}}$$

$$\boxed{\frac{d}{dx}\text{arccsc}x = -\frac{1}{|x|\sqrt{x^2-1}}}$$

19 Related Rates

Example 19.1. An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 m/hr, how fast is the area of the patch increasing when the patch has a radius of 100 m?

Solution. Recall that the relationship between the radius r and the area A of a circle is given by the formula $A = \pi r^2$. Now let t represent the time variable; in this case, both A and r will increase as a function of t (since the oil spill is spreading as time passes). We can write $A = A(t)$ and $r = r(t)$ to emphasize that they are both functions of time.

In this case $r'(t)$ will be the rate of change of the radius of the spill with respect to time, and $A'(t)$ will be the rate of change of the area with respect to time. We will now use implicit differentiation to reveal the relationship between the two derivatives:

$$\begin{aligned} A'(t) &= \frac{d}{dt}[\pi(r(t))^2] \\ &= \pi\frac{d}{dt}(r(t))^2 \\ &= \pi \cdot 2(r(t))r'(t) \\ &= 2\pi r(t)r'(t) \end{aligned}$$

Now we substitute the values specified in the word problem, i.e. we take $r(t) = 100$ and $r'(t) = 30$. Then the rate of change of the area is $A'(t) = 2\pi(100)(30) = 6000\pi$. (This should be interpreted as square meters per hour.) □

Example 19.2. Two airplanes approach an airport, one flying due west at 120 mi/hr and the other flying due north at 150 mi/hr. Assuming they fly at the same constant elevation, how fast is the distance between the planes changing when the westbound plane is 180 mi from the airport and the northbound plane is 225 mi from the airport?

Example 19.3. An observer stands 200 m from the launch site of a hot air balloon. The balloon rises vertically at a constant rate of 4 m/s. How fast is the angle of elevation of the baloon increasing 30 s after the launch?

20 L'Hospital's Rule

Theorem 20.1 (L'Hospital's Rule)**.** *Suppose f and g are functions differentiable at $x = a$, and $\lim_{x\to a}\frac{f(x)}{g(x)}$ is of indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, i.e. either*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0,$$

or else

$$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty.$$

Then

$$\boxed{\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}}$$

provided the limit on the right-hand side exists (or is equal to $\pm\infty$*).*

Example 20.2. Use L'Hospital's rule to verify the limits in the examples above.

Example 20.3. Evaluate the following limits:

(1) $\displaystyle\lim_{x\to 1} \frac{x^3+x^2-2x}{x-1}$
(2) $\displaystyle\lim_{x\to 0} \frac{\sqrt{9+3x}-3}{x}$
(3) $\displaystyle\lim_{x\to 2} \frac{x^3-3x^2+4}{x^4-4x^3+7x^2-12x+12}$

Example 20.4 (L'Hospital's Rule is Inapplicable)**.** Evaluate $\displaystyle\lim_{x\to 1} \frac{x^2+\cos\pi x}{x^2-4x+2}$.

Example 20.5. Evaluate $\displaystyle\lim_{x\to 2} \frac{4-x^2}{\sin\pi x}$.

Example 20.6. Evaluate $\displaystyle\lim_{x\to 0} \frac{e^x-x-1}{\cos x-1}$.

Example 20.7 (Indeterminate Form $\infty-\infty$)**.** Evaluate $\displaystyle\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

Example 20.8 (Indeterminate Form $\infty\cdot 0$)**.** Evaluate $\displaystyle\lim_{x\to\infty} x^2 \sin(\frac{1}{4x^2})$.

Example 20.9 (Indeterminate Form 0^0)**.** Evaluate $\displaystyle\lim_{x\to 0^+} x^x$.

Example 20.10 (Indeterminate Form 1^∞)**.** Evaluate $\displaystyle\lim_{x\to 0} (1+4x)^{\frac{1}{2x}}$.

Definition 20.11. Let f and g be two functions which satisfy $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. We say that $f(x)$ **grows faster** than $g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty, \text{ or equivalently, } \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We denote this relationship by writing $g(x) << f(x)$.

Example 20.12. Which function grows faster, $f(x) = x^2$ or $g(x) = x\ln x$?

Example 20.13. Rank the following functions in order of growth rate:

(1) $\sqrt{x}$;
(2) $(\ln x)^3$;
(3) e^x; and
(4) 1.0001^x.

21 Rolle's Theorem and the Mean Value Theorem

Theorem 21.1 (Rolle's Theorem)**.** *Let f be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Suppose $f(a) = f(b)$. Then there is at least one point c in (a, b) such that $f'(c) = 0$.*

Proof. Since f is continuous on $[a, b]$, by the Extreme Value Theorem 14.4 f must obtain some absolute maximum and absolute minimum value on $[a, b]$. Either f obtains both its maximum and minimum at the endpoints of the interval ($x = a$ and $x = b$), or else f obtains one of its extreme inside the open interval (a, b). We will consider both cases.

Case 1: Suppose f has its local maximum and local minimum at either $x = a$ or $x = b$. But $f(a) = f(b)$ by our hypothesis, so if f doesn't go above or below these values, then f must in fact be a constant function. So $f(x) = K$ for some number K. Then $f'(x) = 0$ for every point in (a, b).

Case 2: Suppose instead that f obtains a local extreme value at some point $x = c$ in the middle interval (a, b). Then f must have a critical point at c, i.e. $f'(c) = 0$ or $f'(c)$ does not exist. But we assumed f is differentiable on (a, b), so $f'(c)$ DOES exist; therefore $f'(c) = 0$.

In either case 1 or case 2 we have produced a point c in (a, b) with $f'(c) = 0$, so the theorem is proved. □

We will now use Rolle's Theorem to prove the Mean Value Theorem, a stronger result.

Theorem 21.2 (Mean Value Theorem)**.** *Let f be continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Restated, there is some point c in (a, b) where the slope of the tangent line at $(c, f(c))$ is the same as the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$.

Proof. Let $\ell(x)$ be the linear function whose graph is the line passing through $(a, f(a))$ and $(b, f(b))$. Define a new function $g(x)$ by the rule $g(x) = f(x) - \ell(x)$, for all x in $[a, b]$. The function $g(x)$ is just a difference of continuous differentiable functions and hence it is continuous and differentiable as well. Notice that $g(a) = f(a) = \ell(a) = f(b) - f(b) = 0$ and $g(b) = f(b) - \ell(b) = f(b) - f(b) = 0$, since $\ell(a) = f(a)$ and $\ell(b) = f(b)$. Hence $g(a) = g(b)$, and therefore the function $g(x)$ meets the conditions required in Rolle's Theorem 21.1.

In that case, Rolle's Theorem implies that for some c in (a, b), we must have $g'(c) = 0$. But then we have $g'(c) = f'(c) - \ell'(c) = 0$ and hence $f'(c) = \ell'(c)$. Since the graph of $\ell(x)$ is just the line between $(a, f(a))$ and $(b, f(b))$, we have $\ell'(c) = \frac{f(b)-f(a)}{b-a}$; so $f'(c) = \frac{f(b)-f(a)}{b-a}$ as required. □

Remark 21.3. Notice that the Mean Value Theorem obvious implies Rolle's Theorem, since if $f(a) = f(b)$, then the slope $\frac{f(b)-f(a)}{b-a}$ is just 0, so there is some point c in (a, b) with $f'(c) = 0$. However, since we can use Rolle's Theorem to prove the Mean Value Theorem, the two really have basically the same content; the Mean Value Theorem is just Rolle's Theorem adjusted for alteration by some linear function $\ell(x)$.

Why do we care about the Mean Value Theorem? Let's investigate some immediate consequences.

First suppose $f(x)$ is any continuous function for which $f'(x) = 0$ at every point x. Then given any two numbers a and b (with $a < b$), choosing any point c in (a, b) yields $f'(c) = 0$. Hence for the Mean Value Theorem to be true, we must have that the slope $\frac{f(b)-f(a)}{b-a}$ is just 0. This only happens if $f(b) - f(a) = 0$, i.e. if $f(a) = f(b)$. Since this is true for ANY a and b, f must be a CONSTANT function, i.e. $f(x) = C$ for some number C.

Then what can we say about two functions which have the same derivative? Suppose f and g are continuous functions such that $f'(x) = g'(x)$ for every x. Then $f'(x) - g'(x) = 0$ everywhere, and hence the above paragraph applies that $f(x) - g(x) = C$ for some constant C. Then $f(x) = g(x) + C$, i.e. f and g differ only by some constant.

This is relevant because we wish to define the notion of an **antiderivative** in the near future, i.e. we wish to define a process which is the inverse of differentiation. Now we know that if f is some continuous function, then any two antiderivatives of f must be the same, except for some constant term. This fact is crucial to understanding antidifferentiation.

22 Antiderivatives

Definition 22.1. A function F is an **antiderivative** of f on an interval I provided $F'(x) = f(x)$ for all x in I.

Theorem 22.2. *Let F be any antiderivative of f. Then all antiderivatives of f have the form $F + C$, where C is some constant.*

Proof. We proved this in our discussion of the Mean Value Theorem above. □

Example 22.3. Find all antiderivatives of the following functions:

(1) $f(x) = 1$

(2) $f(x) = x$

(3) $f(x) = x^2$

(4) $f(x) = x^3$

Fact 22.4 (Power Rule for Antiderivatives)**.** *Let p be any rational number (with $p \neq= -1$). Then all antiderivatives of x^p have the form*

$$\boxed{\tfrac{1}{p+1}x^{p+1} + C,}$$

where C is some constant.

Example 22.5. Find all antiderivatives of $f(x) = 3x^5 + 2 - 5x^{-\frac{3}{2}}$.

Definition 22.6. To denote the operation "find all antiderivatives of f," we write the following:

$$\int f(x)dx$$

We refer to this string of symbols as the **indefinite integral** of f, and it refers to an infinite collection of functions (all of which differ from one another by just a constant). We use this term indefinite integral interchangeably with antiderivative.

We call the function $f(x)$ the **integrand**. The term dx in the notation indicates that we are antidifferentiating with respect to the variable x.

Remark 22.7. This notation, like $\frac{d}{dx}$ and $\frac{dy}{dx}$, is also old-timey and wrapped up in the notion of "infinitesimals." We will simply take the symbols at face value for now, and try to motivate their meaning later on when we start talking about **definite integrals**.

Example 22.8. Find $\int \cos(3x)dx$.

Fact 22.9 (Indefinite Integrals of Trigonometric Functions)**.** *The following all hold for any constant $a \neq 0$:*

(1) $\boxed{\int \cos(ax)dx = \frac{1}{a}\sin(ax) + C}$

(2) $\boxed{\int \sin(ax)dx = -\frac{1}{a}\cos(ax) + C}$

(3) $\boxed{\int \sec^2(ax)dx = \frac{1}{a}\tan(ax) + C}$

(4) $\boxed{\int \csc^2(ax)dx = -\frac{1}{a}\cot(ax) + C}$

(5) $\boxed{\int \sec(ax)\tan(ax)dx = \frac{1}{a}\sec(ax) + C}$

(6) $\boxed{\int \csc(ax)\cot(ax)dx = -\frac{1}{a}\csc(ax) + C}$

Example 22.10. Find the following:

(1) $\int \sec^2(7x)dx$

(2) $\int \cos(\frac{x}{2})dx$

Fact 22.11 (Indefinite Integrals of Exponentials and Logarithms)**.** *The following hold for any constants* $a \neq 0$*,* $b > 1$*:*

(1) $\boxed{\int e^{kx}dx = \frac{1}{k}e^{kx} + C}$

(2) $\boxed{\int b^{kx}dx = \frac{1}{k\ln b}b^{kx} + C}$

(3) $\boxed{\int \frac{1}{x}dx = \ln|x| + C}$

Example 22.12. Find the following:

(1) $\int(\frac{5}{x} - 3x^{-10})dx$

(2) $\int(3e^x - 4)dx$

(3) $\int(12e^{7-3x})dx$

Example 22.13. Find the function y which satisfies $y' = 4x^7$ and $y(0) = 4$.

Example 22.14. Solve the equation $\frac{dy}{dx} = \sin(\pi x)$ subject to the condition $y(2) = 2$.

23 Approximating Areas Under Curves Using Riemann Sums

Example 23.1. Consider a car moving at a constant rate of 60 mi/hr. How far does the car move in exactly 3 hours? We know immediately that the answer is (60 mi/hr) $\cdot$ (3 hr) = 180 miles. How can we visualize this solution geometrically? Consider the graph of the velocity equation $v(t) = 60$, i.e. the graph of the horizontal line at height 60. The distance traveled after t seconds is always given by the equation $60 \cdot t$, which is exactly the *area* under the graph between $t = 0$ and $t = 3$. (Recall that the derivative of a distance function gives a velocity function, and in this case the area under the curve of a velocity function gives a distance function; this foreshadows the inverse relationship between the derivative and the integral!)

Now if a graph is given by a straight line, it is easy to compute the area under the curve by a simple geometric formula; however we wish to compute the exact areas under graphs of more complicated functions.

Example 23.2. Suppose the velocity in m/s of a moving object is given by the equation $v(t) = t^2$, for $0 \leq t \leq 8$. Estimate the displacement of the object after 8 seconds.

Solution. One approach we may take is to divide the interval $[0, 8]$ into, say, four equal subintervals and try to estimate the displacement in each subinterval, i.e. make an estimate for every 2 seconds that pass and then add them together. For instance, in the first two seconds the velocity of the object increases from $v(0) = 0$ to $v(2) = 4$; we can perhaps get a decent approximation of the object's velocity over this interval by taking $v(1) = 1$ m/s. Then we estimate that the object moves approximately (1 m/s) $\cdot$ (2 s) = 2 m in the first 2 seconds.

Then we can make similar estimates for the remaining intervals: say the object moves about $v(3) = 9$ m/s from $t = 2$ to $t = 4$; about $v(5) = 25$ m/s from $t = 4$ to $t = 6$; and about $v(7) = 49$ m/s from $t = 6$ to $t = 8$. Then we may compute an estimate of the displacement of the object as follows:

$$v(1) \cdot 2 + v(3) \cdot 2 + v(5) \cdot 2 + v(7) \cdot 2 = (1 + 9 + 25 + 49) \cdot 2 = 168 \text{ m}.$$

Notice that we can visualize the product $v(1) \cdot 2$ as the area of the rectangle with base $[0, 2]$ and height $v(1) = 1$. Likewise the next product $v(3) \cdot 2$ corresponds to the area of the rectangle with base $[2, 4]$ and height $v(3)$; and so forth with the others. So what we are really computing is the sum of the areas of these four rectangles; geometrically, we are approximating the area under the curve of the graph.

What happens if we repeat this process by dividing the interval into, say, 8 subintervals instead of 4? What about 16, or 100, or 1000 subintervals? □

Definition 23.3. Suppose $[a, b]$ is a closed interval. We may break up $[a, b]$ into n distinct subintervals $[x_0, x_1]$, $[x_1, x_2]$,...,$[x_{n-1}, x_n]$ of equal length $\Delta x = \frac{b-a}{n}$ with $a = x_0$ and $b = x_n$. The endpoints $x_0, x_1, ..., x_n$ are called **grid points** and they create a **regular partition** of $[a, b]$. In general, the k-th grid point is $x_k = a + k\Delta x$, for $k = 0, 1, ..., n$.

Example 23.4. (1) Find a regular partition of $[1, 9]$ into 4 subintervals.

(2) Find a regular partition of $[5, 7]$ into 9 subintervals.

Definition 23.5. Suppose f is a function defined on a closed interval $[a, b]$, and let $x_0, ..., x_n$ give a regular partition of $[a, b]$ into n subintervals. Let $\overline{x}_k$ be any point in the k-th subinterval $[x_{k-1}, x_k]$, for each $k = 1, 2, ..., n$. Then the sum

$$f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + ... + f(\overline{x}_k)\Delta x$$

is called a **Riemann sum** for f on $[a, b]$.

Furthermore, we call this sum a **left** Riemann sum if $\overline{x}_k$ is always the left endpoint of $[x_{k-1}, x_k]$; a **right** Riemann sum of $\overline{x}_k$ is always the right endpoint; and a **midpoint** Riemann sum of $\overline{x}_k$ is always the midpoint.

Example 23.6. Find a regular partition for $[0, 1]$ into 4 subintervals; then compute a left, right, and midpoint Riemann sum for $f(x) = x^3$ using this partition. How do the area estimates compare to the actual area under the curve?

Definition 23.7 (Sigma Notation for Sums)**.** Now that we are working with Riemann sums of n terms, where n may be extremely large, it is in our interest to develop some new notation to describe when we are adding together a large finite number of terms.

Suppose we wish to represent the sum of the first 1000 perfect squares, i.e. $1^2 + 2^2 + 3^2 + ... + 999^2 + 1000^2$. We introduce the following string of symbols to represent this sum:

$$\textstyle\sum_{k=1}^{1000} k^2$$

The Greek letter $\sum$, i.e. the capital *sigma*, is an S for "Sum." The "$k = 1$" below the sigma and the "1000" above the sigma tell us how many terms we wish to add together: in particular, we should start counting terms from $k = 1$ all the way up to $k = 1000$. Now we should interpret the "k^2" term to the right of the capital sigma as a type of function; it means we should take each whole number k (from 1 up to 1000) and spit out its square k^2, and then add them all up.

This type of notation is easier learned through practice than through extensive explication. The following examples illustrate the use of sigma notation to describe large finite sums:

$$\sum_{k=1}^{99} k = 1 + 2 + 3 + ... + 98 + 99 = 4950$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + ... + (n-1) + n$$

$$\sum_{i=0}^{3} i^3 = 0 + 1 + 8 + 27 = 36$$

$$\sum_{j=1}^{4} (2j+1) = 3 + 5 + 7 + 9 = 24$$

$$\sum_{k=-1}^{2} (k^2 + k) = [(-1)^2 + (-1)] + [0^2 + 0] + [1^2 + 1] + [2^2 + 2] = 8$$

Now, with this new sigma-notation for finite sums, we can rewrite any Riemann sum in a very compact form:

$$f(\overline{x}_1)\Delta x + f(\overline{x}_2)\Delta x + ... + f(\overline{x}_n)\Delta x = \sum_{k=1}^{n} f(\overline{x}_k)\Delta x$$

Example 23.8 (Negative Area Under a Curve)**.** Evaluate and interpret a midpoint Riemann sum for $f(x) = 1 - x^2$ on the interval $[1, 3]$, using a regular partition into 4 subintervals.

Remark 23.9. We should know that the idea of **area**, in the geometric sense, is a concrete quantity and can only take nonnegative values. However, for our purposes, when we are working with some function $f(x)$ whose graph lies below the x-axis on a closed interval $[a, b]$ (as in the previous example), then we consider the region bounded by the graph of f and the x-axis, on $[a, b]$, to be a region with "negative area." This way, the Riemann sums make sense as an area approximation.

Example 23.10. Evaluate and interpret a midpoint Riemann sum for $f(x) = 1 - x^2$ on the interval $[0, 3]$, using a regular partition into 6 subintervals.

Definition 23.11. Let R be the region bounded by the graph of a continuous function f and the x-axis between $x = a$ and $x = b$. The **net area** of R is the area of the parts of R that lie above the x-axis minus the area of the parts of R that lie below the x-axis on $[a, b]$.

24 The Definite Integral

Remark 24.1. Now we wish to define the definite integral of a function $f(x)$ on an interval $[a, b]$. Recall that the derivative, which was supposed to compute the slope of the tangent line to the graph of f at a point, was defined by looking at all possible *approximations*, i.e. the slopes of all nearby secant lines, and then taking a limit to find an exact value. Now we wish to define the definite integral, which will compute the net area under the graph of $f(x)$ bounded by $[a, b]$. By analogy, we will do so by looking at all possible *approximations*, i.e. all possible Riemann sums on $[a, b]$, and taking the limit to get an exact number.

Recall that not every function is differentiable... also by analogy, we will see that not every function is integrable. We will restrict our attention to situations where taking a Riemann integral makes sense.

Definition 24.2. Let f be a function defined on a closed interval $[a, b]$. For any given n, there is a unique regular partition of $[a, b]$ into n subintervals, given by $x_0, x_1, ..., x_n$. If $\lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x$ exists, AND does not depend on the choices of $\overline{x}_k$ in each subinterval, then we say f is **integrable** on $[a, b]$. Furthermore, we define this limit to be the **definite integral** of f on $[a, b]$, and we write:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\overline{x}_k)\Delta x$$

Remark 24.3 (Remarks on Notation)**.** Recall from when we were working on derivatives, we can use the fraction $\frac{\Delta y}{\Delta x}$ to represent the slope of a secant line determined by some small change Δx in the input variable x. Then when we take the limit to define the derivative, we get $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$. Here dx is supposed to represent an "infinitesimal" change in x and dy is supposed to represent a corresponding "infinitesimal" change in y.

Now consider our definition of the definite integral: as we take $n \to \infty$, i.e. as we let n grow arbitrarily large, the length of each subinterval in the regular partition shrinks down to 0, i.e. $\lim_{n \to \infty} \Delta x = \lim_{n \to \infty} \frac{b-a}{n} = 0$. So as $n \to \infty$, we have $\Delta x \to 0$. Then in our definite integral notation, the symbol $\int$ is supposed to be an elongated "S" for "sum," which represents the finite sum $\sum_{k=1}^{n} f(\overline{x}_k)$ taken to an infinite limit, and the symbol dx is supposed to be the "infinitesimal" limit of Δx as n tends to infinity, just as in the notation for the derivative.

These notations are old-fashioned but deeply entrenched so we are going to stick with them.

Example 24.4. Which of the following functions are integrable on $[0, 1]$?

(1) $f(x) = \sin x$
(2) $f(x) = 1$ if x is rational, and $f(x) = 0$ if x is irrational
(3) $f(x) = 1 - x^2$ if $0 \le x < \frac{1}{2}$, and $f(x) = 2x - 1$ if $\frac{1}{2} \le x \le 1$

Theorem 24.5. *If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, then f is integrable on $[a, b]$.*

Example 24.6. Compute the following definite integrals.

(1) $\int_2^4 (2x + 3)dx$
(2) $\int_1^6 (2x - 6)dx$
(3) $\int_3^4 \sqrt{1 - (x - 3)^2}dx$

Example 24.7. Graphical example: Sec 5.2 Ex #4.

Fact 24.8 (Some Properties of Integrals)**.** *Let f and g be integrable functions on $[a, b]$, let c be in $[a, b]$, and let k be any constant. Then:*

(1) $\displaystyle\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
(2) $\displaystyle\int_a^b kf(x)dx = k\int_a^b f(x)dx$
(3) $\displaystyle\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

In addition, we also adopt the following two conventions:

(4) $\displaystyle\int_a^a f(x)dx = 0$
(5) $\displaystyle\int_b^a f(x)dx = -\int_a^b f(x)dx$

Remark 24.9. These last two points above we just take as a definition.

Example 24.10. Suppose $\int_0^5 f(x)dx = 3$ and $\int_0^7 f(x)dx = -10$. Evaluate the following integrals:

(1) $\int_0^7 2f(x)dx$

(2) $\int_5^7 f(x)dx$

(3) $\int_5^0 f(x)dx$

(4) $\int_7^0 6f(x)dx$

Example 24.11 (An Explicit Riemann Integral Computation)**.** Find $\int_0^2 (x^3+1)dx$.

Remark 24.12. To solve Example 24.11, we will need to make a one-time use of the summation formula $\sum_{k=1}^n k^3 = \frac{n^2(n+1)^2}{4}$.

25 The Fundamental Theorem of Calculus

Remark 25.1. We have now defined the antiderivative (or indefinite integral) of a function as well as the definite integral of a function, and we have used the same terminology and the same symbols for both concepts, but up to this point in the course we have not made the connection between the two explicit. (Outside of our copious hint-dropping.)

We have also seen in the previous example that although the definition of the definite integral is fairly natural, performing the actual computations involved range from the difficult to the impossible. Thus we desire a much easier process for computing definite integrals. The following theorem shows us the connection between the antiderivative and the definite integral, and as a consequence gives us a powerful computational tool.

Theorem 25.2 (Fundamental Theorem of Calculus)**.** *Let f be a continuous function and let a be any point where f is defined. Define the **area** function of f, centered at a, as follows:*

$$A(x) = \int_a^x f(t)dt$$

(This definition makes sense whenever A is integrable on the closed interval between a and x.) Also let F be any antiderivative of f. Then the following two statements hold:

(1) $A'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$

(2) $\int_a^b f(x)dx = F(b) - F(a)$

Remark 25.3. Statement (1) above can be read as "The definite integral of f from a to x is an antiderivative of f," and (2) can be read as "Any antiderivative of f computes the definite integral of f from a to b." In other words, up to a few details, the antiderivative and the definite integral are exactly the same concept. So it makes sense to regard integration as the "inverse process" of differentiation.

Proof of the FTC. Let f be continuous and let $A(x) = \int_a^x f(t)dt$ be the area function determined by f. By the definition of the derivative, we have

$$A'(x) = \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

Now notice that $A(x+h) - A(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt = \int_x^{x+h} f(t)dt$, i.e. $A(x+h) - A(x)$ represents the net area under the graph of f bounded by the interval $[x, x+h]$ (if $h > 0$; otherwise $[x+h, x]$).

Now since f is continuous, by the Extreme Value Theorem f must attain its absolute minimum and its absolute maximum on this interval. Call these values m and M, respectively. Then it follows that $mh \leq A(x+h) - A(x) \leq Mh$. (Draw the picture to see this!) Thus, dividing by h on each side, we get

$$m \leq \frac{A(x+h) - A(x)}{h} \leq M$$

But since f is continuous, we must have $\lim_{h\to 0} m = f(x)$ and $\lim_{h\to 0} M = f(x)$ (since all the function values on $[x, x+h]$ must become close to $f(x)$ as h shrinks to 0). Therefore, applying the Squeeze Theorem, we get

$$\lim_{x\to h} m \leq \lim_{h\to 0} \frac{A(x+h) - A(x)}{h} \leq \lim_{h\to 0} M$$

Since the left- and right-hand sides above are both $f(x)$, we have $A'(x) = f(x)$. This proves part (1) of the FTC.

To see that part (2) is also true, let F be any antiderivative of f. We just finished showing that A is another antiderivative of f, and thus by our previous discussion on the uniqueness of antiderivatives, we know that $F(x) = A(x) + C$ for some constant term C. It follows then that $F(b) - F(a) = [A(b) - C] - [A(a) - C] = A(b) - A(a) = \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt - 0 = \int_a^b f(x)dx$. (We may make the variable switch in the last step because t, x are just dummy variables in this case.) This shows part (2) and the theorem is proved. □

Example 25.4. Use the FTC to verify that $\int_0^2 (x^3 + 1)dx = 6$ (a la our previous difficult computation).

Example 25.5. Compute the following definite integrals.

(1) $\int_0^{10} (60x - 6x^2)dx$

(2) $\int_0^{2\pi} 3 \sin x dx$

(3) $\int_{\frac{1}{16}}^{\frac{1}{4}} \frac{\sqrt{t} - 2t}{t} dt$

Corollary 25.6. *Every continuous function f has some antiderivative.*

Proof. Just take the area function $A(x) = \int_a^x f(t)dt$ as in the fundamental theorem above. □

Example 25.7. Consider the function $f(x) = x^2$.

(1) Find an explicit formula for the area function $A(x) = \int_2^x t^2 dt$ of f centered at 2.
(2) Find an explicit formula for the area function of f centered at 7.

Example 25.8 (A Simple Function With No Simple Antiderivative)**.** Consider the function $f(x) = e^{-x^2}$, whose graph looks like a bell curve. It is easy to write down, and by the previous corollary, f has an antiderivative, for instance the function $F(x) = \int_0^x e^{-t^2} dt$. Can you write down an explicit formula for F?

Fact 25.9. *There are many continuous functions, like $f(x) = e^{-x^2}$ and $g(x) = \sin x^2$, whose antiderivatives (although they exist by the FTC) are impossible to write down using only the following functions and operations: polynomials, trigonometric functions, exponentials, logarithms, inverse trigonometric functions, hyperbolic trigonometric functions, sums, differences, products, quotients, powers, and compositions.*

The student should contrast this with the situation for derivatives: given any function built up out of the functions and operations above, one can always explicitly compute its derivative using our technology developed earlier.

26 The Substitution Rule

Remark 26.1. In general, the chain rule, product rule, and quotient rule are sufficient for us to compute the derivatives of just about any reasonable functions we run across, even if they are very complicated. On the other hand, right now the family of functions which we can antidifferentiate is extremely small. One of the main goals of the reader's upcoming Calculus II course will be to expand this family as much as possible by introducing a large variety of integration techniques.

There will always be many functions for which finding an explicit formula for an antiderivative is impossible- however, we can get a start here by introducing one very powerful integration technique. Consider the following examples.

Example 26.2. Find $\int \cos(2x)dx$.

Example 26.3. Find $\int 5x^4(x^5+6)^9 dx$.

Theorem 26.4 (Substitution Rule)**.** *If f and g are functions, u is differentiable, and F is any antiderivative of f, then*

$$\boxed{\int f(u(x))u'(x)dx = F(u(x)) + C.}$$

Procedure for the substitution rule:

(1) Given an indefinite integral with a composite function $f(u(x))$ appearing in the integrand, identify the "inner function" $u(x)$ whose derivative $u'(x)$ also appears as a factor in the integrand. (i.e. Look for something of the form $f(u(x))u'(x)$.)

(2) Compute $du = g'(x)dx$.

(3) Evaluate the integral with respect to u.

(4) Un-substitute $u = u(x)$ to finish the problem.

Example 26.5. $\int 2(2x+1)^3 dx$.

Example 26.6. $\int \cos^3 x \sin x dx$.

Example 26.7. $\int \frac{x}{\sqrt{x+1}} dx$.

Theorem 26.8 (Substitution Rule for Definite Integrals (with Change of Parameters))**.** *If f and u are functions, u is differentiable, and F is any antiderivative of f, then*

$$\boxed{\int_a^b f(u(x))u'(x)dx = \int_{u(a)}^{u(b)} f(u)du.}$$

Remark 26.9. In the above theorem we are using a common but perhaps slightly confusing "abuse of notation": we are using u to denote both a function, and a dummy variable inside the integral on the right-hand side of the equality. In practice, this usually doesn't cause confusion, because we typically write $u(x)$ in terms of its explicit formula in x, and u occupies solely its latter role as a dummy variable when we work through problems.

Proof. We know by the chain rule that the composite function $F(u(x))$ is an antiderivative of $f(u(x))u'(x)$. Then if $[a,b]$ is any closed interval, by the Fundamental Theorem of Calculus, it follows that

$$\begin{aligned}\int_a^b f(u(x))u'(x)dx &= [F(u(x))]_a^b \\ &= F(u(b)) - F(u(a)) \\ &= [F(u)]_{u(a)}^{u(b)} \\ &= \int_{u(a)}^{u(b)} f(u)du\end{aligned}$$

Thus the substitution rule applies to definite integrals as well as indefinite ones, as long as apply a vital **change of parameter** from $x = a \to x = b$ to $u = g(a) \to u = g(b)$. □

Example 26.10. Compute $\int_0^2 \frac{dx}{(x+3)^3}$.

Example 26.11. Compute $\int_{-1}^2 \frac{x^2 dx}{(x^3+2)^3}$.

Example 26.12. Compute $\int_0^{\frac{\pi}{2}} \sin^4 x \cos x dx$.