AXIOMATIC GEOMETRY SPRING 2015 (COHEN) LECTURE NOTES

Remark 0.1. These lecture notes are heavily based on John M. Lee's *Axiomatic Geometry* and we work for the most part from his given axioms. Other sources that deserve credit are *Roads to Geometry* by Edward C. Wallace and Stephen F. West and *Elementary Geometry from an Advanced Standpoint* by Edwin Moise.

Remark 0.2. Since we work in several axiom systems (several of which are mutually incompatible and thus produce different theorems), we have labeled theorems and corollaries according to the axiom system under which they can be proved. Thus Incidence Geometry Theorems, Euclidean Geometry Theorems, and Hyperbolic Geometry Theorems correspond to their particular axiom systems. Any geometric theorems simply labeled Theorem are true in neutral geometry and we derive them from the Neutral Geometry Axioms given by John M. Lee. (The neutral theorems are true in both Euclidean geometry and hyperbolic geometry.) Theorems labeled Theorem of Euclid are "pseudo-theorems" in the sense that they were stated and proved in Euclid's *Elements*, but they may or may not actually be provable from Euclid's given postulates (or modern interpretations thereof). Of course they still end up being true in Euclidean geometry.

Remark 0.3. Any discussion of axiom systems and provability necessarily involves some amount of metamathematics. When we state theorems about consistency/independence we are generally stating them informally, but most of the informal examples and statements can be realized in some formal, rigorous way working within the confines of the Zermelo-Fraenkel axioms of set theory. Of course the foundations of set theory are beyond the scope of this course. So we mention it only in passing for sake of the interested (or skeptical) student. We also avoid any rigorous discussion of model theory– for our purposes it is sufficient to use "naive set theory" and "naive models."

1 Euclid's Postulates

The following are the basic terms and premises of Euclid's *Elements* (approx. 300 B.C.E.), which remained the primary text for general mathematical education for over 2000 years, and is the progenitor of the axiomatic method on which modern mathematics is based.

Euclid's Definition 1. A **point** is that which has no part.

Euclid's Definition 2. A **line** is breadthless length.

Euclid's Definition 3. A **straight line** is a line which lies evenly with the points on itself.

In addition to the definitions above, **triangles** and **circles** are defined as you expect.

Euclid's Postulate 1. *To draw a straight line from any point to any point.*

Euclid's Postulate 2. *To produce a finite straight line continuously in a straight line.*

Euclid's Postulate 3. *To describe a circle with any center and distance.*

Euclid's Postulate 4. *That all right angles are equal to one another.*

Euclid's Postulate 5. *That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.*

Euclid's Common Notion 1. *Things which are equal to the same thing are also equal to one another.*

Euclid's Common Notion 2. *If equals be added to equals, the wholes are equal.*

Euclid's Common Notion 3. *If equals be subtracted from equals, the remainders are equal.*

Euclid's Common Notion 4. *Things which coincide with one another are equal to one another.*

Euclid's Common Notion 5. *The whole is greater than the part.*

Question 1.1. Are Euclid's definitions satisfying? Why or why not?

Question 1.2. Are Euclid's postulates and axioms satisfying? Do some seem more or less self-evident than others?

Theorem of Euclid 1. *To construct an equilateral triangle on a given finite straight line.*

Proof...? Let $\overline{AB}$ be a given line segment with endpoints A and B. Let AB denote the length of $\overline{AB}$. By Postulate 3, there is a circle a centered at A with radius AB. Again by Postulate 3, there is a circle b centered at b with radius AB. These two circles must intersect, say at a point C. Then consider the triangle $\triangle ABC$ which has A, B, and C for vertices. If AC denotes the length of side $\overline{AC}$ and BC denotes the length of side $\overline{BC}$, then by the way we constructed our circles a and b we have $AB = AC = BC$. This implies $\triangle ABC$ is equilateral, proving the theorem. □

Question 1.3. Is the proof above sound? What assumptions does Euclid make which do not necessarily follow from his axioms?

Remark 1.4. To answer the latter question above, for one thing Euclid assumes that the circles a and b must intersect at some point C. Although this makes a great deal of intuitive sense (to us as well as Euclid), there is nothing in the Postulates which tells us this should be the case. In fact, we will see later in the course that there is a model of geometry which satisfies all of (modern versions of) Euclid's postulates, but in which the two circles constructed above need not necessarily intersect! (See Example 4.11.)

Theorem of Euclid 2. *To place a straight line equal to a given straight line with one end at a given point.*

Proof...? Let A be the given point and let $\overline{BC}$ denote the given line segment. By Theorem 1, there exists an equilateral triangle with B and C as two of its vertices; label this triangle $\triangle ABD$ (so D is the third vertex).

By Postulate 3, let b be the circle with center B and radius BC. By Postulate 2, the line segment $\overline{DB}$ may be extended to a longer line segment $\overline{DE}$ far enough that $\overline{DE}$ intersects b, say at the point F.

Now again by Postulate 3, we construct a circle d with center D and radius equal to the length of $\overline{DF}$. Finally, by Postulate 2, we extend $\overline{DA}$ to a longer line segment $\overline{DG}$ so that $\overline{DG}$ intersects d at a point H.

This completes the construction. Note that $\overline{DH}$ and $\overline{DF}$ are the same length, as are $\overline{DA}$ and $\overline{DB}$. Since $\overline{DH}$ is comprised of $\overline{DA}$ and $\overline{AH}$, and $\overline{DF}$ is comprised of $\overline{DB}$ and $\overline{DF}$, we conclude that $\overline{AH}$ must be the same length as $\overline{BF}$, which in turn is the same length as $\overline{BC}$. (Note we are using some Common Notions here.) So $\overline{AH}$ is a segment with length equal to the given segment $\overline{BC}$, and with one endpoint at the given point A. □

Question 1.5. Are there gaps present in the argument above? For instance, Postulate 2 appears to guarantee that we can extend line segment $\overline{DB}$ to a longer line segment $\overline{DE}$. But what axiom guarantees this can be done in such a way that $\overline{DE}$ intersects the circle b?

Question 1.6. What if the given point A lies on the given line segment $\overline{BC}$ in the first place? Does Euclid's argument still work? What about if $A = B$?

Question 1.7. Is there anything in Euclid's axioms that guarantees that there exist any points or any lines in the first place?

2 Modern Axiom Systems

A **modern axiom system** consists of four things:

(I) **Primitives** or **undefined terms**. Terms which are deliberately left undefined, and are therefore left open to interpretation. The properties of the undefined terms may only be formally understood by means of the system's axioms.

(II) **Definitions**. These are technical terms which are defined in terms of the primitives and/or previously introduced definitions.

(III) **Axioms**. Unambiguous statements which deal with the primitives and definitions of the system, and are assumed to be true as the basic hypotheses of the system. Formally, it should be possible to formulate any axiom by using only the primitives and definitions, variables, the equals sign, the conjunctions "and," "or," and "implies," the negation "not," and the quantifiers "there exists" and "for all."

(IV) **Theorems**. Unambiguous statements which are proved logically from the axioms and/or previously proven theorems.

We know that the axiomatic system of Euclid's *Elements*, although millenia ahead of its time, is not exactly rigorous by modern standards. Still, if we attempted to interpret it as a modern axiomatic system, we might take *point*, *line*, and *lies on* to be our undefined primitives. In that case, Euclid's Postulate 1 might be interpreted as the statement "For all points A, for all points B, there exists a line ℓ such that A and B lie on ℓ."

Let us now give an example of an abstract axiomatic system (borrowed from Wallace and West's *Road to Geometry*.

Theory of Fe-Fo's.

For our undefined primitves, we take *Fe's*, *Fo's*, and the relation *belongs to*.

Fe-Fo Axiom 1. *There exist exactly three distinct Fe's.*

Fe-Fo Axiom 2. *Any two distinct Fe's belong to exactly one Fo.*

Fe-Fo Axiom 3. *Not all Fe's belong to the same Fo.*

Fe-Fo Axiom 4. *Any two distinct Fo's contain at least one Fe that belongs to both.*

Fe-Fo Theorem 1. *For any two distinct Fo's X and Y, there exists a unique Fe a so that a belongs to both X and Y.*

Proof. Let X and Y be distinct Fo's. By Fe-Fo Axiom 4, there exists a Fe a such that a belongs to both X and Y. So to prove the theorem, we need only show that a is the unique Fe belonging to both X and Y.

Suppose b is a Fe distinct from a. If b belongs to X, then by Fe-Fo Axiom 2, X is the only Fo to which a and b both belong; it follows that b does not belong to Y, since a already belongs to Y. So no such b distinct from a belongs to both X and Y, i.e. a is the only Fe belonging to both X and Y, showing a is the unique Fe with this property and proving the theorem. □

Exercise 2.1. Prove that there exist exactly three Fo's.

Exercise 2.2. Prove that each Fo has exactly two Fe's belonging to it.

In this course, we eventually hope to give a rigorous axiomatic treatment of both Euclidean and hyperbolic geometry. In both cases, it will be helpful to consider an axiom system with a bit more structure than just the given axioms to help facilitate our proofs. For this reason, we will often consider an **axiom system together with set theory and the theory of real numbers**. That is, we will postulate an axiom system just as in the above example, but we will supplement the system by allowing the use of set-theoretic terms and symbols (sets, $\in$, $\subseteq$, $\cup$, $\cap$, $\setminus$, etc.) as well as the basic technology associated with the set of real numbers ($\mathbb{R}$, $+$, $-$, $\cdot$, $/$, exponentiation, logarithms, etc.). When working in such a system, we specify for each primitive whether it should be interpreted as an *object* (or set), a *function*, or a *relation*.

The following is a very natural example which should be familiar to the student from linear algebra.

Theory of a Vector Space (with Set Theory and the Real Numbers).

For undefined primitives, we take *vectors*, *vector spaces*, $\vec{+}$ ("vector plus"), and $\vec{\cdot}$ ("scalar product with"). Vectors and vector spaces are to be interpreted as objects (sets), while $\vec{+}$ is to be interpreted as a function of two vectors and $\vec{\cdot}$ as a function of a real number and a vector.

We have used the grotesque symbols $\vec{+}$ and $\vec{\cdot}$ purely to differentiate them (formally) from the operations $+$ and $\cdot$ which are already defined on the real numbers. But since the meanings of $+$ and $\cdot$ may always be disambiguated from the given context in the theory of vector spaces, we give the following definitions in order to dispense with the cumbersome notation.

Definition 2.3. If $\vec{v}$ and $\vec{w}$ are vectors, we define the string of symbols $\vec{v} + \vec{w}$ to mean $\vec{v} \mathbin{\vec{+}} \vec{w}$. If a is a real number and $\vec{v}$ is a vector, we define the string of symbols $a \cdot \vec{v}$ to mean $a \mathbin{\vec{\cdot}} \vec{v}$.

Now we postulate our axioms.

Vector Space Axiom 1. *There exists a vector space, and all vectors are elements of this vector space.*

Vector Space Axiom 2. *For all vectors $\vec{v}$ and $\vec{w}$, $\vec{v} + \vec{w} = \vec{w} + \vec{v}$.*

Vector Space Axiom 3. *For all vectors $\vec{v}$, $\vec{w}$, and $\vec{u}$, $(\vec{v} + \vec{w}) + \vec{u} = \vec{v} + (\vec{w} + \vec{u})$.*

Vector Space Axiom 4. *There exists a vector $\vec{0}$ with the property that $\vec{0} + \vec{v} = \vec{v}$ for all vectors $\vec{v}$.*

Vector Space Axiom 5. *For every vector $\vec{v}$, there exists a vector $\vec{w}$ for which $\vec{v} + \vec{w} = \vec{0}$.*

Vector Space Axiom 6. *For every vector $\vec{v}$, $1 \cdot \vec{v} = \vec{v}$.*

Vector Space Axiom 7. *For every vector $\vec{v}$, for any two real numbers a and b, $(a+b) \cdot \vec{v} = a \cdot \vec{v} + b \cdot \vec{v}$.*

Vector Space Axiom 8. *For every real number a, for all vectors $\vec{v}$ and $\vec{w}$, $a \cdot (\vec{v} + \vec{w}) = a \cdot \vec{v} + a \cdot \vec{w}$.*

Vector Space Axiom 9. *For every vector $\vec{v}$, for any two real numbers a and b, $a \cdot (b \cdot \vec{v}) = (ab) \cdot \vec{v}$.*

Exercise 2.4. Prove that $\vec{v} + \vec{v} = 2 \cdot \vec{v}$ for each vector $\vec{v}$.

Exercise 2.5. Prove that $\vec{v} + \vec{0} = \vec{v}$ for each vector $\vec{v}$.

Exercise 2.6. Prove that $\vec{0}$ is *unique*, in the sense that if there are vectors $\vec{v}$ and $\vec{w}$ for which $\vec{v} + \vec{w} = \vec{w}$, then $\vec{v} = \vec{0}$.

Exercise 2.7. Prove that for each $\vec{v}$, there is a unique vector $\vec{w}$ for which $\vec{v} + \vec{w} = \vec{0}$. (Therefore it makes sense to denote this unique vector $\vec{w}$ by $-\vec{v}$.)

Exercise 2.8. Prove that $(-1) \cdot \vec{v} = -\vec{v}$ for any vector $\vec{v}$. (See the definition of $-\vec{v}$ in previous exercise.)

3 Incidence Geometry

Here we define our first barebones axiom system for proving geometric theorems. Our primitive terms for **incidence geometry** are *point*, *line*, and *lies on.*

Definition 3.1. We say a line ℓ **contains** a point A if A lies on ℓ. Two lines are said to **intersect** or **meet** if there is a point that lies on both lines. Two lines are **parallel** if they do not meet. A collection of points is **collinear** if there is a line that contains them all (and **noncollinear** otherwise).

Incidence Geometry Axiom 1. *There exist at least three distinct noncollinear points.*

Incidence Geometry Axiom 2. *Given any two distinct points, there is at least one line that contains both of them.*

Incidence Geometry Axiom 3. *Given any two distinct points, there is at most one line that contains both of them.*

Incidence Geometry Axiom 4. *Given any line, there are at least two distinct points that lie on it.*

Incidence Geometry Theorem 1. *If two distinct lines intersect, then there is exactly one point which lies on both lines.*

Proof. Let ℓ and m be distinct lines which intersect. Then by definition, there is a point A which lies on both ℓ and m. If we assume that there is a second point B lying on both ℓ and m, then we would have two distinct lines containing two distinct points, in violation of Axiom 3. So A is the unique point lying on both ℓ and m. □

Incidence Geometry Theorem 2. *For each point, there exist at least two distinct lines containing it.*

Proof. Let A be a point. By Axiom 1, there are at least two points B and C, distinct from one another and distinct from A, so that A, B, and C are noncollinear. By Axiom 3, there is a line $\overleftrightarrow{AB}$ containing A and B, and a line $\overleftrightarrow{AC}$ containing A and C. If $\overleftrightarrow{AB} = \overleftrightarrow{AC}$, then we would have C lying on $\overleftrightarrow{AB}$, which would contradict the noncollinearity of A, B, and C. So we must have $\overleftrightarrow{AB} \neq \overleftrightarrow{AC}$, and thus $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are distinct lines containing A. □

Incidence Geometry Theorem 3. *There exist three distinct lines which do not share a common point.*

Proof. Exercise. □

4 Models of Axiom Systems

A **model** of an axiom system (informally) is a specific interpretation of its undefined primitives, which makes the axioms all true.

Example 4.1. Let the Fe's be the numbers 1, 2, and 3, and let the Fo's be the sets $\{1,2\}$, $\{2,3\}$, and $\{1,3\}$. Let "belongs to" be interpreted as the set theoretic containment symbol $\in$. Check that this defines a model for the theory of Fe-Fo's.

Example 4.2. Let the Fo's be the cities of Dallas, Fort Worth, and Denton. Let the Fe's be the highways I-635, I-35W, and I-35E. Let "belongs to" be interpreted as "passes through." Check that this is another model of the theory of Fe-Fo's.

Example 4.3. For any positive integer n, the set $\mathbb{R}^n$ becomes a model for the theory of a vector space, if we allow the "vectors" to be the elements of $\mathbb{R}^n$, and interpret the symbols $\vec{+}$ and $\vec{\cdot}$ as coordinate-wise addition and coordinate-wise multiplication, respectively.

Example 4.4. Let $C(\mathbb{R})$ denote the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$. Call the elements of $C(\mathbb{R})$ "vectors", and define $\vec{+}$ and $\vec{\cdot}$ as follows: $f \vec{+} g$ is the function which gives $(f \vec{+} g)(x) = f(x) + g(x)$ for all $x \in \mathbb{R}$, and $a \vec{\cdot} f$ is the function which gives $(a \vec{\cdot} f)(x) = af(x)$ for all $x \in \mathbb{R}$. Then $C(\mathbb{R})$ becomes a model for the theory of a vector space.

Two models of a given theory are called **isomorphic** if there exists a one-to-one correspondence (a **bijection**) between the elements of the one model and the elements of the other, which preserves the respective models' interpretations of all function primitives and all relation primitives. (This definition is informal but can be made rigorous– think of isomorphic groups or isomorphic vector spaces; it means that the models are essentially the same.)

Exercise 4.5. Check that the two models of Fe-Fo theory given above are isomorphic.

Exercise 4.6. Conversely, check that $\mathbb{R}^n$ and $\mathbb{R}^m$ are not isomorphic as models of the theory of a vector space, if $n \neq m$. Likewise $C(\mathbb{R})$ is not isomorphic to $\mathbb{R}^n$. (Hint: Consider a vector space basis for each space in question.)

Example 4.7 (Finite Models of Incidence Geometry)**.** The three-point plane; the four-point plane; the five-point plane; the Fano plane. (Lee pp. 26–29.)

Exercise 4.8. Show that any model of Fe-Fo theory is also a model of incidence geometry.

Example 4.9 (Finite Non-Models of Incidence Geometry)**.** The empty plane; the one-point plane; the three-point line; three-ring geometry; one-two geometry; the square. (Lee pp. 30–31.)

Example 4.10 (Infinite Models of Incidence Geometry)**.** The Cartesian plane; spherical geometry; the hyperboloid model.

Example 4.11 (A Model of Euclid's Postulates With Non-Intersecting Circles: The Rational Plane)**.** Define the **rational plane** as follows: let the points be all pairs (p, q) of numbers where p and q are both rational. Let the lines be subsets of the rational plane, consisting of all points (p, q) which satisfy $q = mp + b$ for some rational numbers m and b.

Check that the rational plane is a model for all of Euclid's postulates (and also a model for incidence geometry). However, if we follow along the proof of the Theorem of Euclid 1, and take the points $A = (0, 0)$ and $B = (1, 0)$ in the rational plane, we see that the circles a and b of radius 1 centered at A and B do not intersect at any point in the rational plane (since these circles would normally intersect at the Cartesian points $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ which are irrational on the second coordinate and thus not points in the plane). Thus Euclid's argument fails in this model, and indeed there is no equilateral triangle in the rational plane which has $(0, 0)$ and $(1, 0)$ for two of its vertices.

Example 4.12 (The Empty Model of Euclid's Postulates)**.** Note that the empty set is a model of Euclid's postulates, since none of the postulates specify the existence of any point or line, and thus all the postulates are satisfied vacuously.

5 Properties of Axiom Systems

Definition 5.1. An axiom system is called **consistent** is it is not possible to use the axioms to prove two theorems which directly contradict one another.

An axiom system which is not consistent is mathematically useless. This common wisdom is exemplified by the following fact.

Fact 5.2. *Let $\mathcal{A}$ be an inconsistent axiom system. Then every statement is a theorem of $\mathcal{A}$.*

Proof. Let Q be any statement; we will show $\mathcal{A}$ proves Q.

Since $\mathcal{A}$ is inconsistent, it proves a contradiction, i.e. a statement of the form "P and not P" for some statement P. Since $\mathcal{A}$ proves "not P," it also proves "not P, or Q." It follows that $\mathcal{A}$ proves "P, and also (not P, or Q)," and hence $\mathcal{A}$ proves "Q." □

So an inconsistent axiom system thinks every statement is true. Since we are interested in distinguishing truth from falsehood, we are really only interested in consistent axiom systems. On the other hand, the consistency of a system is a slippery fish. How can one *prove* that an axiom system can never prove a contradiction? The question is *metamathematical*, because to prove something about an axiom system, one needs a "larger" axiom system in which it is possible to prove statements about the "smaller" axiom system. This leads to the notion of relative consistency.

Definition 5.3. If $\mathcal{A}$ and $\mathcal{B}$ are axiom systems, then $\mathcal{A}$ is called **consistent relative to** $\mathcal{B}$ if it is a theorem of $\mathcal{B}$ that $\mathcal{A}$ is consistent.

Considerations of this type have lead to a lot of deep and beautiful mathematics, especially in the 20th century. We will only make note of the following, which is a special formulation of a fundamental theorem of Kurt Gödel, and helps us understand the significance of the existence of a model.

Theorem 5.4 (Gödel's Completeness Theorem (1929))**.** *An axiom system with only finitely many axioms is consistent (relative to the axioms of set theory), if and only if there exists a (set-theoretic) model of the system.*

Students who are not interested in the foundations of set theory, but who want to understand the relationship between consistency and existence of a model on an intuitive level, may omit the parenthetical sections in the theorems above and below.

We gave several examples of models of Fe-Fo theory and vector space theory in the previous section. Thus we deduce the following as a corollary of Gödel's theorem.

Corollary 5.5. *Both Fe-Fo theory and the theory of a vector space are consistent (relative to set theory).*

Definition 5.6. Let $\mathcal{A}$ be an axiom system. An individual statement P is called **independent** of $\mathcal{A}$ if it is not possible to prove or disprove P using only the axioms of $\mathcal{A}$. An axiom system $\mathcal{A}$ is called **independent** if each of its axioms is independent from the others in $\mathcal{A}$.

The following is another corollary of Gödel's completeness theorem.

Theorem 5.7. *Suppose $\mathcal{A}$ is a consistent axiom system (relative to set theory). Then a statement P is independent of $\mathcal{A}$ only if there exists a model in which $\mathcal{A}$ is true and P is true, and also there exists a model in which $\mathcal{A}$ is true but P is false.*

Exercise 5.8. In the theory of a vector space, give an appropriate definition (in terms of the primitives) of the term *linearly independent*. Then show that for each positive integer n, the statement "There exists a linearly independent set with at least n elements" is independent of the theory of a vector space.

Definition 5.9. An axiom system is called **categorical** if all of its models are mutually isomorphic (i.e. it essentially has only one model).

Exercise 5.10. Show that Fe-Fo theory is categorical. Show that incidence geometry is not categorical.

Example 5.11. Let $\mathcal{IG}$ denote the axioms of incidence geometry. Then the axiom system $\mathcal{IG}+$"There exist exactly four distinct points" is categorical. On the other hand, the axiom system $\mathcal{IG}+$"There exist exactly seven distinct points" is not categorical (consider the 7-point plane vs. the Fano plane).

Definition 5.12. Two statements P and Q are called **equivalent** relative to an axiom system $\mathcal{A}$, if $\mathcal{A}$ together with P implies Q and $\mathcal{A}$ together with Q implies P.

6 Historical Notes: Independence of the Euclidean Parallel Postulate

Euclid (approx. 300 BCE). Wrote the *Elements*, history's first and most influential attempt at a purely axiomatic development of mathematics. Introduced Euclid's Postulate 5 as a self-evident axiom despite its dramatically long and cumbersome formulation. This postulate (together with its many equivalent formulations) is now known as the **Euclidean parallel postulate**. Euclid may have anticipated criticism of this postulate, as he did not invoke its use in his proofs until the 29th theorem of Book I of the *Elements*. In these notes we follow Euclid's tradition by delaying involvement of the parallel postulate for as long as possible.

Proclus (412–485 CE). A Greek philosopher-mathematician who wrote a very influential commentary and critique on Euclid's *Elements*. Proclus believed that the Euclidean parallel postulate was certainly true, but should be a *theorem* rather than an axiom, i.e. he believed it should be provable from Euclid's first four postulates together with the common notions. He attempted to do this in his commentaries, but there was a gap in his proof: he tacitly assumed (without justification from an axiom) that parallel lines are everywhere *equidistant*, i.e. the distance between them is always the same. (See Theorem 13.3.) In fact this assumption turns out to be equivalent to the Euclidean parallel postulate.

Omar Khayyam (1048–1123 CE). A Persian philosopher-mathematician-poet. Author of the book *Explanations of the Difficulties in the Postulates of Euclid's Elements*. Khayyam was the first critic of Euclid who did not attempt to prove the parallel postulate from the other axioms, but instead offered a *substitute* axiom which he considered more natural. The new axiom was based on one of Aristotle's principles, and essentially amounted (in modern terms) to the assumption that parallel lines are equidistant. (Khayyam's assumption was a big improvement on Proclus's, because Khayyam's was intentional and Proclus's was by accident.) Khayyam also considered geometric objects of the following nature:

Definition 6.1. A **Saccheri quadrilateral** is a quadrilateral with two congruent opposite sides, which are simultaneously perpendicular to a third side.

Khayyam proved that under his new axiom, every Saccheri quadrilateral is in fact a rectangle. From this he also deduced the Euclidean parallel postulate. It is now known that the statement "Every Saccheri quadrilateral is a rectangle" is equivalent to the parallel postulate.

Giovanni Girolamo Saccheri (1667–1733 CE). An Italian Jesuit priest and philosopher-mathematician. Author of the proudly-named *Euclides ab omni naevo vindicatus*, which translates to *Euclid freed of every flaw*! Saccheri resumed Proclus's efforts by setting out to prove the parallel postulate from Euclid's other axioms.

Saccheri proceeded in a proof by contradiction: he assumed the parallel postulate was false, and intended to deduce some logical absurdity. His arguments were quite rigorous for his day, and he proved a number of interesting theorems based on the negation of the parallel postulate, for instance that (1) Saccheri quadrilaterals are *not* rectangles and (2) the sum of the three angle measures of a triangle is always *strictly less* than 180°. At the end of his treatise, he proved that if the Euclidean parallel postulate is false, then there must exist parallel lines which approach closer and closer together but never meet. At this point he declared his proof by contradiction complete, because such a phenomenon is **"...repugnant to the nature of straight lines!"**

Of course this does not constitute a contradiction in any mathematically rigorous sense, unless one assumes that parallel lines are everywhere equidistant a la Proclus and Khayyam, which assumption turns out to be equivalent to the parallel postulate in the first place.

John Playfair (1748–1819 CE). A Scottish mathematician. Published an edition of Euclid's *Elements* in which he changed Euclid's Postulate 5 to the following formulation:

Playfair's Postulate. *Two straight lines cannot be drawn through the same point, parallel to the same line, without coinciding with one another.*

In other words, gives a line and a point not on the line, there is at most one line through the given point which is parallel to the given line. This axiom is highly intuitive and places the emphasis on the uniqueness of parallel lines. Playfair showed it is equivalent to the original Euclidean parallel postulate.

Karl Friedrich Gauss (1777-1855 CE). The *Princeps mathematicorum*, or *prince of mathematicians*. In his private work prior to 1830, he investigated the mathematical consequences of assuming the negation of the parallel postulate in geometry (i.e. he assumed that given a point and a line, there are *at least two* lines passing through the given point parallel to the given line). This leads to the development of what he termed **non-Euclidean geometry**. Perhaps fearing backlash from the academic and religious community of the time, he never published his results.

Nicolai Lobachevsky (1792–1856 CE). A Russian mathematician. The first to publish a paper (1829) with a fully fleshed out development of non-Euclidean geometry based on the assumption that the Euclidean parallel postulate is false. Lobachevsky worked from the following assumption, which is now known as the **hyperbolic parallel postulate** (though the terminology came about many years later):

Hyperbolic Parallel Postulate. *Given a point and a line, there are at least two lines which pass through the given point and are parallel to the given line.*

Lobachevsky was the first mathematician both brilliant enough to rigorously study the consequences of the hyperbolic parallel postulate, and brave enough to publish his results. For this reason non-Euclidean geometry is often called **Lobachevskian geometry**.

Janos Bolyai (1802–1860 CE). A Hungarian mathematician who developed the non-Euclidean geometry completely independently of Lobachevski and Gauss. His father, the mathematician Farkas Bolyai, once wrote to Janos regarding Janos's obsession with the Euclidean parallel postulate: *"For God's sake, I beseech you, give it up. Fear it no less than sensual passions because it too may take all your time and deprive you of your health, peace of mind and happiness in life."*

Janos published his results in 1832 as an appendix to one of his father's books. He wrote to his father on the non-Euclidean geometry, *"I created a new, different world out of nothing."*

Farkas Bolyai was a friend of Gauss and wrote Gauss on Janos's new geometry. Gauss replied, *"To praise it would amount to praising myself. For the entire content of the work ... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years."* (In another letter to a different friend, however, Gauss said of Janos Bolyai, *"I regard this young geometer Bolyai as a genius of the first order."*

Eugenio Beltrami (1835–1900 CE). An Italian mathematician. He delivered the *coup de grace* for attempts to prove the fifth postulate, by demonstrating that there is a model for Euclidean geometry (i.e. geometry with the assumption of the Euclidean parallel postulate) if and only if there is a model for Lobachevskian geometry (i.e. geometry with the assumption of the negation of the Euclidean parallel postulate). Therefore Euclidean geometry is consistent if and only if Lobachevskian geometry is.

We also deduce from Beltrami's work that both the Euclidean parallel postulate and the hyperbolic parallel postulate are fully independent of Euclid's other axioms.

Felix Klein (1849–1925 CE). A German mathematician and founder of the *Erlangen program*, the program to study different types of geometry by understanding their underlying symmetry groups. He coined the term **hyperbolic geometry** as a synonym for Lobachevskian geometry.

7 Historical Notes: Modern Axiomatizations of Euclidean Geometry

The discovery that the Euclidean parallel postulate is independent of the other usual axioms of geometry was instrumental in revolutionizing the way mathematicians practice mathematics. Shortly after these developments, Georg Cantor (1845–1914 CE) proved his theorems of set theory (including the famous diagonal argument) which was another massive shock to the system. But Cantor's naive theorem-proving led to paradoxes and contradictions (like Russell's paradoxical set $\{x : x \notin x\}$), which spurred leading scholars to carefully and rigorously axiomatize set theory. This program eventually resulted in the modern ZFC (Zermelo-Fraenkel plus Axiom of Choice) axioms for set theory which are generally accepted as a suitable foundation for mathematics.

Simultaneously, since Euclid's *Elements* was the ancient progenitor of axiomatic mathematics, there was a program to re-axiomatize Euclidean geometry in a modern and rigorous way.

David Hilbert (1862–1943 CE). Possibly the most reputable mathematician of the turn of the century. He gave the first collection of modern axioms for geometry (*Gründlagen der Geometrie* 1899) which are sufficient to prove all of the theorems of Euclid's *Elements*.

To have a sense of Hilbert's axioms, consider first Euclid's postulates. Note that although Euclid clearly had length and angle measurements in mind, he never once mentions such things, nor makes any reference to the real numbers at all as a frame of reference. This is probably largely because Euclid's development of geometry predates the concept of irrational numbers, and thus there was no reasonable way to assign a number to the length of every line segment. Thus Euclid's postulates, which avoid any reference to numbers or measurements, is an axiomatization of what we now call **synthetic geometry**.

Hilbert followed Euclid's tradition and gave a synthetic axiomatization of geometry. In place of length and angle measurements, he introduced the notions of *congruence* and *betweenness*. Although his axioms were elegant, the difficulty of working with these synthetic notions made Hilbert's development for geometry somewhat cumbersome: there were 20+ given axioms, and many theorems were devoted to spelling out intuitively clear notions about congruence and betweenness before more interesting theorems could be approached.

George Birkhoff (1884-1944 CE). In 1932 this American mathematician proposed a different approach to axiomatizing geometry from Hilbert's synthetic approach. Birkhoff developed an axiom system which allows for the use of real numbers as tools of measurement (see the remarks in Section 2 on axiom systems with set theory and the real numbers). In other words he formalized the notions that we can measure lengths and angles using a ruler and a protractor. This approach may be called **metric geometry**. An advantage of Birkhoff's approach is that he was able to reduce Hilbert's 20+ axioms down to just *four*. A disadvantage is that with such a minimalist set of axioms, it can become a lengthy and tedious process to prove all necessary theorems from scratch.

The School Mathematics Study Group (1960s CE). A committee formed to revise mathematics curricula in the US. They sought an axiomatic approach to geometry which was both (1) rigorously based in metric ("ruler and protractor") geometry a la Birkhoff, and (2) rich enough in axioms that one can start teaching interesting theorems to students without spending a long tedious development on relatively basic notions. The committee developed a list of 22 axioms, called the SMSG axioms, which became the predominant axiom system used by US teachers in the latter half of the twentieth century. Although the SMSG axioms are pedagogically very useful, they have a weakness: the axioms are not all mutually independent, i.e. some of the 22 axioms may actually be proven from the other given axioms. This sacrifice in minimality is made in order to speed up the development of more advanced geometric theorems.

The axioms which we will now use to begin developing geometry are given by John M. Lee, and they are essentially a trimmed down variation of the SMSG postulates. Lee proves in his book that each of his axioms is independent of the previous axioms (respecting the order in which they are introduced).

8 Axioms for Neutral Geometry: Ruler Postulates and Lemmas

We are ready to begin axiomatizing geometry. We will build an axiom system which utilizes set theory and the real numbers (see Section 2). Our undefined primitives are **points** and **lines**, **distance** (to be interpreted as a function of pairs of points), and **measure** (to be interpreted as a function of an angle, a term we will define later on).

We will eventually give nine axioms, not including any parallel postulate. Following Euclid's example, we will delay the involvement of a parallel postulate for as long as possible. The theorems we prove without invoking any parallel postulate are called theorems of **neutral geometry**, because they are truth in both Euclidean and non-Euclidean geometry.

Neutral Geometry Postulate 1 (Set Postulate)**.** *Every line is a set of points, and there is a set of all points called the **plane**.*

Eventually once all our axioms are postulated, we will see that any model of neutral geometry is also a model of incidence geometry. Bearing this in mind, we give a quick definition.

Definition 8.1. A point A **lies on** a line ℓ if $A \in \ell$.

The previous definition, combined with Definition 3.1, now gives meaning to the terms **contains**, **intersect**, **meet**, **parallel**, **collinear**, and **noncollinear**.

Definition 8.2. If two lines ℓ and m are parallel we will write $\ell \parallel m$.

Neutral Geometry Postulate 2 (Existence Postulate). *There at exist at least three distinct noncollinear points.*

Neutral Geometry Postulate 3 (Unique Line Postulate). *Given any two distinct points, there is a unique line containing both of them.*

Definition 8.3. If A and B are points, then the unique line containing A and B is denoted $\overleftrightarrow{AB}$.

Neutral Geometry Postulate 4 (Distance Postulate). *For every pair of points A and B, the distance from A to B is a nonnegative real number determined by A and B.*

Definition 8.4. If A and B are points, the distance between A and B is denoted AB.

Let us briefly recall some set-theoretic definitions.

Definition 8.5. If X and Y are sets, a function $f : X \to Y$ is called **injective** or **one-to-one** if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. A function $f : X \to Y$ is called **surjective** or **onto** if for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$. A function $f : X \to Y$ is called a **bijection** if it is both an injection and a surjection.

Neutral Geometry Postulate 5 (Ruler Postulate). *For every line ℓ, there is a bijection $f : \ell \to \mathbb{R}$ with the property that for any two points $A, B \in \ell$, we have $AB = |f(B) - f(A)|$.*

Definition 8.6. If ℓ is a line, a function $f : \ell \to \mathbb{R}$ is called **distance-preserving** if $AB = |f(B) - f(A)|$ for all points $A, B \in \ell$.

For each line ℓ, a bijective distance-preserving function is called a **coordinate function** for ℓ. (Thus existence of coordinate functions is guaranteed by the Ruler Postulate.)

Proposition 8.7. *Every line contains infinitely many distinct points.*

Proof. If ℓ is a line, then by the Ruler Postulate there is a bijection $f : \ell \to \mathbb{R}$. Since f is a surjection, for each positive integer n we can find a point $A_n \in \ell$ with the property that $f(A_n) = n$. If $n \neq m$, then since f is an injection, $A_n \neq A_m$. It follows that the set $\{A_1, A_2, A_3, ...\}$ is an infinite collection of pairwise distinct points in ℓ. □

Corollary 8.8. *Every model of neutral geometry is a model of incidence geometry.*

Proof. Suppose $\mathcal{M}$ is a model of neutral geometry; so $\mathcal{M}$ is a set together with some interpretations of the neutral geometry primitives *point*, *line*, *distance*, and *measure*. To view $\mathcal{M}$ as a model of incidence geometry, we need to have some interpretations of the incidence geometry primitives *point*, *line*, and *lies on*. We do this in the obvious way: we let *points* be the given points of $\mathcal{M}$, *lines* be the given lines of $\mathcal{M}$, and we interpret *lies on* as the set theoretic symbol $\in$.

Since $\mathcal{M}$ satisfies the Existence Postulate 2, it also satisfies Incidence Geometry Postulate 1. Since $\mathcal{M}$ satisfies the Unique Line Postulate 3, it also satisfies Incidence Geometry Postulates 2 and 3. Lastly, by Proposition 8.7, each line in $\mathcal{M}$ contains at least two distinct points, so Incidence Geometry Postulate 4 is satisfied as well. Thus $\mathcal{M}$ is a model of incidence geometry as claimed. □

Lemma 8.9 (Ruler Sliding Lemma). *Suppose ℓ is a line and $f : \ell \to \mathbb{R}$ is a coordinate function for ℓ. Given a real number c, define a new function $f_1 : \ell \to \mathbb{R}$ by $f_1(X) = f_1(X) + c$ for all $X \in \ell$. Then f_1 is also a coordinate function for ℓ.*

Proof. We need to show f_1 is both a bijection and distance-preserving.

f_1 is injective: Let $A, B \in \ell$ with $A \neq B$. Since f is injective, $f(A) \neq f(B)$. It follows that $f(A) + c \neq f(B) + c$, and hence $f_1(A) \neq f_1(B)$. So f_1 is one-to-one.

f_1 is surjective: Let $y \in \mathbb{R}$. Then $y - c \in \mathbb{R}$, and hence since f is surjective, there is a point $A \in \ell$ for which $f(A) = y - c$. Then $f_1(A) = f(A) + c = y - c + c = y$. So f_1 is onto.

f_1 is distance-preserving: Let $A, B \in \ell$. Then $|f_1(B) - f_1(A)| = |(f(B) + c) - (f(A) + c)| = |f(B) + c - f(A) - c| = |f(B) - f(A)| = AB$ (the latter equality holds because f is distance-preserving). This shows f_1 is distance-preserving and proves the lemma. □

Lemma 8.10 (Ruler Flipping Lemma). *Suppose ℓ is a line and $f : \ell \to \mathbb{R}$ is a coordinate function for ℓ. If we define $f_2 : \ell \to \mathbb{R}$ by $f_2(X) = -f(X)$ for all $X \in \ell$, then f_2 is also a coordinate function for ℓ.*

Proof. Exercise. □

Theorem 8.11 (Ruler Placement Theorem). *Suppose ℓ is a line, and A, B are two distinct points on ℓ. Then there exists a coordinate function $f : \ell \to \mathbb{R}$ such that $f(A) = 0$ and $f(B) > 0$.*

Proof. By the Ruler Postulate 5, there is a coordinate function $g : \ell \to \mathbb{R}$. Then $g(A) \in \mathbb{R}$, and hence by the Ruler Sliding Lemma 8.9, the function $g_1 : \ell \to \mathbb{R}$ defined by $g_1(X) = g(X) - g(A)$ is also a coordinate function, and it clearly satisfies $g_1(A) = 0$.

Because $B \neq A$ and g_1 is injective, we have $g_1(B) \neq 0$. So either $g_1(B) > 0$ or $g_1(B) < 0$. If $g_1(B) > 0$, then set $f = g_1$ and we have found our desired f. Otherwise if $g_1(B) < 0$, then let f be the function defined by $f(X) = -g_1(X)$ for each $X \in \ell$. By the Ruler Flipping Lemma 8.10, f is a coordinate function, and $f(B) = -g_1(B) > 0$, so again we have found our desired f. □

Proposition 8.12. *If A and B are any two points, then (1) $AB = BA$, and (2) $AB = 0$ if and only if $A = B$.*

Exercise 8.13. Prove Proposition 8.12 above.

9 Segments, Midpoints, and Rays

Definition 9.1. Let A, B, C be collinear points, so A, B, C all lie on some line ℓ. We say B is **between** A and C, and we write $A * B * C$, if there exists a coordinate function $f : \ell \to \mathbb{R}$ for which $f(A) < f(B) < f(C)$.

Proposition 9.2. *If A, B, C are points lying on a line ℓ, and there exists a coordinate function $f : \ell \to \mathbb{R}$ satisfying $f(C) < f(B) < f(A)$, then $A * B * C$.*

Proof Sketch. This is an easy application of the Ruler Flipping Lemma 8.10. □

Corollary 9.3. *If A, B, C are points, then $A * B * C$ if and only if $C * B * A$.*

Theorem 9.4. *If A, B, C are points and $A * B * C$, then $AB + BC = AC$.*

Proof. Since $A * B * C$, there exist a line ℓ containing A, B, C and a coordinate function $f : \ell \to \mathbb{R}$ with $f(A) < f(B) < f(C)$. Since f is distance-preserving, we have $AB = |f(B) - f(A)| = f(B) - f(A)$, $BC = |f(C) - f(B)| = f(C) - f(B)$, and $AC = |f(C) - f(A)| = f(C) - f(A)$. The theorem now follows from a direct computation. □

Theorem 9.5. *Given three distinct collinear points, exactly one of them lies between the other two.*

Proof. Let A, B, C be distinct points lying on a line ℓ, and let $f : \ell \to \mathbb{R}$ be a coordinate function. Since f is one-to-one, $f(A)$, $f(B)$, and $f(C)$ are distinct real numbers; by relabeling points if necessary, we may assume $f(A) < f(B) < f(C)$. Then we have $A * B * C$, i.e. one of the points is between the other two. We still need to show it is not the case that $B * A * C$ or $A * C * B$. But this follows easily from Theorem 9.4: if we suppose for a contradiction that $B * A * C$, then $AC = AB + BC$ and also $BC = AB + AC$, whence we deduce $AC = AB + AB + AC > AC$, an absurdity. (Here we are also using the fact that $AB > 0$ since A, B are distinct.) By a similar argument, it cannot be the case that $A * C * B$; so B is the only point lying between the other two. □

Definition 9.6. Given two points A and B, the **line segment** connecting A and B is the set $\overline{AB} = \{C \in \overleftrightarrow{AB} : A * C * B\} \cup \{A\} \cup \{B\}$.

Two line segments $\overline{AB}$ and $\overline{CD}$ are called **congruent** if $AB = CD$, and we write $\overline{AB} \cong \overline{CD}$.

The fact that segment congruence is an equivalence relation is obvious from the definition and the fact that equality of real numbers is an equivalence relation.

Lemma 9.7 (Segment Extension Lemma (Euclid's Postulate 2))**.** *If $\overline{AB}$ is any segment, there exist points $C, D \in \overleftrightarrow{AB}$ such that $C * A * B$ and $A * B * D$.*

Proof. By Postulate 5 let $f : \overleftrightarrow{AB} \to \mathbb{R}$ be a coordinate function. Assume $f(A) < f(B)$; we may do this without loss of generality by the Ruler Flipping Lemma 8.10. Find real numbers $x, y \in \mathbb{R}$ satisfying $x < f(A) < f(B) < y$. Since f is a surjection, there exist points $C, D \in \overleftrightarrow{AB}$ satisfying $f(C) = x$ and $f(D) = y$. Then $C * A * B$ and $A * B * D$ as claimed. ☐

Definition 9.8. Given distinct points A and B, the **ray from** A **through** B is the set $\overrightarrow{AB} = \{X \in \overleftrightarrow{AB} : X = A \text{ or } X = B \text{ or } A * X * B \text{ or } A * B * X\}$. The point A is called the **starting point** of $\overrightarrow{AB}$. If $\overrightarrow{r}$ is any ray, we always denote the unique line containing $\overrightarrow{r}$ by $\overleftrightarrow{r}$.

Note it is obvious from the definitions that $\overline{AB} \subseteq \overrightarrow{AB} \subseteq \overleftrightarrow{AB}$ and $\overline{AB} \subseteq \overrightarrow{BA} \subseteq \overleftrightarrow{AB}$.

Lemma 9.9 (Segment Construction Lemma)**.** *Let $\overrightarrow{r}$ be a ray starting at a point O and let x be a nonnegative real number. Then there exists a unique point $A \in \overrightarrow{r}$ such that $OA = x$.*

Proof. Let B be a point on $\overrightarrow{r}$ distinct from O, and by Theorem 8.11 let $f : \overleftrightarrow{r} \to \mathbb{R}$ be a coordinate function satisfying $f(O) = 0$ and $f(B) > 0$. Since f is a bijection, there is a unique $A \in \overleftrightarrow{r}$ satisfying $f(A) = x$. Since $f(O) = 0$ and $f(B) > 0$, and $f(A) \geq 0$, we have either $O = A$ or $O * A * B$ or $A = B$ or $O * B * A$; in any of these four cases $A \in \overrightarrow{r}$. And $OA = |f(A) - f(O)| = |x - 0| = x$. ☐

Definition 9.10. A point M is said to be a **midpoint** of a segment $\overline{AB}$ if $A * M * B$ and $\overline{AM} \cong \overline{MB}$.

Theorem 9.11 (Midpoint Existence)**.** *Every segment has a unique midpoint.*

Exercise 9.12. Prove Theorem 9.11 above.

10 Axioms for Neutral Geometry: Plane Separation, Angles, and Measures

Neutral Geometry Postulate 6 (Plane Separation Postulate)**.** *For any line ℓ, the set of all points not on ℓ is the disjoint union of two subsets of the plane called the **sides** of ℓ. If A and B are two distinct points not on ℓ, then A and B are on the same side of ℓ if and only if $\overline{AB} \cap \ell = \emptyset$.*

Definition 10.1. For any line ℓ and any point P not on ℓ, let $\mathrm{CHP}(\ell, P)$ denote the union of ℓ with the side of ℓ containing P. (The notation stands for **closed half-plane**.) If ℓ is a line, $O \in \ell$, and $\overrightarrow{r}$ is a ray starting at O which does not lie entirely on ℓ, then we also denote by $\mathrm{CHP}(\ell, \overrightarrow{r})$ the closed half-plane $\mathrm{CHP}(\ell, P)$, where P is any point on $\overrightarrow{r}$ distinct from O. (Exercise: check that this definition does not depend on the choice of point P.)

Definition 10.2. An **angle** is the union of two rays with a common endpoint. If the two rays are denoted $\overrightarrow{a}$ and $\overrightarrow{b}$, we denote the angle by $\angle ab$. If the two rays are denoted by $\overrightarrow{OA}$ and $\overrightarrow{OB}$, we denote the angle by $\angle AOB$. The common endpoint is called the **vertex** of the angle, and the rays are called the **sides**.

Definition 10.3. Two rays are called **collinear** if they are both subsets of the same line.

Proposition 10.4. *If two rays $\overrightarrow{a}$ and $\overrightarrow{b}$ share a common vertex O, then either (1) $\overrightarrow{a} = \overrightarrow{b}$, or else (2) $\overrightarrow{a} \cap \overrightarrow{b} = \{O\}$.*

Definition 10.5. If $\overrightarrow{a}$ and $\overrightarrow{b}$ are collinear rays which share a common vertex O, and $\overrightarrow{a} \cap \overrightarrow{b} = \{O\}$, then we say $\overrightarrow{a}$ and $\overrightarrow{b}$ are **opposite rays**.

Definition 10.6. If the two rays $\overrightarrow{a}$ and $\overrightarrow{b}$ of an angle $\angle ab$ are opposite, then $\angle ab$ is called a **straight angle**. If $\overrightarrow{a} = \overrightarrow{b}$, then $\angle ab$ is called a **zero angle**. An angle which is neither a straight angle nor a zero angle is a **proper angle**.

Neutral Geometry Postulate 7 (Angle Measure Postulate)**.** *For every angle $\angle ab$, the measure of $\angle ab$ is a real number lying in the closed interval $[0, 180]$ determined by $\angle ab$.*

Definition 10.7. If the measure of an angle $\angle ab$ is the real number $x \in [0, 180]$, we say that the measure of $\angle ab$ is x **degrees**, and we write $m\angle ab = x^\circ$.

Two angles $\angle ab$ and $\angle cd$ are **congruent**, denoted $\angle ab \cong \angle cd$, if $m\angle ab = m\angle cd$.

Definition 10.8. Given a ray $\overrightarrow{r}$ and a point P not on the line $\overleftrightarrow{r}$ containing $\overrightarrow{r}$, the **half-rotation** determined by $\overrightarrow{r}$ and P, denoted by $\mathrm{HR}(\overrightarrow{r}, P)$, is the set of all rays which share a common vertex with $\overrightarrow{r}$ and which are contained in the closed half-plane $\mathrm{CHP}(\overleftrightarrow{r}, P)$.

Neutral Geometry Postulate 8 (Protractor Postulate)**.** *For every ray $\overrightarrow{r}$ and every point P not on the line $\overleftrightarrow{r}$ containing $\overrightarrow{r}$, there is a bijection $g : \mathrm{HR}(\overrightarrow{r}, P) \to [0, 180]$ which satisfies: (1) $g(\overrightarrow{r}) = 0$; (2) if $\overrightarrow{s}$ is opposite to $\overrightarrow{r}$ then $g(\overrightarrow{s}) = 180$; and (3) if $\overrightarrow{a}$ and $\overrightarrow{b}$ are any two rays in $\mathrm{HR}(\overrightarrow{r}, P)$ then* $m\angle ab = |g(\overrightarrow{b}) - g(\overrightarrow{a})|$.

Definition 10.9. Every such function g which satisfies the conditions in the Protractor Postulate 8 for some ray $\overrightarrow{r}$ and point P is called a **coordinate function** for $\mathrm{HR}(\overrightarrow{r}, P)$.

We say a ray $\overrightarrow{b}$ is **between** two rays $\overrightarrow{a}$ and $\overrightarrow{c}$ if there exists a half-rotation containing all three of $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$, and a coordinate function g for this half-rotation, such that $g(\overrightarrow{a}) < g(\overrightarrow{b}) < g(\overrightarrow{c})$, and we write $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$.

(*Note:* Our definition of betweenness of rays differs slightly from Lee's, who makes the additional requirement that no two of the rays $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ be collinear.)

Proposition 10.10. *If $\angle ab$ is an angle, then (1) $m\angle ab = m\angle ba$; (2) $m\angle ab = 0^\circ$ if and only if $\angle ab$ is a zero angle; and (3) $m\angle ab = 180^\circ$ if and only if $\angle ab$ is a straight angle.*

Exercise 10.11. Prove Proposition 10.10 above.

Theorem 10.12 (Congruence of Vertical Angles)**.** *If $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, and $\overrightarrow{d}$ are four distinct rays starting from a common point O, and $\overrightarrow{a}$ is opposite to $\overrightarrow{c}$ and $\overrightarrow{b}$ is opposite to $\overrightarrow{d}$, then $\angle ab \cong \angle cd$.*

Proof omitted.

Lemma 10.13 (Angle Construction Lemma)**.** *Let O be any point, let $\overrightarrow{a}$ be a ray with vertex O, and let $x \in [0, 180]$. On each side of $\overleftrightarrow{a}$, there is a unique ray $\overrightarrow{b}$ with vertex O such that $m\angle ab = x$.*

Proof. Choose a point P on either side of $\overrightarrow{r}$, and by Postulate 8 let $g : \mathrm{HR}(\overrightarrow{r}, P)$ be a coordinate function. Since g is a bijection, there is a unique ray $\overrightarrow{b} \in \mathrm{HR}(\overrightarrow{r}, P)$ for which $g(\overrightarrow{b}) = x$. Then $m\angle ab = |g(\overrightarrow{b}) - g(\overrightarrow{a})| = |x - 0| = x$. □

Proposition 10.14. *If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are rays no two of which are collinear, then $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$ if and only if $\overrightarrow{c} * \overrightarrow{b} * \overrightarrow{a}$. In either case we have $m\angle ac = m\angle ab + m\angle bc$.*

Exercise 10.15. Prove Proposition 10.14 above. (*Hint:* Formulate and prove a "Protractor Flipping Lemma" analogous to Lemma 8.10.)

Proposition 10.16. *If $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ are rays satisfying $\overrightarrow{a} * \overrightarrow{b} * \overrightarrow{c}$, then for every half-rotation $\mathrm{HR}(\overrightarrow{r}, P)$ which contains $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$, and for every coordinate function $g : \mathrm{HR}(\overrightarrow{r}, P) \to [0, 180]$, either $g(\overrightarrow{a}) < g(\overrightarrow{b}) < g(\overrightarrow{c})$ or $g(\overrightarrow{c}) < g(\overrightarrow{b}) < g(\overrightarrow{a})$.*

Exercise 10.17. Prove Proposition 10.16 above.

Definition 10.18. An angle $\angle ab$ is called a **right angle** if $\mathrm{m}\angle ab = 90°$, an **acute angle** if $\mathrm{m}\angle ab < 90°$, and a **obtuse angle** if $\mathrm{m}\angle ab > 90°$. Two angles $\angle ab$ and $\angle cd$ are called **congruent** if $\mathrm{m}\angle ab = \mathrm{m}\angle cd$.

Definition 10.19. A ray $\vec{r}$ is called an **angle bisector** of an angle $\angle ab$ if $\vec{a} * \vec{r} * \vec{b}$ and $\mathrm{m}\angle ar = \mathrm{m}\angle rb$.

Lemma 10.20. *Every proper angle has a unique angle bisector.*

Proof. This is a direct corollary of Lemma 10.13. □

Definition 10.21. Two proper angles $\angle ab$ and $\angle cd$ are called **complementary** if $\mathrm{m}\angle ab + \mathrm{m}\angle cd = 90°$, and **supplementary** if $\mathrm{m}\angle ab + \mathrm{m}\angle cd = 180°$.

Two angles form a **linear pair** if they share a side, and their non-shared sides are opposite rays.

Proposition 10.22. *If two angles form a linear pair, then they are supplementary.*

Exercise 10.23. Prove Proposition 10.22.

Corollary 10.24. *If two angles in a linear pair are congruent, then they are both right angles.*

Lemma 10.25. *If $\angle ab$, $\angle bc$, and $\angle cd$ are angles such that $\vec{a} * \vec{b} * \vec{c}$, $\vec{b} * \vec{c} * \vec{d}$, and $\vec{a}, \vec{d}$ are opposite rays (i.e. the three angles form a **linear triple**) then $\mathrm{m}\angle ab + \mathrm{m}\angle bc + \mathrm{m}\angle cd = 180°$.*

Exercise 10.26. Prove Lemma 10.25.

Definition 10.27. The **interior** of an angle $\angle AOB$ is the set of all points which are on the same side of $\overleftrightarrow{OB}$ as A, and on the same side of $\overleftrightarrow{OA}$ as B. We denote the interior by $\mathrm{Int}(\angle AOB)$.

The following lemma is of fundamental importance and may seem "obviously true," but it requires careful proof. The proof is not difficult but it is a fairly long and tedious digression; for this reason, we omit it from these notes. If the student is interested in the details of the proof, we recommend consulting the very clear and well-written arguments leading up to the corresponding Lemma 4.23 in John M. Lee's book.

Lemma 10.28 (Interior Lemma)**.** *Suppose $\vec{a}$, $\vec{b}$, and $\vec{c}$ are rays starting from a common point O, no two of which are collinear. Then $\vec{a} * \vec{b} * \vec{c}$ if and only if $\vec{b} \setminus \{O\} \subseteq \mathrm{Int}(\angle ac)$.*

Corollary 10.29 (Betweenness vs. Betweenness Theorem)**.** *Suppose ℓ is a line, O is a point not on ℓ, and $A, B, C \in \ell$. Then $A * B * C$ if and only if $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.*

Proof. First assume $A * B * C$. Then the line $\overleftrightarrow{AB}$ meets $\overleftrightarrow{OC}$ only at the point C (Postulate 3, and hence $\overline{AB}$ does not meet $\overleftrightarrow{OC}$ since $C \notin \overline{AB}$. Thus B is on the same side of $\overleftrightarrow{OC}$ as A by Postulate 6.

Let X be any point in $\overrightarrow{OB}$ distinct from O, so either $O * X * B$ or $X = B$ or $O * B * X$. In any case, $O \notin \overline{XB}$. Since $\overleftrightarrow{XB}$ meets $\overleftrightarrow{OC}$ only at the point O (Postulate 3), we see that $\overline{XB}$ does not meet $\overleftrightarrow{OC}$. So X is on the same side of $\overleftrightarrow{OC}$ as B by Postulate 6, and therefore X lies on the same side of $\overleftrightarrow{OC}$ as A.

A similar argument, with the roles of A and C interchanged, shows that X lies on the same side of $\overleftrightarrow{OA}$ as C. So $X \in \mathrm{Int}(\angle AOC)$. Since X was chosen arbitrarily from $\overrightarrow{OB} \setminus \{O\}$, we have $\overrightarrow{OB} \setminus \{O\} \subseteq \mathrm{Int}(\angle AOC)$. Now Lemma 10.28 implies $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$.

Conversely, assume $\overrightarrow{OA} * \overrightarrow{OB} * \overrightarrow{OC}$. By Theorem 9.5, exactly one of the collinear points A, B, C lies between the other two. Now Lemma 10.28 implies that $B \in \mathrm{Int}(\angle AOC)$. In other words, B is on the same side of $\overleftrightarrow{OC}$ and A and B is on the same side of $\overleftrightarrow{OA}$ as C. So $\overline{AB}$ does not intersect $\overleftrightarrow{OC}$ and $\overline{BC}$ does not intersect $\overleftrightarrow{OA}$. It follows that C is not between A and B and A is not between B and C; so by process of elimination, it must be the case that $A * B * C$. □

11 Axioms for Neutral Geometry: Triangles and Congruence

Definition 11.1. A **triangle** is the union of three segments $\overline{AB}$, $\overline{BC}$, and $\overline{AC}$, called **sides** or **edges**, formed by three noncollinear points A, B, C, called **vertices**. We denote such a triangle $\triangle ABC$. The **angles of the triangle** are the angles $\angle BAC$, $\angle ABC$, and $\angle ACB$. Whenever no ambiguity can result, we simply denote them $\angle A$, $\angle B$, and $\angle C$.

Lemma 11.2 (The Crossbar Theorem)**.** *Suppose $\triangle ABC$ is a triangle and $\overrightarrow{AD}$ is a ray between $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Then $\overrightarrow{AD}$ intersects $\overline{BC}$ at a point Y in such a way that $B * Y * C$.*

Proof. First, we claim that B and C are on opposite sides of $\overleftrightarrow{AD}$. To see this, suppose for the sake of a contradiction that they are on the same side. Then $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are both elements of the half-rotation $\text{HR}(\overrightarrow{AD}, B)$, and hence there is a coordinate function $g : \text{HR}(\overrightarrow{AD}, B)$ with $g(\overrightarrow{AB}) > 0$ and $g(\overrightarrow{AC}) > 0$. But since $g(\overrightarrow{AD}) = 0$, we have either $g(\overrightarrow{AD}) < g(\overrightarrow{AB}) < g(\overrightarrow{AC})$ or else $g(\overrightarrow{AD}) < g(\overrightarrow{AC}) < g(\overrightarrow{AB})$, which violates Proposition 10.16. This proves the claim.

So in fact B and C are on opposite sides of $\overleftrightarrow{AD}$, whence $\overline{BC}$ intersects $\overleftrightarrow{AD}$ at some point Y by Postulate 6.

It remains to check that $Y \in \overrightarrow{AD}$, and that $B * Y * C$. It is obvious that $Y \neq A$ since A, B, C are not collinear. Thus, the Interior Lemma 10.28 implies that $D \in \text{Int}(\angle BAC)$. In particular D is on the same side of $\overleftrightarrow{AB}$ as C. In addition, $\overline{CY}$ does not intersect $\overleftrightarrow{AB}$, since B is the unique point of intersection between $\overleftrightarrow{CY}$ and $\overleftrightarrow{AB}$ and $B \notin \overline{CY}$ (Why?). Hence C and Y are on the same side of $\overleftrightarrow{AB}$. So D and Y are on the same side of $\overleftrightarrow{AB}$. We deduce then from Postulate 6 that $A \notin \overline{DY}$, and hence it is not the case that $D*A*Y$. This means either $A*Y*D$ or $D = Y$ or $A*D*Y$, and in any case $Y \in \overrightarrow{AD}$ as claimed.

It also follows that $\overrightarrow{AY} = \overrightarrow{AD}$, and now the Betweenness vs. Betweenness Theorem 10.29 implies that $B * Y * C$. ☐

Lemma 11.3 (Pasch's Theorem)**.** *Suppose $\triangle ABC$ is a triangle and ℓ is a line that does not contain any of the points A, B, or C. If ℓ intersects one of the sides of $\triangle ABC$, then it also intersects another side.*

Proof. Assume by relabeling points if necessary that ℓ intersects $\overline{AB}$. If it also intersects $\overline{BC}$, we are done, so assume the opposite. Then B and C are on the same side of ℓ by Postulate 6, while A and B are on opposite sides. So A and C are on opposite sides, whence $\overline{AC}$ intersects ℓ by Postulate 6. ☐

Definition 11.4. Two triangles are **congruent** if there is a one-to-one correspondence between the the vertices of one triangle with the vertices of the other, in such a way that all three pairs of corresponding angles are congruent and all three pairs of corresponding sides are congruent. If $\triangle ABC$ and $\triangle DEF$ are triangles and we write $\triangle ABC \cong \triangle DEF$, it means that not only are the triangles congruent, but the congruence is given by the specific correspondence $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$.

Neutral Geometry Postulate 9 (SAS Postulate)**.** *If there is a correspondence between the vertices of two triangles such that two sides and the included angle of one triangle are congruent to the corresponding sides and angle of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 11.5 (ASA Congruence)**.** *If there is a correspondence between the vertices of two triangles such that two angles and the included side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent under that correspondence.*

Proof. Suppose $\triangle ABC$ and $\triangle DEF$ are triangles such that $\angle A \cong \angle D$, $\angle B \cong \angle E$, and $\overline{AB} \cong \overline{DE}$. If $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$ by the SAS Postulate 9, and we are done. So assume for the sake of contradiction that $\overline{AC} \ncong \overline{DF}$. Thus either $AC > DF$ or $DF > AC$; assume without loss of generality that $AC > DF$.

By the Ruler Placement Theorem 8.11, find a coordinate function $f : \overleftrightarrow{AC} \to \mathbb{R}$ satisfying $f(A) = 0$ and $f(C) > 0$. Let $C' \in \overleftrightarrow{AC}$ be such that $f(C') = DF$; then since $f(A) = 0$, $f(C') = DF$, and $f'(C) = AC$, we have $A * C' * C$ and thus $C' \in \overline{AC}$. The two triangles $\triangle ABC'$ and $\triangle DEF$ satisfy the conditions in the SAS Posulate 9, so they are congruent. Thus $\angle ABC' \cong \angle E$. Since $A * C' * C$, the Betweenness vs. Betweenness Theorem 10.29 implies that $\overrightarrow{BA} * \overrightarrow{BC'} * \overrightarrow{BC}$, and therefore $\text{m}\angle ABC > \text{m}\angle ABC'$. Thus we get $\text{m}\angle ABC > \text{m}\angle E$, contradicting our hypothesis, and ruling out the possibility that $\overline{AC}$ and $\overline{DF}$ are not congruent. □

Exercise 11.6. Prove that relative to Postulates 1–8 of neutral geometry, the SAS Postulate 9 is logically equivalent to the ASA Theorem 11.5. In other words, prove that if we assume Postulates 1–8 plus the ASA Theorem 11.5 as an axiom, then it is possible to prove the SAS Postulate 9 as a theorem. (*Hint:* Mimic the proof of Theorem 11.5, but exchange segments for angles and vice versa.)

The student should remember the following five theorems from their middle school geometry course. We will omit the proofs here.

Theorem 11.7 (SSS Congruence). *If there is a correspondence between the vertices of two triangles such that all three sides of one triangle are congruent to the corresponding sides of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 11.8 (SAA Congruence). *If there is a correspondence between the vertices of two triangles such that two angles and a nonincluded side of one triangle are congruent to the corresponding angles and side of the other triangle, then the triangles are congruent under that correspondence.*

Theorem 11.9 (Triangle Inequality). *If A, B, and C are any three points (not necessarily distinct), then $AC \leq AB + BC$, and equality holds if and only if $A = B$, $B = C$, or $A * B * C$.*

Theorem 11.10 (Isosceles Triangle Theorem). *If $\triangle ABC$ is a triangle, then $\overline{AB} \cong \overline{AC}$ if and only if $\angle B \cong \angle C$. (Such a triangle is called **isosceles**.)*

Theorem 11.11 (Scalene Inequality). *If $\triangle ABC$ is a triangle, then $AB > AC$ if and only if $\text{m}\angle C > \text{m}\angle B$.*

Theorem 11.12 (Hinge Theorem). *Suppose two triangles $\triangle ABC$ and $\triangle DEF$ satisfy $\overline{AB} \cong \overline{DE}$, $\overline{AC} \cong \overline{DF}$, and $\text{m}\angle A > \text{m}\angle D$. Then $BC > EF$.*

Proof. By Lemma 10.13, there is a ray $\overrightarrow{r}$ starting at A on the same side of $\overleftrightarrow{AB}$ as C such that the angle formed between $\overrightarrow{OB}$ and $\overrightarrow{r}$ is congruent to $\angle D$. By Lemma 9.9, let P be a point on $\overrightarrow{r}$ satisfying $\overline{AP} \cong \overline{DF}$. Then P is on the same side of $\overleftrightarrow{AB}$ as C, and $\triangle ABP \cong \triangle DEF$ by SAS Posulate 9. In particular $EF = BP$.

Now let $\overrightarrow{s}$ be an angle bisector of $\angle CAP$ (Lemma 10.20), and by the Crossbar Theorem 11.2, find a point S at the intersection of $\overrightarrow{s}$ and $\overline{BC}$. Now $\angle CAS \cong \angle SAP$, $\overline{AC} \cong \overline{DF} \cong \overline{AP}$, and $\overline{AS} = \overline{AS}$, so SAS Postulate 9 implies $\triangle ACS \cong \triangle APS$. In particular $CS = PS$. So by Theorem 9.4 and the triangle inequality (Theorem 11.9), $EF = BP < BS + PS = BS + CS = BC$. □

Lemma 11.13 (Exterior Angle Inequality). *Let $\triangle ABC$ be a triangle and let P, Q be points such that $A * B * P$ and $C * B * Q$. Then $\angle ABQ \cong \angle CBP$, and both angles are strictly greater in measure than either of the angles $\angle A$ or $\angle C$.*

Proof. The fact that $\angle ABQ \cong \angle CBP$ is a direct consequence of Theorem 10.12. So we only need to show $\text{m}\angle CBP > \text{m}\angle A$ and $\text{m}\angle CBP > \text{m}\angle C$.

We begin with the second inequality. Let M be the midpoint of $\overline{BC}$ (Theorem 9.11) and let X be the point on the ray opposite $\overrightarrow{MA}$ such that $MX \cong AM$ (Lemma 9.9). Then $\angle BMX \cong \angle CMA$ by Theorem 10.12, and hence the SAS Postulate 9 implies that $\triangle BMX \cong \triangle CMA$. In particular, $\angle MBX \cong \angle MCA$.

Now we wish to show that $\overrightarrow{BX}$ is between $\overrightarrow{BM}$ and $\overrightarrow{BP}$. Because $\overline{AP}$ intersects $\overleftrightarrow{BM}$ at B, we know that A and P are on opposite sides of $\overleftrightarrow{BM}$. Likewise since $\overline{AX}$ intersects $\overleftrightarrow{BM}$ at M, we know A and X

are on opposite sides of $\overleftrightarrow{BM}$. It follows that P and X are on the same side of $\overleftrightarrow{BM}$, and hence $\overrightarrow{BX}$ and $\overrightarrow{BP}$ are subsets of the same side of $\overleftrightarrow{BM}$. On the other hand $\overline{MX}$ is disjoint from $\overleftrightarrow{BP}$ (since $A \notin \overline{MX}$), so M and X are on the same side of $\overleftrightarrow{BP}$, and hence $\overrightarrow{BM}$ and $\overrightarrow{BX}$ are subsets of the same side of $\overleftrightarrow{BP}$. It then follows from Lemma 10.28 that $\overrightarrow{BM} * \overrightarrow{BX} * \overrightarrow{BP}$ as claimed.

Thus we conclude that $\text{m}\angle MBP = \text{m}\angle MBX + \text{m}\angle XBP$, whence $\text{m}\angle CBP = \text{m}\angle MBP > \text{m}\angle MBX = \text{m}\angle MCA = \text{m}\angle C$. A similar argument shows $\text{m}\angle ABQ > \text{m}\angle A$ and concludes the proof of the lemma. □

Corollary 11.14. *The sum of the measures of any two angles of a triangle is strictly less than* $180°$*.*

Proof. Let $\triangle ABC$ be any triangle; we consider the quantity $\text{m}\angle A + \text{m}\angle B$. Use Lemma 9.7 to find a point D such that $A * B * D$. The angle $\angle CBD$ is exterior, so Lemma 11.13 implies $\text{m}\angle CBD > \text{m}\angle A$. But $\angle ABC$ and $\angle CBD$ form a linear pair, so $\text{m}\angle CBD = 180° - \text{m}\angle ABC$ by Proposition 10.22. Substituting yields the inequality

$$180° - \text{m}\angle B > \text{m}\angle A,$$

which is equivalent to the inequality we want to prove. □

Definition 11.15. Recall that two lines ℓ and m are **parallel** if they do not intersect.

Suppose ℓ and m are lines (not necessarily parallel) and t intersects both in distinct points; then t is called **transversal** to ℓ and m. In this situation, the intersections of t with ℓ and m respectively make eight distinct angles. Say $t \cap \ell = \{A\}$ and $t \cap m = \{B\}$ where $A \neq B$. Then the **interior angles** are the two angles made by t and ℓ on the same side of ℓ as B, and the two angles made by t and m on the same side of m as A. The **exterior angles** are the other four angles. Any two of these angles are called **corresponding** if they are either both interior or both exterior and lie on the same side of t; they are called **alternate** if they are both interior or both exterior and lie on opposite sides of t. Two angles are **corresponding** if one is interior and the other is exterior, and they both lie on the same side of the transversal.

Theorem 11.16 (Alternate Interior Angles Theorem)**.** *If two lines are cut by a transversal making a pair of alternate interior angles congruent, then the lines are parallel.*

Proof. Let ℓ and m be lines cut by a transversal t which meets them at A and B respectively. We will proceed with the proof by *contrapositive*: that is, we will assume that t and m are in fact *not* parallel, and then show that neither pair of alternate interior angles can be congruent.

So assume t and m are not parallel; thus they meet at some point C and we have a triangle $\triangle ABC$. Choose points D and E so that $D * A * C$ and $E * B * C$ (Lemma 9.7). By the exterior angle inequality (Lemma 11.13), $\text{m}\angle DAB > \text{m}\angle ABC$ and $\text{m}\angle EBA > \text{m}\angle BAC$. Thus neither pair of alternate interior angles is congruent. □

Corollary 11.17 (Corresponding Angles Theorem)**.** *If two lines are cut by a transversal making a pair of corresponding angles congruent, then the lines are parallel.*

Exercise 11.18. Use Theorem 11.16 to prove Corollary 11.17 above.

Definition 11.19. Two distinct lines ℓ and m are called **perpendicular** if they intersect at some point O, and one of the rays in ℓ starting at O makes an angle of $90°$ with one of the rays in m starting at O. In this case we write $\ell \perp m$.

Lemma 11.20. *If two lines ℓ and m are perpendicular, then all four of the angles they form are right.*

Exercise 11.21. Prove Lemma 11.20.

Lemma 11.22 (Dropping a Perpendicular)**.** *Let ℓ be a line and let P be a point not on ℓ. Then there exists a unique line m that is perpendicular to ℓ at P.*

Proof. We need to show two things: first that there exists such a line m, and second that this line is unique.

First we show existence. Find three points $B, C, D \in \ell$ satisfying $B * C * D$. By Lemma 10.13, there is a unique ray $\overrightarrow{r}$ on the other side of ℓ from P, which makes an angle with $\overrightarrow{CD}$ congruent to $\angle PCD$. Let P' be the point on $\overrightarrow{r}$ satisfying $CP' = CP$ (Lemma 9.9). Let $M = \overleftrightarrow{PP'}$; then $P \in m$, and we claim $m \perp \ell$.

To see this, note that P and P' are on opposite sides of ℓ, and hence there is a point E where $\overline{PP'}$ intersects ℓ. A slightly subtle point here is that we don't know whether or not $C = E$, or whether $E \in \overrightarrow{CD}$ or $E \in \overrightarrow{CB}$, so we must check several cases.

First suppose $C = E$. Note then that $\angle PED = \angle PCD$ and $\angle P'ED = \angle P'CD$, and therefore $\angle PED \cong \angle P'ED$ since $\angle PCD \cong \angle P'CD$. But also $\angle PED$ and $\angle P'ED$ form a linear pair; so $\angle PED$ and $\angle P'ED$ are both right by Corollary 10.24. This shows $\ell \perp m$.

On the other hand, suppose $C \neq E$, and assume $E \in \overrightarrow{CD}$. Then since $\angle PCD \cong \angle P'CD$, $\overline{PC} \cong \overline{P'C}$, and $\overline{CE} = \overline{CE}$, we may conclude by SAS Postulate 9 that $\triangle PCE \cong \triangle P'CE$. It follows that $\angle PEC \cong \angle P'EC$, but these latter angles also form a linear pair, so they are both right by Corollary 10.24. Hence $\ell \perp m$.

Next assume $C \neq E$ but $E \in \overrightarrow{CB}$. Note that $\angle PCD$ and $\angle PCB$ form a linear pair, so they are supplementary by Proposition 10.22. Likewise $\angle P'CD$ and $\angle P'CB$ are supplementary. So since $\angle PCD \cong \angle P'CD$, it follows from these observations that $\mathrm{m}\angle PCB = 180° - \mathrm{m}\angle PCD = 180° - \mathrm{m}\angle P'CD = \mathrm{m}\angle P'CB$, i.e. $\angle PCB \cong \angle P'CB$. Again $\overline{PC} \cong \overline{P'C}$ and $\overline{CE} = \overline{CE}$, so again SAS Postulate 9 implies $\triangle PCE \cong \triangle P'CE$, and we may conclude the argument as in the previous paragraph. This covers all the cases, so indeed we have found a line m through P satisfying $\ell \perp m$.

Next we need to show m is the unique line perpendicular to ℓ through P. Suppose for the sake of contradiction that there is another such line n distinct from m. Let E be the point where ℓ meets m and let F be the point where n meets m. E and F are distinct points since $m \neq n$. Thus there is a triangle $\triangle PEF$. But then $\mathrm{m}\angle E = 90°$ and $\mathrm{m}\angle F = 90°$ by Lemma 11.20, contradicting Corollary 11.14. □

12 Surprising Neutral Theorems: Angle Defects and Saccheri Quadrilaterals

Theorem 12.1 (Saccheri-Legendre)**.** *The sum of the measures of all three angles of a triangle is less than or equal to* $180°$.

Proof. Let $\triangle ABC$ be an arbitrary triangle, and set $\sigma = \mathrm{m}\angle A + \mathrm{m}\angle B + \mathrm{m}\angle C$. We claim the following: there exists a triangle $\triangle A_1B_1C_1$ with the property that $\mathrm{m}\angle A_1 + \mathrm{m}\angle B_1 + \mathrm{m}\angle C_1 = \sigma$, but such that $\mathrm{m}\angle A_1 \leq \frac{1}{2}\mathrm{m}\angle A$.

To see this, let M be the midpoint of $\overline{BC}$ (Theorem 9.11), and let P be a point so that $A * M * P$. Let D be the point on the ray $\overrightarrow{MP}$ satisfying $AM = MD$ (Lemma 9.9. Now $\angle AMC \cong \angle DMB$ by Theorem 10.12, $\overline{AM} \cong \overline{MD}$, and $\overline{BM} \cong \overline{MC}$, so $\triangle AMC \cong \triangle DMB$ by the SAS Postulate 9. It follows that $\angle BCA \cong \angle DBM$ and $\angle CAM \cong \angle MDB$.

Since $B * M * C$ and $A * M * D$, Theorem 10.29 implies that $\overrightarrow{AB} * \overrightarrow{AM} * \overrightarrow{AC}$ and $\overrightarrow{BA} * \overrightarrow{BM} * \overrightarrow{BD}$. From this, Proposition 10.14 tells us $\mathrm{m}\angle BAC = \mathrm{m}\angle BAM + \mathrm{m}\angle CAM$ and $\mathrm{m}\angle ABD = \mathrm{m}\angle ABM + \mathrm{m}\angle DBM$. Using all the equalities we have derived, we get

$$\begin{aligned}\sigma &= \text{m}\angle BAC + \text{m}\angle ABC + \text{m}\angle BCA \\ &= \text{m}\angle BAM + \text{m}\angle CAM + \text{m}\angle ABC + \text{m}\angle BCA \\ &= \text{m}\angle BAM + \text{m}\angle MDB + \text{m}\angle ABM + \text{m}\angle DBM \\ &= \text{m}\angle BAM + \text{m}\angle MDB + \text{m}\angle ADB \\ &= \text{m}\angle BAD + \text{m}\angle ADB + \text{m}\angle ADB.\end{aligned}$$

So triangle $\triangle ABD$ has the same angle sum as $\triangle ABC$. Moreover, since $\text{m}\angle BAD + \text{m}\angle ADB = \text{m}\angle BAM + \text{m}\angle CAM = \text{m}\angle A$, we have either $\text{m}\angle BAD \leq \frac{1}{2}\text{m}\angle A$ or else $\text{m}\angle ADB \leq \frac{1}{2}\text{m}\angle A$. In the first case we take $A_1 = A$, $B_1 = B$, and $C_1 = D$; in the second case we take $A_1 = D$, $B_1 = B$, and $C_1 = A$. In either case the claim is proved.

Now it follows, by repeating the construction that there exists a triangle $\triangle A_2B_2C_2$ with the property that $\text{m}\angle A_2 + \text{m}\angle B_2 + \text{m}\angle C_2 = \text{m}\angle A_1 + \text{m}\angle B_1 + \text{m}\angle C_1 = \sigma$, and $\text{m}\angle A_2 \leq \frac{1}{2}\text{m}\angle A_1 \leq \frac{1}{4}\text{m}\angle A$.

In fact, repeating the construction as many times as needed, we get that for each positive integer n, there exists a triangle $\triangle A_nB_nC_n$ satisfying $\text{m}\angle A_n + \text{m}\angle B_n + \text{m}\angle C_n = \sigma$, and $\text{m}\angle A_n \leq \frac{1}{2^n}\text{m}\angle A$.

Now assume for a contradiction that $\sigma > 180°$. Let $\epsilon = \sigma - 180$, so $\epsilon > 0$. Let N be a positive integer so large that $\dfrac{1}{2^N} < \dfrac{\epsilon}{\text{m}\angle A}$. Then find the triangle $\triangle A_NB_NC_N$. Note that since $\text{m}\angle A_N \leq \frac{1}{2^N}\text{m}\angle A < \epsilon$ and $\text{m}\angle A_N + \text{m}\angle B_N + \text{m}\angle C_N = (180 + \epsilon)°$, we must have $\text{m}\angle B_N + \text{m}\angle C_N > 180°$, contradicting Corollary 11.14. Thus we are assured that $\sigma \leq 180°$.

□

Definition 12.2. The **defect** of a triangle $\triangle ABC$ is the quantity $\delta(\triangle ABC) = 180° - (\text{m}\angle A + \text{m}\angle B + \text{m}\angle C)$. By the Saccheri-Legendre Theorem, the defect of any triangle is a nonnegative number (possibly equal to zero).

Definition 12.3. A **quadrilateral** is the union of four line segments of the form $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, where none of these segments intersect except at A, B, C, or D, and where no three of the points A, B, C, D are collinear. We denote such a quadrilateral by $ABCD$ (note that following the middle school tradition, we list all vertices in sequential order, so $CDBA$ is an acceptable notation but $CDAB$ is not).

The segments $\overline{AC}$ and $\overline{BD}$ in a quadrilateral $ABCD$ are called the **diagonals**. $ABCD$ is called **convex** if its diagonals intersect. A convex quadrilateral has four **interior angles**: $\angle DAB$, $\angle ABC$, $\angle BCD$, and $\angle CDA$.

A quadrilateral is called a **rectangle** if all of its interior angles are right.

Theorem 12.4. *If $ABCD$ is a convex quadrilateral, then the sum of the measures of its four interior angles is less than or equal to* $360°$.

Proof. Since $ABCD$ is convex, let M be the point where $\overline{AC}$ and $\overline{BD}$ intersect. Since $B * M * D$, Theorem 10.29 implies that $\overrightarrow{AD} * \overrightarrow{AM} * \overrightarrow{AB}$ and $\overrightarrow{CD} * \overrightarrow{CA} * \overrightarrow{CB}$. It follows then from Proposition 10.14 that $\text{m}\angle CAB + \text{m}\angle DAC = \text{m}\angle DAB$ and $\text{m}\angle BCA + \text{m}\angle ACD = \text{m}\angle BCD$. Then

$$\begin{aligned}\text{m}\angle A + \text{m}\angle B + \text{m}\angle C + \text{m}\angle D &= (\text{m}\angle CAB + \text{m}\angle DAC) + \text{m}\angle ABC + (\text{m}\angle BCA + \text{m}\angle ACD) + \text{m}\angle CDA \\ &= (\text{m}\angle ABC + \text{m}\angle CBA + \text{m}\angle CAB) + (\text{m}\angle DAC + \text{m}\angle ACD + \text{m}\angle CDA) \\ &\leq 180° + 180° = 360°\end{aligned}$$

by the Saccheri-Legendre Theorem 12.1 applied to triangles $\triangle ABC$ and $\triangle ACD$. □

Definition 12.5. The **defect** of a convex quadrilateral $ABCD$ is the quantity $\delta(ABCD) = 360° - (\text{m}\angle DAB + \text{m}\angle ABC + \text{m}\angle BCD + \text{m}\angle CDA)$.

Theorem 12.6. *If $ABCD$ is a convex quadrilateral, then $\delta(ABCD) = \delta(\triangle ABC) + \delta(\triangle BCD)$.*

Exercise 12.7. Prove Theorem 12.6.

Definition 12.8. Recall from Definition 6.1 that a quadrilateral $ABCD$ is called a **Saccheri quadrilateral** if it has two congruent sides $\overline{AD}$ and $\overline{BC}$, and the angles $\angle A$ and $\angle B$ are right angles.

The segments $\overline{AD}$ and $\overline{BC}$ are called the **legs**, the segment $\overline{AB}$ is called the **base**, and the segment $\overline{DC}$ is the **summit**. The angles $\angle A$ and $\angle B$ are the **base angles** and the angles $\angle C$ and $\angle D$ are the **summit angles**.

Proposition 12.9. *The summit angles of a Saccheri quadrilateral are congruent.*

Exercise 12.10. Prove the previous proposition.

Corollary 12.11. *A Saccheri quadrilateral is a rectangle if and only if its defect is* $0°$.

Exercise 12.12. Prove the previous corollary.

Proposition 12.13. *The summit angles of a Saccheri quadrilateral are not obtuse, and thus they are either both acute or both right.*

Exercise 12.14. Prove the previous proposition.

Proposition 12.15. *The line joining the midpoints of the base and summit of a Saccheri quadrilateral are perpendicular to both.*

Exercise 12.16. Prove the previous proposition.

Corollary 12.17. *The summit and the base of a Saccheri quadrilateral are parallel.*

Proof. As a consequence of the preceding proposition together with Theorem 11.16. □

Theorem 12.18. *In any Saccheri quadrilateral, the length of the summit is greater than or equal to the length of the base.*

Proof. Let $ABCD$ be a Saccheri quadrilateral with base $\overline{AB}$; we wish to show $CD \geq AB$. Consider the angles $\angle ADB$ and $\angle CBD$; either they are congruent, or $\text{m}\angle ADB < \text{m}\angle CBD$, or $\text{m}\angle ADB > \text{m}\angle CBD$.

If they are congruent, then $\triangle ABC \cong \triangle CBD$ by the SAS Postulate 9, in which case $\overline{CD} \cong \overline{AB}$ and we are done.

Alternatively, if $\text{m}\angle ADB < \text{m}\angle CBD$, then $CD > AB$ by the Hinge Theorem 11.12 and we are done again.

Lastly, suppose $\text{m}\angle ADB > \text{m}\angle CBD$. Now $\text{m}\angle ABD + \text{m}\angle CBD = \text{m}\angle ABC = 90°$, and this implies $\text{m}\angle ADB + \text{m}\angle ABD + \text{m}\angle BAD > \text{m}\angle CBD + \text{m}\angle ABD + 90° = 180°$, contradicting the Saccheri-Legendre Theorem 12.1 and showing that this last scenario is impossible. □

13 Euclidean Geometry: Triangles and Rectangles

We are finally ready to assume some form of the Euclidean parallel postulate, and see what kind of theorems we can deduce as a result. For the next six sections including this one, we officially assume the following axiom.

Playfair's Postulate. *For each line ℓ and each point P that does not lie on ℓ, there is a unique line that contains P and is parallel to ℓ.*

Later we will show that relative to the axioms of neutral geometry, Playfair's Postulate is equivalent to Euclid's Postulate 5. For this reason, theorems proved with this additional assumption will be labeled Euclidean Geometry Theorems.

Euclidean Geometry Theorem 13.1 (Converse to the Alternate Interior Angles Theorem). *If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent.*

Proof. Suppose ℓ and m are cut by a transversal t, and let A and B denote the points where t meets ℓ and m, respectively. Choose either pair of alternate interior angles, and choose points $C \in \ell$ and $D \in m$ so that the chosen angles are $\angle CAB$ and $\angle ABD$ respectively.

By Lemma 10.13, there is a ray $\overrightarrow{AE}$ on the same side of t as D that makes an angle with BA congruent to $\angle CAB$. By the Alternate Interior Angles Theorem 11.16, the line $\overleftrightarrow{AE}$ is parallel to ℓ. By Playfair's Postulate, $\overleftrightarrow{AE} = m$. So $E \in \overleftrightarrow{BD}$ and hence $\angle ABD = \angle ABE$. Therefore $\angle CAB \cong \angle ABD$. □

Definition 13.2. Let ℓ be a line and P a point not on ℓ. By Lemma 11.22, there is a unique point $Q \in \ell$ such that $\overleftrightarrow{PQ} \perp \ell$. We define the **distance** between P and ℓ to be the distance PQ.

We call two lines ℓ and m **everywhere equidistant** if for any two points A and B on m, the distance from A to ℓ is equal to the distance from B to ℓ.

Euclidean Geometry Theorem 13.3 (Proclus's Tacit Assumption). *Parallel lines are everywhere equidistant.*

Proof. Let ℓ and m be parallel lines and let $A, B \in m$ be arbitrary. Drop perpendiculars to find points $C, D \in \ell$ with $\overleftrightarrow{AC} \perp \overleftrightarrow{\ell}$ and $\overleftrightarrow{BD} \perp \ell$. Now by Euclidean Theorem 13.1, the angles $\angle BAD$ and $\angle ADC$ are congruent, and the angles $\angle BAC$, $\angle ABC$, $\angle ACD$, $\angle DBC$ are all congruent as well. It follows then from the SAA Theorem 11.8 that $\triangle ABD \cong \triangle DCA$. So $\overline{AC} \cong \overline{BD}$, and $AC = BD$. This shows ℓ and m are everywhere equidistant. □

Euclidean Geometry Theorem 13.4 (There Is No Triangle Defect). *Every triangle has angle sum equal to* $180°$. *(Equivalently, the defect of every triangle is* $0°$.*)*

Proof. Let $\triangle ABC$ be a triangle, and by Playfair's postulate let ℓ be the unique line parallel to $\overleftrightarrow{BC}$ passing through A. Let D, E be points on ℓ so that $D * A * E$. By Euclidean Theorem 13.1, we have $\angle DAB \cong \angle ABC$ and $\angle EAC \cong \angle ACB$. But $\text{m}\angle DAB + \text{m}\angle CAB + \text{m}\angle EAC = 180°$ since the angles form a linear triple (Lemma 10.25). Substituting congruent angles, we get $\text{m}\angle ABC + \text{m}\angle CAB + \text{m}\angle ACB = 180°$, as claimed. □

Euclidean Geometry Corollary 13.5 (There Is No Quadrilateral Defect). *Every quadrilateral has angle sum* $360°$. *(Equivalently, the defect of every quadrilateral is* $0°$.*)*

Proof. By Euclidean Theorem 13.4 and Theorem 12.6. □

Euclidean Geometry Corollary 13.6. *Every Saccheri quadrilateral is a rectangle.*

Proof. By Euclidean Corollary 13.5 and Corollary 12.11. □

Euclidean Geometry Corollary 13.7. *A rectangle exists.*

Exercise 13.8. Prove Euclidean Corollary 13.7.

Definition 13.9. A **square** is a rectangle with four congruent sides.

Euclidean Geometry Corollary 13.10. *A square exists.*

Exercise 13.11. Prove Euclidean Corollary 13.10.

Euclidean Geometry Theorem 13.12 (Euclid's Postulate 5). *If* ℓ *and* m *are two lines cut by a transversal* t *in such a way that the measures of two consecutive interior angles add up to less than* $180°$, *then* ℓ *and* m *intersect on the same side of* t *as those two angles.*

Proof. Note that ℓ and m are not parallel, for if they were then Euclidean Theorem 13.1 would imply that their consecutive interior angles would have measures adding up to exactly $180°$, which is not the case. So ℓ and m intersect, say at a point C.

Let A and B denote the points where t meets ℓ and m respectively, and let D, E, F, and G be points on ℓ and m respectively satisfying $D * A * E$ and $F * B * G$. Assume (by relabeling if necessary) that $\text{m}\angle EAB + \text{m}\angle BGA < 180°$. It follows immediately from Proposition 10.22 that $\text{m}\angle DAB + \text{m}\angle FBA > 180°$. Suppose for the sake of contradiction that C is on the opposite side of t from $\angle EAB$ and $\angle BGA$; so C is on the same side as $\angle DAB$ and $\angle FBA$. But then $\triangle ABC$ is a triangle, and $\text{m}\angle CAB + \text{m}\angle CBA = \text{m}\angle DAB + \text{m}\angle FBA > 180°$, contradicting Corollary 11.14. $\square$

14 Euclidean Geometry: Remarks on Area

Before we can proceed to some of our most interesting theorems, we would like to rigorously formulate the intuitive concept of **area**. There are a few necessary intermediate steps for this to be done naturally: first one formulates the notion of a **polygon**, and a **polygonal region** is defined to be the union of the polygon itself together with the **interior** of the polygon. To define the interior of a polygon, however, one must first segue into the related notions of **convexity** and **nonconvexity** and carefully formulate what it means to be an **interior angle** of a polygon. Afterwards one defines a **general polygonal region** to be any finite union of polygonal regions, and clarifies what it means for two polygonal regions to be **congruent**, and to be **overlapping** or **non-overlapping**.

Unfortunately due to the unalterable finiteness of time, we are forced to make some sacrifices. We will forego the rigorous development of these concepts for the time being. But we still wish to work with them to prove many of our more interesting theorems, and thus henceforth we will rely on the student's intuitive notion of what the emboldened terms in the previous paragraph are *supposed* to mean. The student may rest assured that the terms *can be made* rigorous, and the especially interested student should take a look at Chapter 8 and Chapter 11 (especially the closing comments on page 209) of Lee's book to get a sense of how it is done.

We will now take the following definition:

Definition 14.1. Let $\mathcal{P}$ denote the set of all general polygonal regions (see previous remarks). An **area function** is a function $\alpha : \mathcal{P} \to (0, \infty)$ which satisfies (1) if $\mathcal{R}_1$ and $\mathcal{R}_2$ are congruent polygonal regions, then $\alpha(\mathcal{R}_1) = \alpha(\mathcal{R}_2)$, and (2) if $\mathcal{R}_1, \mathcal{R}_2, ..., \mathcal{R}_n$ are mutually non-overlapping polygonal regions, then $\alpha(\mathcal{R}_1 \cup \mathcal{R}_2 \cup ... \cup \mathcal{R}_n) = \alpha(\mathcal{R}_1) + \alpha(\mathcal{R}_2) + ... + \alpha(\mathcal{R}_n)$.

Technically we are taking an area function α to be a function of polygonal regions, not polygons. In general, however, to make our notation easier we will not really distinguish between the two; for instance if $\triangle ABC$ is a triangle, we will write $\alpha(\triangle ABC)$ to mean the area of the polygonal region determined by $\triangle ABC$.

Henceforth in Euclidean geometry we adopt the following as a postulate. The reader should note immediately that the notion of area given by the postulate below is a *strictly Euclidean* concept, as it depends on the notion of a square, which may or may not exist without the assumption of a Euclidean parallel postulate.

Euclidean Area Postulate. *There exists a unique area function α with the property that $\alpha(\mathcal{R}) = 1$ whenever $\mathcal{R}$ is a square region with sides of length 1.*

Although we are choosing to assume the above as an axiom, the reader should be aware of the following (astounding!) fact: the Euclidean Area Postulate is actually a *theorem* of Euclidean geometry, i.e. it may be proven from Neutral Geometry Postulates 1–9 together with Playfair's Postulate. So it is *not independent* and we are not genuinely expanding our domain of assumptions.

We also state the following familiar theorems without proof.

Euclidean Geometry Theorem 14.2. *The area of a square of side length x is x^2.*

Euclidean Geometry Theorem 14.3. *The area of a rectangle is the product of the lengths of any two adjacent sides.*

Definition 14.4. An **altitude** of a triangle $\triangle ABC$ is a line segment of the form $\overline{AF}$ where F is the foot of the perpendicular from A to $\overleftrightarrow{BC}$ (See Lemma 11.22). So a triangle has exactly three distinct altitudes.

A **height** of a triangle is the length of an altitude. If the height is the length of the altitude $\overline{AF}$, we say that the height corresponds to the **base** $\overline{BC}$.

Euclidean Geometry Theorem 14.5. *The area of a triangle is one-half the length of any base multiplied by the corresponding height.*

15 Euclidean Geometry: Triangle Similarity

Definition 15.1. Two triangles $\triangle ABC$ and $\triangle DEF$ are called **similar** if $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$, and $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$. We denote this by $\triangle ABC \sim \triangle DEF$.

Proposition 15.2. *If two triangles are congruent, then they are similar.*

Proof. By definition. □

Euclidean Geometry Theorem 15.3 (AA Similarity)**.** *If there is a correspondence between the vertices of two triangles such that two pairs of corresponding angles are congruent, then the triangles are similar under that correspondence.*

Proof. Let $\triangle ABC$ and $\triangle DEF$ be two triangles for which $\angle A \cong \angle D$ and $\angle B \cong \angle E$. By Euclidean Theorem 13.4, we also have $\angle C \cong \angle F$. We wish to prove the equality

$$\tfrac{AB}{DE} = \tfrac{BC}{EF} = \tfrac{AC}{DF}.$$

If any of the ratios above is equal to 1, then we get $\triangle ABC \cong \triangle DEF$ by the ASA Theorem 11.5, and hence $\triangle ABC \sim \triangle DEF$ and we are done. So let us assume each of the ratios is different from 1; without loss of generality, suppose $DE > AB$.

Find a point P on $\overline{DE}$ satisfying $DP = AB$, so $D * P * E$. By Playfair's postulate, let ℓ be the unique line parallel to $\overleftrightarrow{EF}$ passing through P. Now ℓ intersects $\overline{DF}$ at some point Q by Pasch's Theorem satisfying $D * Q * F$ (Lemma 11.3). Now $\overline{DP} \cong \overline{AB}$, $\angle B \cong \angle DPQ$ (by Euclidean Theorem 13.1), and $\angle C \cong \angle PQD$ (again by Euclidean Theorem 13.1), so $\triangle ABC \cong \triangle DPQ$ by the SAA Congruence Theorem 11.8.

Drop a perpendicular from Q to $\overleftrightarrow{DE}$ (Lemma 11.22) and let Y denote the foot of the perpendicular, so $Y \in \overleftrightarrow{DE}$ and $\overleftrightarrow{QY} \perp \overleftrightarrow{DE}$. Consider $\triangle PQE$. By Euclidean Theorem 14.5, the following equality holds:

$$\frac{\alpha(\triangle PQE)}{\alpha(\triangle DPQ)} = \frac{(1/2)(PE)(QY)}{(1/2)(DP)(QY)} = \frac{PE}{DP}.$$

Next drop a perpendicular from from P to $\overleftrightarrow{DF}$, and let Z be the foot, so $Z \in \overleftrightarrow{DF}$ and $\overleftrightarrow{PZ} \perp \overleftrightarrow{DF}$. We consider $\triangle PQF$ and apply Euclidean Theorem 14.5 again to get the following ratio:

$$\frac{\alpha(\triangle PQF)}{\alpha(\triangle DPQ)} = \frac{(1/2)(QF)(PZ)}{(1/2)(DQ)(PZ)} = \frac{QF}{DQ}.$$

To finish the argument, we claim that $\alpha(\triangle PQE) = \alpha(\triangle PQF)$. To see this, drop perpendiculars from E and F to $\overleftrightarrow{PQ}$; let R and S denote the feet of these perpendiculars respectively. Now Euclidean Theorem 13.3 implies that $ER = FS$ since $\overleftrightarrow{EF} \parallel \overleftrightarrow{PQ}$. So $\alpha(\triangle PQE) = \frac{1}{2}(PQ)(ER) = \frac{1}{2}(PQ)(FS) = \alpha(\triangle PQF)$

as claimed.

Now putting together all of the equalities we have deduced, we get $\frac{PE}{DP} = \frac{QF}{DQ}$. Hence $1+\frac{PE}{DP} = 1+\frac{QF}{DQ}$; hence $\frac{DE}{DP} = \frac{DP+PE}{DP} = \frac{DQ+QF}{DQ} = \frac{DF}{DQ}$; hence $\frac{DE}{AB} = \frac{DF}{AC}$.

This confirms equality of two of the ratios in the definition of similarity; to get equality of the third ratio, one just repeats the same argument as above with respect to different vertices. So the proof is complete. □

Proposition 15.4. *Two triangles $\triangle ABC$ and $\triangle DEF$ are similar if and only if* $\dfrac{AB}{AC} = \dfrac{DE}{DF}$ *and* $\angle A \cong \angle D$.

Exercise 15.5. Prove Proposition 15.4.

16 Euclidean Geometry: The Theorems of Menelaus and Ceva

Definition 16.1. Let $\triangle ABC$ be a triangle. A triple of points X, Y, Z distinct from one another, and distinct from each of A, B, C, are called **Menelaus points** if they lie on the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$, and $\overline{AC}$ respectively, and either exactly two of them lie on $\triangle ABC$ or none of them do.

The following theorem is due to the Greek astronomer-geometer Menelaus of Alexandria (approx. 70 CE), and did not appear in Euclid's *Elements*!

Euclidean Geometry Theorem 16.2 (Menelaus's Theorem)**.** *Let $\triangle ABC$ be a triangle and let X, Y, Z be a triple of Menelaus points for $\triangle ABC$. Then X, Y, Z are collinear if and only if*

$$\left(\frac{AX}{XB}\right)\left(\frac{BY}{YC}\right)\left(\frac{CZ}{ZA}\right) = 1.$$

Proof. First assume that X, Y, Z are collinear and thus mutually lie on some line ℓ. By Playfair's postulate, let m be the unique line passing through B and parallel to $\overleftrightarrow{AC}$. Now note that m must intersect $\overleftrightarrow{XZ}$; for if not, then $\overleftrightarrow{XZ}$ and $\overleftrightarrow{AC}$ would be two distinct lines parallel to m and passing through Z, in violation of Playfair's postulate. So let W denote the point of intersection of $\overleftrightarrow{XZ}$ with m.

By Euclidean Theorem 13.1, $\angle BWX \cong \angle AZX$ and $\angle WBX \cong \angle ZAX$, and so the AA Similarity Theorem 15.3 tells us that $\triangle XBW \sim \triangle XZA$. In particular,

$$\frac{AX}{XB} = \frac{AZ}{BW}.$$

A very similar argument shows that $\triangle CYZ \sim \triangle BYW$, and hence

$$\frac{BY}{YC} = \frac{BW}{CZ}.$$

When we multiply the two equalities above, we get $(\frac{AX}{XB})(\frac{BY}{YC}) = \frac{AZ}{CZ}$, which is equivalent to what we are trying to prove.

For the converse, suppose that X, Y, Z are Menelaus points satisfying the ratio $\left(\frac{AX}{XB}\right)\left(\frac{BY}{YC}\right)\left(\frac{CZ}{ZA}\right) = 1$. At least one of the points X, Y, Z does not lie on $\triangle ABC$; assume by relabeling points if necessary that Z is the point.

Suppose for the sake of contradiction that $\overleftrightarrow{XY} \parallel \overleftrightarrow{AC}$. Then by Euclidean Theorem 13.1, $\angle BXY \cong \angle BAC$ and $\angle BYX \cong \angle BCA$, whence $\triangle XBY \sim \triangle ABC$ by AA Similarity Theorem 15.3. In this case $\frac{AX}{XB} = \frac{CY}{YB}$. We deduce from our hypothesis that $\frac{CZ}{ZA} = 1$, i.e. $CZ = ZA$, i.e. Z is the midpoint of $\overline{AC}$, contradicting the fact that Z does not lie on $\triangle ABC$. This contradiction assures that $\overleftrightarrow{XY}$ is not parallel to $\overleftrightarrow{AC}$.

So let Z' be the point where the two lines meet. Since X, Y, Z' are collinear, the "only if" part of Menelaus's Theorem (which we already proved) implies that $\left(\frac{AX}{XB}\right)\left(\frac{BY}{YC}\right)\left(\frac{CZ'}{Z'A}\right) = 1$. Thus it must be the case that $\frac{CZ}{ZA} = \frac{CZ'}{Z'A}$.

Now either $A * C * Z$ or $C * A * Z$; without loss of generality, assume the former. Then $ZA > CZ$ implies $Z'A > CZ'$ implies $A * C * Z'$. Thus $AC + CZ = ZA$ and $AC + CZ' = Z'A$. Since $\frac{CZ}{ZA} = \frac{CZ'}{Z'A}$, subtracting 1 from both sides yields $\frac{CZ-ZA}{ZA} = \frac{CZ'-Z'A}{ZA'}$, and then we deduce $\frac{-AC}{ZA} = \frac{-AC}{Z'A}$. The only way this can be true is if $ZA = Z'A$ and $Z = Z'$. So X, Y, Z are collinear. ☐

Definition 16.3. If $\triangle ABC$ is a triangle, a **Cevian line** or just **Cevian** is a line of the form $\overleftrightarrow{AX}$ where $X \in \overline{BC}$, but $X \neq B$ and $X \neq X$.

Definition 16.4. A set of lines, rays, or line segments is called **concurrent** if they all meet in a single point.

The next theorem was actually first proved by the Arab mathematician Al-Mu'taman ibn Hud in the 11th century. It was re-proved independently by Giovanni Ceva in 1678, and Western tradition has ascribed the theorem's name to the latter author.

Euclidean Geometry Theorem 16.5 (Ceva's Theorem)**.** *If $\overleftrightarrow{AX}$, $\overleftrightarrow{BY}$, and $\overleftrightarrow{CZ}$ are Cevians for a triangle $\triangle ABC$, then the Cevians are concurrent if and only if*

$$\left(\frac{AY}{YC}\right)\left(\frac{CX}{XB}\right)\left(\frac{BZ}{ZA}\right) = 1.$$

Proof. We will use Menelaus's Theorem 16.2 to prove Ceva's Theorem. First suppose the Cevians are all concurrent, and let P denote the point where they meet. Consider $\triangle BCY$. Then X, P, A are collinear Menelaus points for $\triangle BCY$, and thus they satisfy the ratio

$$\left(\frac{BP}{PY}\right)\left(\frac{YA}{YC}\right)\left(\frac{CX}{XB}\right) = 1.$$

Similarly, the points Z, P, C are collinear Menelaus points for $\triangle BYA$, and so

$$\left(\frac{AC}{CY}\right)\left(\frac{YP}{PB}\right)\left(\frac{BZ}{ZA}\right) = 1.$$

Multiplying the two equalities above and simplifying yields the desired ratio from Ceva's Theorem!

Conversely, suppose the Cevians satisfy the ratio $\left(\frac{AY}{YC}\right)\left(\frac{CX}{XB}\right)\left(\frac{BZ}{ZA}\right) = 1$. Let P be the point where $\overleftrightarrow{AX}$ and $\overleftrightarrow{CZ}$ intersect, and find the additional Cevian $\overleftrightarrow{BP}$. This latter must intersect $\overline{AC}$ in a point Y'. Then since $\overleftrightarrow{AX}$, $\overleftrightarrow{BY'}$, and $\overleftrightarrow{CZ}$ are concurrent Cevians, the "only if" part of Ceva's Theorem (which we have already proved) implies that

$$\left(\frac{AY'}{Y'C}\right)\left(\frac{CX}{XB}\right)\left(\frac{BZ}{ZA}\right) = 1.$$

From this we deduce that $\frac{AY'}{Y'C} = \frac{AY}{YC}$, and now we proceed exactly as in the last paragraph of the proof of Menelaus's Theorem 16.2 to show that in fact $Y' = Y$. So the three Cevians are concurrent and the theorem is proved. ☐

17 Euclidean Geometry: Triangle Centers

Definition 17.1. A **median** of a triangle $\triangle ABC$ is a line segment of the form $\overline{AM}$ where M is the midpoint of $\overline{BC}$. So a triangle has exactly three distinct medians.

Theorem 17.2. *The three medians of a triangle are concurrent, and meet at a point called the* ***centroid****.*

Exercise 17.3. Use Ceva's Theorem 16.5 to prove Theorem 17.2.

Remark 17.4. Although we have heavily relied on the Euclidean parallel postulate (by way of Ceva's Theorem) to indicate a proof of Theorem 17.2, it turns out there is a proof which does not rely on the Euclidean parallel postulate at all. So, remarkably, it is a theorem of neutral geometry, and we have labeled it as such.

The following theorem, however, is genuinely Euclidean.

Euclidean Geometry Theorem 17.5. *If $\triangle ABC$ is a triangle, G is the centroid, and M is the midpoint of $\overline{BC}$, then $\dfrac{AG}{AM} = \dfrac{2}{3}$.*

Proof. Let N, P denote the midpoints of $\overline{AC}$ and $\overline{AB}$ respectively. First we will show that $\overleftrightarrow{NP} \parallel \overleftrightarrow{BC}$. To see this, construct a point Q on $\overleftrightarrow{PN}$ so that $P * N * Q$ and $PN = QN$ (Lemma 9.9). Since N is the midpoint of $\overline{AC}$, $\overline{AN} \cong \overline{NC}$. And since $\angle PNA$, $\angle QNC$ are vertical angles, they are congruent as well (Theorem 10.12). So $\triangle PNA \cong \triangle CNQ$ by SAS Postulate 9.

In particular $\angle PAN \cong \angle QCN$. But these are alternate interior angles for the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CQ}$, so we conclude by Euclidean Theorem 13.1 that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CQ}$. Now since $\overleftrightarrow{PQ}$ and $\overleftrightarrow{BC}$ are also transversals to these parallel lines, the Alternative Interior Angles Theorem 11.16 together with the Corresponding Angles Theorem 11.17 imply that $\angle CQN \cong \angle ABC$. Since these angles are also congruent to $\angle APN$, Euclidean Theorem 13.1 implies that $\overleftrightarrow{PN} \parallel \overleftrightarrow{BC}$. In addition, by the AA Similarity Theorem 15.3, we have $\triangle ABC \sim \triangle CQN$, and hence $\dfrac{PN}{BC} = \dfrac{AD}{AB} = \dfrac{1}{2}$.

Consider triangles $\triangle NGP$ and $\triangle BGC$. Since $\overleftrightarrow{BN}$ and $\overleftrightarrow{CP}$ are both transversal to the parallel lines $\overleftrightarrow{BC}$ and $\overleftrightarrow{PN}$, the Alternate Interior Angles Theorem 11.16 and the AA Similarity Theorem 15.3 together imply that $\triangle NGP \sim \triangle BGC$. Let R denote the point where $\overleftrightarrow{AG}$ intersects $\overline{NP}$ (Crossbar Theorem 11.2) and consider the triangles $\triangle PGR$ and $\triangle CGM$: by the Vertical Angles Theorem 10.12, $\angle PGR \cong \angle CGM$, so these triangles are similar as well. It follows from these similarity observations and Proposition 15.4 that

$$\frac{PR}{PN} = \frac{PG \cdot \left(\frac{MC}{CG}\right)}{PG \cdot \left(\frac{BC}{CG}\right)} = \frac{MC}{BC} = \frac{1}{2}$$

and therefore

$$\frac{RG}{GM} = \frac{PR \cdot \left(\frac{GM}{MC}\right)}{MC \cdot \left(\frac{RG}{PR}\right)} = \frac{PR}{MC} = \frac{PR}{PN} = \frac{1}{2}.$$

So $\dfrac{RG}{RM} = \dfrac{RM - GM}{RM} = 1 - \dfrac{GM}{RM} = 1 - 2\dfrac{RG}{RM}$; solving this equation for RG yields $RG = \dfrac{1}{3}RM$.

Lastly, since $\overleftrightarrow{AC}$ is transversal to the parallel lines $\overleftrightarrow{PN}$ and $\overleftrightarrow{BC}$, use standard Euclidean arguments to check that $\triangle ARN \sim \triangle AMC$. It follows that $\dfrac{AR}{AN} = \dfrac{AM}{AC}$, and hence $\dfrac{AR}{AM} = \dfrac{AN}{AC} = \dfrac{1}{2}$. We conclude by computing:

$$\begin{aligned}\frac{AG}{AM} &= \frac{AR}{AM} + \frac{RG}{AM} \\ &= \frac{AR}{AM} + \frac{RG}{2RM} \\ &= \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{2}{3}.\end{aligned}$$

□

Definition 17.6. If $\overline{AB}$ is a line segment, the **perpendicular bisector** of $\overline{AB}$ is the line ℓ which is perpendicular to $\overleftrightarrow{AB}$ and meets $\overline{AB}$ in its midpoint. (It is easy to see from Theorem 9.11 and Lemma 10.13 that such a line exists and is unique.)

Lemma 17.7. *A point P is on the perpendicular bisector of a segment $\overline{AB}$ if and only if $PA = PB$.*

Proof. If P lies on $\overleftrightarrow{AB}$ then the lemma is obvious. So assume P is not on $\overleftrightarrow{AB}$.

First suppose P lies on the perpendicular bisector of $\overline{AB}$. Let M denote the midpoint of $\overline{AB}$, so $\overleftrightarrow{PM}$ is the perpendicular bisector. Then $\overline{AM} \cong \overline{BM}$, $\angle PMA \cong \angle PMB$, and $\overline{PM} = \overline{PM}$, so SAS Postulate 9 immediately implies $\triangle PAM \cong \triangle PBM$. Thus $PA = PB$.

Conversely, suppose $PA = PB$. Use Lemma 10.20 to construct a ray $\overrightarrow{PQ}$ which bisects $\angle APB$. By the Crossbar Theorem 11.2, this ray meets $\overline{AB}$ at a point M. Now $\overline{PA} \cong \overline{PB}$, $\angle APM \cong \angle BPM$, and $\overline{PM} = \overline{PM}$, so SAS Postulate 9 implies $\triangle PAM \cong \triangle PBM$. In particular, $AM = MB$ and thus M is the midpoint of $\overline{AB}$. So $\overleftrightarrow{PM}$ is the perpendicular bisector. □

Euclidean Geometry Theorem 17.8. *The three perpendicular bisectors of the sides of a triangle are concurrent, and meet at a point called the **circumcenter**.*

Proof. Let $\triangle ABC$ be a triangle. Let ℓ and m be the perpendicular bisectors of $\overline{AB}$ and $\overline{BC}$ respectively, and let M and N be their points of intersection respectively. (So M is the midpoint of $\overline{AB}$ and N is the midpoint of $\overline{BC}$.)

Suppose for the sake of a contradiction that $\ell \parallel m$. Find a point D where m intersects $\overleftrightarrow{AB}$. D is not equal to B since m intersects $\overline{BC}$ at N, and D is not equal to N because A, B, C are not collinear. So we can consider $\triangle BND$. Since $\overline{AB}$ is transversal to ℓ and m, we get that $\angle BDN$ is right by Euclidean Theorem 13.1. But $\angle DNB$ is also right. So two angles of the triangle have measures adding up to $180°$; this violates Theorem 11.14.

We conclude that ℓ and m are not parallel and thus meet at some point P. Now the "only if" part of Lemma 17.7 implies that $PA = PB$ and $PB = PC$, whence $PA = PC$. Let m be the perpendicular bisector of $\overline{AC}$. The "if" part of Lemma 17.7 now shows that P lies on m, and proves the theorem. □

Definition 17.9. A **circle** is a set of the form $\mathcal{C}(O, r) = \{A : OA = r\}$ for some point O and some positive real number r. The point O is called the **center** and the number r is called the **radius**.

We say that a triangle $\triangle ABC$ can be **circumscribed** if there exists a point O and a positive real number r such that A, B, and C all lie on $\mathcal{C}(O, r)$.

Euclidean Geometry Corollary 17.10. *Every triangle can be circumscribed, by a circle centered at the circumcenter.*

Proof. Let $\triangle ABC$ be a triangle and let O be the circumcenter (Euclidean Theorem 17.8). Let $r = OA$. Since O lies on all three perpendicular bisectors of $\triangle ABC$, Lemma 17.7 implies $OA = OB = OC$, i.e. $A, B, C \in \mathcal{C}(O, r)$. □

The reader should note that the following theorem also does *not* require the Euclidean parallel postulate, i.e. it is a theorem of neutral geometry.

Theorem 17.11. *The three angle bisectors of the interior angles of a triangle are concurrent, and meet at a point called the* ***incenter****.*

Proof. Let $\triangle ABC$ be a triangle. Let $\overrightarrow{AQ}$ be the angle bisector of $\angle A$ with $B * Q * C$ (such a point Q exists by the Crossbar Theorem 11.2). Let $\overrightarrow{BP}$ be the angle bisector of $\angle B$ with $A * P * Q$ (Crossbar Theorem 11.2 again).

Now drop perpendiculars (Lemma 11.22 from P to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$; let X, Y, Z be the feet of the perpendiculars respectively. Use SAA Congruence (Theorem 11.8) and the fact that P lies on the bisector of $\angle A$ to note that $\triangle APY \cong \triangle APZ$, and in particular $\overline{PY} \cong \overline{PZ}$. Since P also lies on the bisector of $\angle B$, we can similarly deduce that $\overline{PX} \cong \overline{PZ}$. So $\overline{PX} \cong \overline{PZ}$.

We also have $\angle PXC \cong \angle PYC$ since both are right, and therefore the SAS Postulate 9 implies that $\triangle CXP \cong \triangle CYP$. In particular $\angle XCP \cong \angle YCP$, and therefore P lies on the angle bisector of $\angle C$. □

Definition 17.12. We say that a triangle can be **inscribed**, if there exists a circle which meets each of the sides of the triangle in exactly one point.

Corollary 17.13. *Every triangle can be inscribed, by a circle centered at the incenter.*

Proof. Let O be the incenter (Theorem 17.11). Drop a perpendicular from O to $\overleftrightarrow{AB}$, and let M be the point of intersection. Set $r = OM$. The proof of Theorem 17.11 implies that $\mathcal{C}(O, r)$ contains M as well as the feet of the other perpendiculars from O to $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$. We leave it as an exercise to the reader to check that no other points of $\triangle ABC$ lie on $\mathcal{C}(O, r)$. □

Euclidean Geometry Theorem 17.14. *The three altitudes of a triangle are concurrent, and meet at a point called the* ***orthocenter****.*

Exercise 17.15. Prove Euclidean Theorem 17.14. (*Hint:* Given a triangle $\triangle ABC$, construct a triangle $\triangle XYZ$ so that A is the midpoint of $\overline{YZ}$, B is the midpoint of $\overline{XZ}$, and C is the midpoint of $\overline{XY}$. Consider the altitudes of $\triangle ABC$; what role do they play for $\triangle XYZ$?)

Euclidean Geometry Theorem 17.16. *The distance from any vertex of a triangle to the orthocenter of the triangle is twice the distance from the circumcenter to the midpoint of the side opposite the vertex.*

Proof. Let $\triangle ABC$ be a triangle, H the orthocenter, and O the circumcenter. Let M denote the midpoint of $\overline{AC}$. We wish to show that $AH = 2OM$.

By the Angle Construction Lemma 9.9, find a ray $\overrightarrow{r}$ starting at B, on the same side of $\overleftrightarrow{BC}$ as O, and making a right angle with $\overleftrightarrow{BC}$. Now note that $\angle OMC$ is right, and therefore Theorem 11.14 implies that $\angle OCM$ is acute. Thus by Euclid's Postulate 5, $\overrightarrow{r}$ intersects $\overrightarrow{CO}$; let D denote the point of intersection. Now the AA Similarity Theorem 15.3 immediately shows that $\triangle BCD \sim \triangle MCO$, and in particular $\dfrac{DB}{OM} = \dfrac{BC}{MC} = 2$.

Next we claim that $\angle DAC$ is right. To see this, note that D, B, and C are each equidistant from O by Euclidean Corollary 17.10. It follows that $\triangle OAD$ and $\triangle OAC$ are both isosceles. Hence Theorem 11.10 implies that $\angle DAO \cong \angle ADO$ and $\angle OAC \cong \angle OCA$. Then Euclidean Theorem 13.4 implies that $\text{m}\angle DAC = \text{m}\angle DAO + \text{m}\angle OAC = \text{m}\angle ADO + \text{m}\angle OCA = 180 - \text{m}\angle DAC$, whence $\text{m}\angle DAC = 90°$.

Now since $\overleftrightarrow{AH}$ and $\overleftrightarrow{DB}$ are both perpendicular to $\overleftrightarrow{BC}$, the Alternate Interior Angles Theorem 11.16 tells us that $\overleftrightarrow{AH} \parallel \overleftrightarrow{DB}$. Similarly, since $\overleftrightarrow{BH}$ and $\overleftrightarrow{DA}$ are both perpendicular to $\overleftrightarrow{AC}$, we have $\overleftrightarrow{BC} \parallel \overleftrightarrow{DA}$. Since $\overleftrightarrow{DH}$ is transversal to both pairs of parallel lines, Euclidean Theorem 13.1 implies that $\angle ADH \cong \angle BHD$ and $\angle AHD \cong \angle BDH$. So $\triangle AHD \cong \triangle BDH$ by the ASA Theorem 11.5. In particular $DB = AH$. So $AH = DB = 2OM$ as claimed. □

Euclidean Geometry Theorem 17.17. *The orthocenter, circumcenter, and centroid of a triangle are collinear, and lie on a line called the* ***Euler line*** *for the triangle.*

Proof. Let $\triangle ABC$ be a triangle, let O be its circumcenter, and G its centroid.

First consider the case $O = G$. Let M be the midpoint of $\overline{AB}$. Then the median $\overline{AM}$ contains the circumcenter O, so $\overleftrightarrow{AM}$ is also the perpendicular bisector of $\overline{BC}$. This immediately implies that $\overline{AM}$ is the altitude from A to $\overleftrightarrow{BC}$. So O lies on this altitude. Repeating these observations two more times with the other two medians will show that O lies on all three altitudes. So O is also the orthocenter, and thus the orthocenter, circumcenter, and centroid are trivially collinear. (It is fairly easy to see that this happens if and only if $\triangle ABC$ is equilateral, and in this case we don't really distinguish any particular line as being an Euler line for the triangle.)

So consider the non-trivial case where $O \neq G$. By the Segment Construction Lemma 9.9, find a point H so that $O * G * H$ and $GH = 2OG$. We claim that H is actually the orthocenter of $\triangle ABC$. To see this, let M be the midpoint of $\overline{AC}$, and consider triangles $\triangle BGH$ and $\triangle MOG$. Since G is the centroid, $\dfrac{BG}{BM} = \frac{2}{3}$ by Euclidean Theorem 17.5, and hence an easy computation shows that $\dfrac{BG}{GM} = 2$. Likewise $\dfrac{GH}{OG} = 2$, and $\angle BGH \cong \angle MGO$ by the Vertical Angles Theorem 10.12. So Proposition 15.4 implies that $\triangle BGH \sim \triangle MOG$. In particular, $\angle HBG \cong \angle OMG$.

By Playfair's Postulate, let ℓ be the unique line parallel to $\overleftrightarrow{AC}$ passing through B, and let $X \in \ell$ be any point lying on the opposite side of $\overleftrightarrow{BH}$ from G. By Euclidean Theorem 13.1, $\angle XBG \cong \angle CMG$. But then $m\angle XBH = m\angle XBG - m\angle HBG = m\angle CMG - m\angle OMG = m\angle CMG = 90°$. So $\angle XBH$ is right. Then if we let F denote the point where $\overleftrightarrow{BH}$ intersects $\overleftrightarrow{AC}$, we get that $\angle BFC$ is also right by the Alternate Interior Angles Theorem 11.16. So F is actually the foot of the perpendicular from B to $\overline{AC}$, i.e. $\overline{BF}$ is an altitude containing H. Repeating the argument two more times reveals that H lies on all three altitudes, i.e. H is the orthocenter as claimed. □

18 Euclidean Geometry: The Nine-Point Circle

Euclidean Geometry Theorem 18.1 (The Nine-Point Circle)**.** *For any triangle, the nine points consisting of (1) the midpoint of each of the sides of the triangle, (2) the feet of the three altitudes of the triangle, and (3) the midpoints of the segments formed by joining the orthocenter to the vertices of the triangle all lie on a single circle. The center of this circle is the midpoint of the segment connecting the orthocenter and the circumcenter.*

Proof. Let $\triangle ABC$ be any triangle. Let H denote the orthocenter, O the circumcenter, and S the midpoint of $\overline{OH}$. Let M_1, M_2, M_3 denote the midpoints of $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively; let F_1, F_2, F_3 denote the feet of the perpendiculars from A, B, C respectively; and let P_1, P_2, P_3 be the midpoints of $\overline{AH}, \overline{BH}, \overline{CH}$ respectively.

Since $\overleftrightarrow{P_1H}$ and $\overleftrightarrow{M_1O}$ are both perpendicular to $\overline{BC}$, we have $\overleftrightarrow{P_1H} \parallel \overleftrightarrow{M_1O}$ by the Alternate Interior Angles Theorem 11.16. Since $\overleftrightarrow{OH}$ and $\overleftrightarrow{P_1M_1}$ are both transversal to these parallel lines, it follows from Euclidean Theorem 13.1 that $\angle HP_1S \cong \angle OM_1S$ and $\angle P_1HS \cong \angle M_1OS$. By our choice of P_1 and by Euclidean Theorem 17.16, we have $HP_1 = \frac{1}{2}HA = OM_1$, so $\overline{HP_1} \cong \overline{OM_1}$. Therefore by the ASA Theorem 11.5, $\triangle P_1SO \cong \triangle M_1SO$. In particular $P_1S = M_1S$.

Next consider the right triangle $\triangle P_1F_1M_1$. By the Angle Construction Lemma 10.13, let $\overrightarrow{r}$ be a ray starting at F_1, on the same side of $\overleftrightarrow{F_1M_1}$ as P_1, which makes an angle with $\overrightarrow{F_1P_1}$ congruent to $\angle F_1P_1S$. By the Crossbar Theorem 11.2, $\overrightarrow{r}$ intersects $\overline{P_1M_1}$ at some point Q. Now $\triangle P_1F_1Q$ has two congruent angles $\angle QP_1F_1 \cong \angle QF_1P_1$ and is thus isosceles. So $\overline{P_1Q} \cong \overline{F_1Q}$. Moreover, $m\angle QF_1M_1 = 90 - m\angle QF_1P_1 = 180 - 90 - m\angle QP_1F_1 = 180 - m\angle P_1F_1M_1 - m\angle M_1P_1F_1 = m\angle P_1M_1F_1$. So $\triangle QF_1M_1$ is also isosceles, whence $\overline{M_1Q} \cong \overline{F_1Q}$ as well. So Q is the midpoint of $\overline{P_1M_1}$; and thus it must be the case that $Q = S$!

Taking stock, we have shown $P_1S = M_1S = F_1S$. Similar arguments, repeated twice, will show that $P_2S = M_2S = F_2S$ and $P_3S = M_3S = F_3S$. So to finish the proof, it suffices to check, for instance, that $P_1S = P_2S = P_3S$. To see this, note that since P_1 is the midpoint of $\overline{AH}$ and S is the midpoint of $\overline{OH}$, it must be the case that $\triangle P_1HS$ and $\triangle AHO$ are similar, and that $P_1S = \frac{1}{2}AO$. Similarly $P_2S = \frac{1}{2}BO$ and $P_3S = \frac{1}{2}CO$. But O is equidistant from A, B, and C by Euclidean Corollary 17.10, and thus $P_1S = P_2S = P_3S$, as claimed. □

19 Equivalent Formulations of the Euclidean Parallel Postulate

Theorem 19.1. *Relative to neutral geometry, the following statements are equivalent:*

(I) For every line ℓ and every point P not on ℓ, there is a unique line parallel to ℓ and containing P (Playfair's Postulate).

(II) For every line ℓ and every point P not on ℓ, such that is at most one line parallel to ℓ and containing P.

(III) If ℓ and m are two lines cut by a transversal t in such a way that the measures of two consecutive interior angles add up to less than $180°$, then ℓ and m intersect on the same side of t as those two angles (Euclid's Postulate 5).

(IV) Parallel lines are everywhere equidistant (Proclus's Tacit Assumption).

(V) If two parallel lines are cut by a transversal, then both pairs of alternate interior angles are congruent (Converse to the Alternate Interior Angles Theorem).

(VI) The sum of the measures of the three angles of any triangle is $180°$ (Euclidean Theorem 13.4).

(VII) There exists a triangle whose three angle measures sum to $180°$.

(VIII) The sum of the measures of the three angles of any triangle is always the same.

(IX) Every Saccheri quadrilateral is a rectangle (Euclidean Corollary 13.5).

(X) A rectangle exists (Clairaut's Postulate).

(XI) A square exists.

(XII) There exist a line ℓ and a point P not on ℓ such that there is at most one line parallel to ℓ and containing P (Negation of the Hyperbolic Parallel Postulate).

(XIII) If $\triangle ABC$ is a right triangle with right angle $\angle C$, then $(AB)^2 = (BC)^2 + (AC)^2$ (Pythagorean Theorem).

(XIV) There exist triangles with arbitrarily large area (Wallis's Axiom). In other words, for every real number $K > 0$ there exists a triangle $\triangle ABC$ with $\alpha(\triangle ABC) > K$.

(XV) Every triangle can be circumscribed. (Euclidean Corollary 17.10).

(XVI) There exist triangles which are similar but not congruent.

Proof. **Roadmap of the Proof:** We have already seen that Statement (I) implies statements (III), (IV), (V), (VI), (VIII), (IX), (X), and (XI). We will begin by showing that (III) $\Longrightarrow$ (V) $\Longrightarrow$ (IV) $\Longrightarrow$ (II) $\Longrightarrow$ (I), and therefore (I)–(V) are all equivalent. Then we will show (VIII) $\Longrightarrow$ (VII) $\Longrightarrow$ (X) $\Longrightarrow$

(XI) $\implies$ (VI) $\implies$ (IX) $\implies$ (X), and (VI) $\implies$ (VIII), so (VII)–(XI) are equivalent to one another. Next, we show (VI) $\implies$ (II), and therefore (VI)–(XII) are equivalent to (I)–(VI) as well. Lastly we will show the equivalence of with (XII), (XIII), and (XIV) each individually, and we leave the equivalence of (XV) and (XVI) as Exercises 19.2 and 19.3.

(III) $\implies$ (V). Suppose ℓ and m are two parallel lines cut by a transversal t. Find and label points $A, B, C \in \ell$ and $D, E, F \in m$ so that $A * B * C$, $D * E * F$, $\ell \cap t = \{B\}$, and $m \cap t = \{E\}$. Since $\overrightarrow{BA}$ and $\overrightarrow{ED}$ do not intersect, Euclid's fifth postulate (which we have assumed as hypothesis) tells us that $\text{m}\angle ABE + \text{m}\angle BED \geq 180°$. Similarly, since $\overrightarrow{BC}$ and $\overrightarrow{EF}$ do not intersect, we have $\text{m}\angle CBE + \text{m}\angle BEF \geq 180°$. Now these four interior angles make two linear pairs, so their total angle sum must be exactly $360°$ (Proposition 10.22). So $\text{m}\angle ABE + \text{m}\angle BED = 360° - \text{m}\angle CBE - \text{m}\angle BEF \geq 180°$ as well, and therefore it must be the case that $\text{m}\angle ABE + \text{m}\angle BED = 180°$. Since $\angle BED$ and $\angle BEF$ are a linear pair, $\text{m}\angle BEF = 180° - \text{m}\angle BED$. Substituting, we get $\text{m}\angle ABE + 180° - \text{m}\angle BEF = 180°$, and thus $\text{m}\angle ABE - \text{m}\angle BEF = 0°$. So $\angle ABE \cong \angle BEF$. Similarly $\angle EAC \cong \angle BED$, and thus the converse to the alternate interior angles theorem is proved.

(V) $\implies$ (IV). Copy down the proof of Euclidean Theorem 13.3 word for word; it relied only on statement (V) (Euclidean Theorem 13.1) in the first place.

(IV) $\implies$ (II). By contrapositive, assume there exists a line ℓ and a point P not on ℓ, so that there are two distinct lines m, n both containing P and both parallel to ℓ. Let F be the foot of the perpendicular from P to ℓ. Let $A, B \in m$ be points so that $A * P * B$, so A and B are on opposite sides of n. It follows that exactly one of these points A, B is on the same side of n as F; without loss of generality assume A is this point. So B is on the other side of n. Let G be the foot of the perpendicular from B to ℓ. Since $n \parallel \ell$, G is on the same side of n as F, and therefore B and G lie on opposite sides of n. So $\overline{BG}$ intersects n, say at a point R. Clearly B and R are on the same side of ℓ, and $R \in \overleftrightarrow{BG}$, so by a homework exercise (Lee 3G) we must have $BG \neq BR$. It follows that either $BG \neq AF$ or $BR \neq AF$; whence either m or n fails to be equidistant from ℓ.

(II) $\implies$ (I). Let ℓ be a line and P a point not on ℓ. Let A and B be any distinct points on ℓ. By the Angle Construction Lemma 10.13, find a point D on the opposite side of $\overleftrightarrow{AP}$ from B so that $\angle APD \cong \angle PAD$. Then by the Alternate Interior Angles Theorem 11.16, $\overleftrightarrow{PD}$ is parallel to ℓ, i.e. there is at least one line parallel to ℓ containing P. By hypothesis (II), this line is unique, and therefore Playfair's Postulate holds.

(VIII) $\implies$ (VII). Let D be the midpoint of $\overline{AB}$, and construct a line ℓ parallel to $\overleftrightarrow{BC}$ passing through D. By Pasch's Theorem 11.3, ℓ intersects $\overline{AC}$, say at a point E distinct from A and C. Now by hypothesis, the angle sum of $\triangle ABC$ is equal to the angle sum of $\triangle ADE$, and hence $\text{m}\angle ABC + \text{m}\angle ACE = \text{m}\angle ADE + \text{m}\angle AED$. But the right-hand side in the previous equation is equal to $360° - \text{m}\angle EDB - \text{m}\angle DEC$ by Proposition 10.22. So the angle sum of quadrilateral $BCED$ is $\text{m}\angle ABC + \text{m}\angle ACE + \text{m}\angle EDC + \text{m}\angle DEC = 360°$. Therefore $BCED$ has zero defect, i.e. $\delta(BCED) = 0°$. Since $\delta(BCED) = \delta(\triangle BCD) + \delta(\triangle BCE)$ by Theorem 12.6, it must be the case that both $\triangle BCD$ and $\triangle BCE$ also have zero defect. So triangles exist with angle sum $180°$.

(VII) $\implies$ (X). Assume there exists a triangle $\triangle ABC$ with angle sum $180°$. At most one angle can be obtuse by the Saccheri-Legendre Theorem 12.1, so assume without loss of generality that $\angle B$ and $\angle C$ are acute. Drop a perpendicular (Lemma 11.22) from A to $\overline{BC}$ and let F denote the foot. It can be shown that since $\angle B$ and $\angle C$ are acute, the foot F must lie on the segment $\overline{BC}$– we have omitted the proof of this fact from these notes, but we hope that the reader can sketch a picture and find it plausible, and perhaps prove it for herself.

Now note that since the angle sum of $\triangle ABC$ is exactly $180°$, we ust have

$$\begin{aligned}
\mathrm{m}\angle ACF + \mathrm{m}\angle CAF + \mathrm{m}\angle CFA &= \mathrm{m}\angle ACB + (\mathrm{m}\angle BAC - \mathrm{m}\angle BAF) + 90^\circ \\
&= (\mathrm{m}\angle ACB + \mathrm{m}\angle BAC) - \mathrm{m}\angle BAF + (180^\circ - \mathrm{m}\angle AFB) \\
&= (180^\circ - \mathrm{m}\angle ABC) - \mathrm{m}\angle BAF + 180^\circ - \mathrm{m}\angle AFB \\
&= 360^\circ - (\mathrm{m}\angle ABF + \mathrm{m}\angle BAF + \mathrm{m}\angle AFB).
\end{aligned}$$

Now the left-hand side above is less than or equal to 180° by the Saccheri-Legendre Theorem 12.1, while the right-hand side is greater than or equal to 180° by the Saccheri-Legendre Theorem 12.1. So the angle sum of the right triangle $\triangle ACF$ must actually be exactly 180°. Now use the Angle Construction Lemma 10.13 to find a ray $\overrightarrow{r}$ starting at A, on the same side of $\overleftrightarrow{AF}$ as C, and making an angle with $\overleftrightarrow{AF}$ congruent to $\angle ACF$, and let D be the foot of the perpendicular from C to $\overleftrightarrow{r}$. Then $\triangle ACD \cong \triangle ACF$ by the SAA Theorem 11.8, and now it is easy to argue that quadrilateral $AFCD$ has four 90° angles and is therefore a rectangle.

(X) $\Longrightarrow$ (XI) Suppose a rectangle $ABCD$ exists. Either $AB \geq BC$ or $BC \geq AB$; without loss of generality assume $AB \geq BC$. By the Segment Construction Lemma 9.9, find a point E on $\overline{AB}$ and a point F on $\overline{DC}$ so that $AE = DF = AB$; we claim $AEFD$ is a square. To see this, note that $\mathrm{m}\angle AEF + \mathrm{m}\angle EFD \leq 180^\circ$ and $\mathrm{m}\angle BEF + \mathrm{m}\angle EFC \leq 180^\circ$ by Theorem 12.4. But $\mathrm{m}\angle AEF + \mathrm{m}\angle EFD = 360^\circ - \mathrm{m}\angle BEF - \mathrm{m}\angle EFC$ by Proposition 10.22, so we also have $\mathrm{m}\angle AEF + \mathrm{m}\angle EFD \geq 360^\circ$. Therefore $\mathrm{m}\angle AEF + \mathrm{m}\angle EFD = 180^\circ$. Now an easy SAS argument shows that $\triangle AED \cong \triangle FDE$, and we conclude that $\angle AEF$ and $\angle EFD$ are both right angles and that $\overline{EF}$ is congruent to the other three sides. So $AEFD$ is a square.

(XI) $\Longrightarrow$ (VI) (Proof in sketch.) Assume there exists a square $ABCD$, with side length $AB = BC = CD = DA = r$ for some positive real number r. Note that by using the Construction Lemmas 9.9 and 10.13, we can copy the square three times in a side-by-side manner to build a new square with sides of length $2r$. By repeating the argument, we may construct squares of side length nr for any positive integer n.

First suppose $\triangle DEF$ is a right triangle, with $\mathrm{m}\angle F = 90^\circ$. Let $K = \max(DF, EF)$, and choose n so large that $nr > K$. As indicated in the previous paragraph, construct a square $FGHI$ with side lengths $FG = GH = HI = IF = nr > K$, in such a way that $F * D * G$ and $F * E * I$. Now drop the perpendicular from D to $\overleftrightarrow{HI}$ and let J denote the foot. Note that $\angle DGH, \angle GHJ, \angle HJD, \angle DJI, \angle JIF, \angle IFD$ are all right angles, so Theorem 12.4 implies that $\mathrm{m}\angle GDJ \leq 90^\circ$ and $\mathrm{m}\angle FDJ \leq 90^\circ$. But the sum of these two angles is exactly 180° since they form a linear pair; the only way this is possible is it $\mathrm{m}\angle GDJ = \mathrm{m}\angle FDJ = 90^\circ$. In particular, $DJIF$ is a rectangle. Similarly, if we drop a perpendicular from E to $\overleftrightarrow{DJ}$ and let K denote the foot, then we can argue that $DKEF$ is a rectangle as well. Then by Theorem 12.6, $0^\circ = \delta(DKEF) = \delta(\triangle DEF) + \delta(\triangle DEK)$, from which we conclude that $\triangle DEF$ has zero defect. Since $\triangle DEF$ was an arbitrary right triangle, we have shown all right triangles have angle sum exactly 180°.

To finish the argument, suppose finally that $\triangle LMN$ is any triangle whatsoever. Assume without loss of generality that $\angle M$ and $\angle N$ are both acute. Drop the perpendicular from L to $\overleftrightarrow{MN}$ and let O denote the foot. Then $\triangle LOM$ and $\triangle LON$ are both right and hence have zero defect. By the Saccheri-Legendre Theorem 12.1, $\mathrm{m}\angle OLM + \mathrm{m}\angle OML \leq 90^\circ$ and $\mathrm{m}\angle OLN + \mathrm{m}\angle ONL \leq 90^\circ$, and therefore $\mathrm{m}\angle MLN + \mathrm{m}\angle NML + \mathrm{m}\angle MNL = \mathrm{m}\angle OLM + \mathrm{m}\angle OLN + \mathrm{m}\angle OML + \mathrm{m}\angle ONL \geq 90^\circ - \mathrm{m}\angle OML + 90^\circ 0 \mathrm{m}\angle OML + \mathrm{m}\angle ONL = 180^\circ$.

(VI) $\Longrightarrow$ (IX). Since any Saccheri quadrilateral can be split into two triangles, each of which have zero defect by hypothesis, the Saccheri quadrilateral itself has zero defect by Theorem 12.6. Therefore it is a rectangle by Corollary 12.11.

(IX) $\implies$ (X). Since we can use the Existence Postulate 2 and the Construction Lemmas 9.9 and 10.13 to construct a Saccheri quadrilateral in the plane, (IX) trivially implies (X).

(VI) $\implies$ (VIII). Trivially.

(VI) $\implies$ (II). Let ℓ be a line and P a point not on ℓ. Suppose contrapositively that there are two distinct lines m, n passing through P each parallel to ℓ; we will construct a triangle with positive defect. Let F be the foot of the perpendicular from P to ℓ. At least one of the lines m and n is not perpendicular to $\overleftrightarrow{PF}$; say m is not. Then one of the rays in m makes an acute angle with $\overleftrightarrow{PF}$; let $\overrightarrow{PA}$ be this ray. So $\text{m}\angle FPA < 90°$.

Construct an infinite sequence of points $T_0, T_1, T_2, T_3, ...$ on ℓ lying on the same side of $\overleftrightarrow{PF}$ as A, as follows. First let let $T_0 = F$. Let T_1 be the point on ℓ on the same side of $\overleftrightarrow{PF}$ as A for which $\overline{FT_1} \cong \overline{PF}$. Let T_2 be the point for which $F * T_1 * T_2$ and $\overline{T_1T_2} \cong \overline{PF}$. In general, if T_k has been defined for a positive integer k, then define T_{k+1} to be the point for which $F * T_k * T_{k+1}$ and $\overline{T_kT_{k+1}} \cong \overline{PF}$. Note that each triangle $\triangle PT_kT_{k+1}$ is isosceles by construction, and therefore $\angle T_kPT_{k+1} \cong \angle PT_{k+1}T_k$. It follows that if we let $\theta_k = \text{m}\angle PT_kF$, then $\text{m}\angle FPT_k = \theta_1 + \theta_2 + ... + \theta_k$.

Now note that for each k, we must have $\overrightarrow{PF} * \overrightarrow{PT_k} * \overrightarrow{PA}$, for if $\overrightarrow{PA}$ lay between $\overrightarrow{PF}$ and $\overrightarrow{PT_k}$, we would get that $\overrightarrow{PA}$ intersects ℓ by the Crossbar Theorem 11.2, which is not the case. So we have $\sum_{i=1}^{k} \theta_i = \text{m}\angle FPT_k < \text{m}\angle FPA$, for every positive integer k. Taking limits as $k \to \infty$, we see that

$$\sum_{i=1}^{\infty} \theta_i \leq \text{m}\angle FPA.$$

In particular, $\lim_{i\to\infty} \theta_i = 0$. Then let $\epsilon = 90° - \text{m}\angle FPA > 0$, and choose a value of i so large that $\theta_i < \epsilon$. Compute the angle sum of $\triangle PFT_i$: we get $\text{m}\angle PFT_i + \text{m}\angle FPT_i + \text{m}\angle FTP_i < 90° + \text{m}\angle FPA + \text{m}\angle FPT_i < 90° + 90° - \epsilon + \epsilon = 180°$. So $\triangle PFT_i$ is a triangle with angle sum strictly less than 180°.

(II) $\implies$ (XII). By Postulate 2 there exists a line $\overleftrightarrow{AB}$ and a point P not on it. Then hypothesis (II) implies the conclusion of (XII) trivially.

(XII) $\implies$ (X). Contrapositively, suppose no rectangle exists. We will show that for every line ℓ, for every point P not on ℓ, there are at least two parallels to ℓ containing P.

So let ℓ be any line and P any point not on ℓ. Let F be the foot of the perpendicular from P to ℓ. By the Angle Construction Lemma 10.13, find a line m passing through P and perpendicular to $\overleftrightarrow{PF}$. Then $m \parallel \ell$ by the Alternate Interior Angles Theorem 11.16. Let G be any point on ℓ other than F. By the Angle Construction Lemma 10.13 again, find a line t containing G and perpendicular to ℓ. Then $t||\overline{PF}$ be the Alternate Interior Angles Theorem 11.16, and therefore $P \notin t$. So let H be the foot of the perpendicular dropped from P to t. Once more by Theorem 11.16, $\overline{PH} \parallel \ell$, so $H \neq G$. Therefore $PFGH$ is a quadrilateral with three right angles $\angle F \cong \angle G \cong \angle H$. Since rectangles do not exist, and the angle sum of a quadrilateral is no more than 360° (Theorem 12.4), it must be that $\angle FPH$ is acute. It follows that $H \notin m$, and hence m and $\overleftrightarrow{PH}$ are two distinct lines containing P and parallel to ℓ.

(VI) $\implies$ (XIII). Let $\triangle ABC$ be a right triangle with right angle $\angle C$, and to ease notation write $c = AB$, $a = BC$, $b = AC$. Drop the perpendicular from C to $\overline{AB}$, and denote the foot by F. Denote $x = AF$ and $y = BF$. Note that by our hypothesis, we have $\text{m}\angle A + \text{m}\angle B = 90°$, and $\text{m}\angle A + \text{m}\angle ACF = 90°$, from which we conclude $\angle B \cong \angle ACF$. Likewise we have $\angle A \cong \angle BCF$. So triangles $\triangle ABC$, $\triangle BCF$, and $\triangle ACF$ are all similar to one another, and thus satisfy the same side length ratios. For instance, $\frac{b}{c} = \frac{x}{b}$, from which we conclude $b^2 = xc$. Also $\frac{a}{c} = \frac{y}{a}$, and hence $a^2 = yc$. It follows that $a^2 + b^2 = xc + yc = (x+y)c = c^2$.

(XIII) $\Longrightarrow$ (X). Assume the Pythagorean Theorem is true; we will show a rectangle exists. First use the usual tricks (Existence Postulate 2, the Construction Lemmas, and dropping a perpendicular) to construct a right triangle $\triangle ABC$ with right angle $\angle C$. Drop a perpendicular from C to $\overline{AB}$ and let F denote the foot. Let $c = AB$, $a = BC$, $b = AC$, $h = CF$, $x = AF$, and $y = BF$, so $x + y = c$ and $y - x = c - 2x$. Then we know $c^2 = a^2 + b^2$, $a^2 = y^2 + h^2$, and $b^2 = x^2 + h^2$. Therefore $a^2 - b^2 = y^2 - x^2 = (y - x)(y + x) = (c - 2x)c$. Solving for x, we get

$$x = \frac{-a^2 + b^2 + c^2}{2c} = \frac{b^2}{c}.$$

A similar argument shows that $y = \dfrac{a^2}{c}$. In addition, we have $h = \sqrt{b^2 - x^2} = \sqrt{\dfrac{b^2c^2 - b^4}{c^2}} = \dfrac{b}{c}\sqrt{c^2 - b^2} = \dfrac{ab}{c}$.

Now let G be the foot of the perpendicular from F to $\overline{AC}$ and H the foot of the perpendicular from F to $\overline{BC}$. Then quadrilateral $CGFH$ has three right angles (all except possibly $\angle GFH$). Let $p = GC$, $q = FH$, $r = CH$, and $s = GF$. Then the same arguments as above yield $p = \dfrac{h^2}{b}$, $r = \dfrac{h^2}{a}$, $q = \dfrac{hy}{a}$, and $s = \dfrac{hx}{b}$. But then $p = \dfrac{h^2}{b} = \dfrac{a^2b}{c^2} = \dfrac{ab}{c} \cdot \dfrac{a^2}{c} \cdot \dfrac{1}{a} = \dfrac{hy}{a} = q$, and $r = \dfrac{h^2}{a} = \dfrac{ab^2}{c^2} = \dfrac{ab}{c} \cdot \dfrac{b^2}{c} \cdot \dfrac{1}{b} = \dfrac{hx}{b} = s$. So $\triangle CFG \cong \triangle GCH$ by the SSS Theorem 11.7. In particular $\angle GFC$ and $\angle HFC$ are complementary, since $\angle GCF$ and $\angle HCF$ are. Therefore $\angle GFH$ is right, and quadrilateral $CGFH$ is in fact a rectangle.

(I) $\Longrightarrow$ (XIV). Since we can build line segments of arbitrary length by the Segment Construction Lemma 9.9, and the area of a triangle is one-half the length of the base times the height by Euclidean Theorem 14.5, it is easy to construct triangles with arbitrarily big areas.

(XIV) $\Longrightarrow$ (I). We will prove this later when we study hyperbolic area functions– see Hyperbolic Theorem 25. □

Exercise 19.2. Show that Statement (XV) in Theorem 19.1 (Every triangle can be circumscribed) is equivalent to the Euclidean parallel postulate.

(*Hint:* One direction is just Euclidean Corollary 17.10. For the other direction, show that (XV) implies Euclid's Fifth Postulate as follows: Let t be a transversal to two lines ℓ and m so that the consecutive interior angles sum up to less than 180°. Let P and Q be the points where t meets ℓ, m respectively. Let A be the midpoint of $\overline{PQ}$. Drop perpendiculars from A to ℓ and m; let F and G denote the feet. Find points B, C such that F is the midpoint of AB and G is the midpoint of AC. Argue that A, B, C are not collinear and thus there is a triangle $\triangle ABC$, which can be circumscribed by hypothesis. Finish the proof by observing that a triangle can be circumscribed if and only if its perpendicular bisectors are concurrent, by Lemma 17.7.)

Exercise 19.3. Show that Statement (XVI) in Theorem 19.1 (There exist similar non-congruent triangles) is equivalent to the Euclidean parallel postulate.

(*Hint:* The fact the EPP $\Longrightarrow$ (XVI) is easy using the Construction Lemmas 10.13 and 9.9 and the AA Similarity Theorem 15.3. To show that (XVI) $\Longrightarrow$ EPP, suppose there are similar triangles $\triangle ABC$ and $\triangle DEF$. By definition the two triangles have the same angle measures. Without loss of generality $\triangle ABC$ has two shorter sides than $\triangle DEF$; use the Construction Lemmas to superimpose a copy of $\triangle ABC$ onto $\triangle DEF$ in such a way that they share a common angle. Now argue that the angle sum of the resulting trapezoid is exactly 360° (see the arguments in (VIII) $\Longrightarrow$ (VII) in the proof of Theorem 19.1) for comparison).

Corollary 19.4. *Relative to neutral geometry, the following statements are equivalent:*

(I) For every line ℓ and every point P not on ℓ, there are at least two lines parallel to ℓ and containing P (Hyperbolic Parallel Postulate).

(II) If ℓ and m are lines cut by a transversal t in such a way that the measures of two consecutive interior angles adds up to less than 180°, then ℓ and m need not necessarily intersect.

(III) Parallel lines are not everywhere equidistant.

(IV) If two parallel lines are cut by a transversal, the alternate interior angles need not be congruent.

(V) The sum of the measures of the three angles of any triangle is strictly less than 180° (The Defect is Positive).

(VI) There exist triangles with different angle sums.

(VII) If two triangles are similar, then they are congruent.

(VIII) If two triangles have all three angles congruent, then the triangles are congruent. (AAA Congruence)

(IX) There is a universal upper bound to the area of any triangle. In other words, there exists a real number $K > 0$ such that if $\triangle ABC$ is any triangle, $\alpha(\triangle ABC) \leq K$.

(X) Rectangles do not exist.

(XI) Squares do not exist.

(XII) Not every triangle can be circumscribed.

(XIII) The Pythagorean Theorem fails to be true.

20 Models of Hyperbolic Geometry

"I mention his talk about angles *because it suggests something Wilcox had told me of his awful dreams. He said that the* geometry *of the dream-place he saw was abnormal, non-Euclidean, and loathsomely redolent of spheres and dimensions apart from ours. Now an unlettered seaman felt the same thing whilst gazing at the terrible reality."* –H. P. Lovecraft, *Call of Cthulhu*

Definition 20.1. A **circle** in $\mathbb{R}^2$ is a set of the form $\mathcal{C}((a,b),r) = \{(x,y) : (x-a)^2 + (y-b)^2 = r^2\}$ for some real numbers a, b, r. We denote $\mathbb{S}^1 = \mathcal{C}((0,0),1)$, the **unit circle**.

A **diameter** of the circle $\mathcal{C}((a,b),r)$ is a set which is either of the form $\{(x,y) : y-b = m(x-a)\}$ for some $m \in \mathbb{R}$, or of the form $\{(x,y) : y = b\}$. Two circles are called **orthogonal** if whenever they intersect at a point (c,d), the diameters of the two circles containing (c,d) are perpendicular to one another.

Definition 20.2 (Inversion in a Circle)**.** Let $\mathcal{S}$ be a circle of radius r centered at a point O. If P is any point other than O, the **inverse** of P (with respect to $\mathcal{S}$) is the unique point P' lying on $\overrightarrow{OP}$ for which $OP \cdot OP' = r^2$.

Remark 20.3 (Constructing Circle Inversions and Orthogonal Circles)**.** If $\mathcal{S}$ is a circle centered at O and P is any point in the interior of the circle other than O, the inverse of P may be constructed as follows: construct a ray $\overrightarrow{r}$ starting at P perpendicular to $\overleftrightarrow{OP}$. This ray intersects $\mathcal{S}$ at a point R. Next construct a ray $\overrightarrow{s}$ starting at R, perpendicular to $\overleftrightarrow{OR}$, and on the same side of $\overleftrightarrow{OR}$ as P. The ray $\int$ will intersect $\overrightarrow{OP}$ at a point. This point, P', is the inverse of P.

Any circle which contains both P and P' is orthogonal to $\mathcal{S}$. Conversely, if a circle contains P and is orthogonal to $\mathcal{S}$, then the circle contains P' as well.

Example 20.4 (The Poincaré Disk)**.** Let $\mathbb{D}$ be the subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ of $\mathbb{R}^2$, i.e. $\mathbb{D}$ is the interior of the unit circle. The **Poincaré disk** is the set $\mathbb{D}$ together with the following interpretations:

(1) A *point* is any element of $\mathbb{D}$.

(2) A *line* is either a diameter of $\mathbb{S}^1$ intersected with $\mathbb{D}$, or else a circle orthogonal to $\mathbb{S}^1$ intersected with $\mathbb{D}$.

(3) Let A and B be two points in $\mathbb{D}$. There is a (Poincaré) line connecting A and B, and this line intersects $\mathbb{S}^1$ in two points P and Q, with P closer to A than B and Q closer to B than A (in the usual Euclidean distance). The (Poincaré) *distance* between two points $A, B \in \mathbb{D}$ is

$$\left|\ln\left(\frac{AQ}{BQ} \cdot \frac{BP}{AP}\right)\right|,$$

where XY denotes the usual Euclidean distance between two points X and Y.

(4) The *measure* of an angle $\angle AOB$ is the ordinary Cartesian angle measure given by the tangent lines at O to the arcs $\overleftrightarrow{AO}$,$\overleftrightarrow{BO}$ respectively.

Example 20.5 (Distances in the Poincaré Disk)**.** To gain some intuition for how distances work in the Poincaré disk, let us define a few points A, B, C, D, E as follows:

$$\begin{aligned}
A &= (0, 0) \\
B &= (\tfrac{e-1}{e+1}, 0) \approx (.4621, 0) \\
C &= (\tfrac{e^2-1}{e^2+1}, 0) \approx (.7615, 0) \\
D &= (\tfrac{e^5-1}{e^5+1}, 0) \approx (.9866, 0) \\
E &= (\tfrac{e^{10}-1}{e^{10}+1}, 0) \approx (.9999, 0)
\end{aligned}$$

All these four points are contained in the horizontal diameter of the Poincaré disk. So here the points P and Q in the definition of the Poincaré distance are just $P = (-1, 0)$ and $Q = (1, 0)$.

Note that in the *Euclidean* distance, B is roughly halfway between A and C, and D and E are extremely close to one another. But when we compute a few *hyperbolic* distances, we get:

distance from A to $B = 1$;
distance from B to $C = 1$;
distance from C to $D = 3$;
distance from D to $E = 5$.

This illustrates an important general phenomenon in the Poincaré disk. As one approaches the boundary of the disk, the distance between points dilates greatly. This is actually easy to see from the definition of the Poincaré distance $\left|\ln\left(\frac{AQ}{BQ} \cdot \frac{BP}{AP}\right)\right|$: if either point A or point B is very close to the boundary of the disk, then either AP or BQ is very small. In that case, since these terms appear in the denominator of the fraction in the logarithm, the the Poincaré distance will be very large.

For this reason lines within the Poincaré disk really are "infinitely long" (or as Euclid would put it, they can be "produced indefinitely"). So even though the geometry of the disk looks bounded from our perspective "outside the disk," inside the disk there is infinite length and breadth just like in the Euclidean plane one is used to visualizing.

As a last remark, check using a logarithm rule that one can actually swap P and Q and the value of the Poincaré distance is unchanged.

Theorem 20.6. *The Poincaré disk is a model of neutral geometry plus the hyperbolic parallel postulate.*

A genuine proof is too technical for our time constraints. Instead of giving a proof, we will give some plausibility arguments that the Poincaré disk satisfies each postulate.

Plausibility Argument. The Poincaré disk satisfies Postulates 1 and 2. Clear.

The Poincaré disk satisfies Postulate 3. We want to see that if A and B are any two points in the Poincaré disk that there is a unique Poincaré line containing both. Consider two cases: either A and B both lie on a diameter of $\mathbb{S}^1$, or they don't. In the first case, the diameter d that they both lie on is a Poincaré line containing both. It is clear that d is the only diameter which contains both points. If there were a circle orthogonal to $\mathbb{S}^1$ containing both A and B, then the circle would also have to contain A', the inverse of A– but A, A', and B are collinear, and thus no circle contains all three. This means d is the unique Poincaré line containing A and B.

In the other case, suppose A and B do not lie on a common diameter of $\mathbb{S}^1$. Then the inverse A' of A is not collinear with A and B, and therefore the three points A, A', and B uniquely determine a circle orthogonal to $\mathbb{S}^1$– the intersection of this circle with the Poincaré disk is a Poincaré line containing both A and B. This has to be the unique Poincaré line containing both.

The Poincaré disk satisfies Postulates 4 and 5. Postulate 4 is satisfied by the definition of distance in the Poincaré disk. So we need to check Postulate 5, i.e. we need to check that every Poincar'e line ℓ admits a coordinate function. In other words, we need to show there exists a distance-preserving bijection between ℓ and $\mathbb{R}$.

So suppose ℓ is any Poincaré line, and let P and Q denote the endpoints of ℓ lying on $\mathbb{S}^1$. Define a function $f : \ell \to \mathbb{R}$ by the rule:

$$f(A) = \ln\left(\frac{AQ}{AP}\right).$$

We claim f is a coordinate function. If A and B are two distinct points on ℓ, then it is easy to see that $\dfrac{AQ}{AP} \neq \dfrac{BQ}{BP}$. Then since the natural logarithm function is injective, it must be that $f(A) \neq f(B)$, i.e. f is injective as well.

If y is any real number, then e^y is some positive real number between 0 and ∞. Then we can locate a unique point A on ℓ satisfying the ratio $\dfrac{AQ}{AP} = e^y$ (a picture helps here). It immediately follows that $f(A) = y$, showing the surjectivity of f.

To see that f is distance-preserving, let A and B be two points on ℓ. Suppose without loss of generality that P is closer to A than B and Q is closer to B than A. Then, using our logarithm rules, we see that

$$\begin{aligned}|f(B)-f(A)| &= \left|\ln\left(\frac{AQ}{AP}\right)-\ln\left(\frac{BQ}{BP}\right)\right| \\ &= \left|\ln\left(\frac{AQ/AP}{BQ/BP}\right)\right| \\ &= \left|\ln\left(\frac{AQ}{BQ}\cdot\frac{BP}{AP}\right)\right| \\ &= \text{the Poincaré distance from } A \text{ to } B.\end{aligned}$$

The Poincaré disk satisfies Postulate 6. Clear.

The Poincaré disk satisfies Postulates 7 and 8. Since the angle measures in the Poincaré disk are essentially inherited directly from the Euclidean plane, one can intuit that the Protractor Postulate is satisfied in this model.

The Poincaré disk satisfies Postulate 9. This is probably the most difficult Postulate to verify. Given two triangles $\triangle ABC$ and $\triangle DEF$ satisfying $m\angle A = m\angle D$, and with the property that the Poincaré distances from A to B and from A to C are the same as the Poincar'e distances from D to E and from D to F, respectively, one wants to prove that $\triangle ABC$ is congruent to $\triangle DEF$, i.e. all corresponding angle measures and side lengths are the same. Traditionally there is a "hard proof" which is just doing the hard analysis to compute side lengths and angle measures, and an "easy proof" involving composing inversions in a circle to get from one triangle to the other. For the details of the latter, we recommend Edwin Moise's book *Elementary Geometry from an Advanced Standpoint*. We will make some comments on the proof in class.

The Poincaré disk satisfied the Hyperbolic Parallel Postulate. Easy to see! □

Example 20.7. Strange phenomena in the Poincaré disk: asymptotic parallel lines; triangles with positive defect; non-rectangular Saccheri quadrilaterals; bounded triangle areas; triangles with no circumcenter; a tesselation by pentagons.

Corollary 20.8. *The hyperbolic parallel postulate is consistent with the axioms of neutral geometry.*

Corollary 20.9. *Both the Euclidean parallel postulate and the hyperbolic parallel postulate are independent of the axioms of neutral geometry.*

Example 20.10. Other models: the Beltrami-Klein disk; the Poincaré half-plane; the hyperboloid.

Remark 20.11 (Remarks on the Categoricity of Euclidean Geometry and the Non-Categoricity of Hyperbolic Geometry)**.** It can be shown that Euclidean geometry is a *categorical* theory in the sense of Definition 5.9. In other words, up to isomorphism, $\mathbb{R}^2$ is the only model of Euclidean geometry.

The proof takes a lot of steps, but the idea is simple: suppose $\mathcal{M}$ is an arbitrary model of Euclidean geometry, and *coordinatize* $\mathcal{M}$! In other words, select arbitrary perpendicular lines ℓ and m in $\mathcal{M}$, and find the point O of intersection. Let $f_\ell : \ell \to \mathbb{R}$ be a coordinate function for ℓ with $f_\ell(O) = 0$, and let $f_m : m \to \mathbb{R}$ be a coordinate function for m with $f_m(O) = 0$. For any point P in $\mathcal{M}$, let X_P denote the foot of the perpendicular from P to ℓ and let Y_P denote the foot of the perpendicular from P to m. Then define a map $F : \mathcal{M} \to \mathbb{R}^2$ by the rule $F(P) = (f_\ell(X_P), f_m(Y_P))$. One can then argue (using in particular the uniqueness of parallels, Proclus's tacit assumption, and the Pythagorean theorem) that this map F is a bijection from $\mathcal{M}$ onto $\mathbb{R}$ which preserves all lines, distances, and angle measures. In other words, this map F turns out to be an isomorphism. Since $\mathcal{M}$ is arbitrary, this shows all models of Euclidean geometry are essentially the same as $\mathbb{R}^2$.

What about hyperbolic geometry? Interestingly, hyperbolic geometry is not categorical. The proof is simple and relies upon the AAA Congruence Theorem, which states that if two triangles have the same angle measures then they are congruent.

The proof sketch goes as follows: Suppose $\mathcal{M}$ is some model of hyperbolic geometry (for instance, the Poincaré disk or half-plane). Find any triangle $\triangle ABC$ in the model $\mathcal{M}$, and denote its three angle measures by α, β, γ. Now note that up to congruence, $\triangle ABC$ is the *only* $\alpha - \beta - \gamma$ triangle in the model– any other triangle with the same angle measures will be congruent to it. Let us measure its side lengths: set $a = BC$, $b = AC$, and $c = AB$.

Now we claim there is another model $\mathcal{M}'$ which is not isomorphic to $\mathcal{M}$. Define it as follows. Let $K \neq 1$ be an arbitrary positive real constant. Let $\mathcal{M}'$ consist of exactly the same points, lines, and angle measures as $\mathcal{M}$, but redefine the distance, so that the distance between two points in $\mathcal{M}'$ is always exactly K times the distance between the same two points in $\mathcal{M}$. One can check that this definition of $\mathcal{M}'$ really gives a model of hyperbolic geometry.

These two models $\mathcal{M}$ and $\mathcal{M}'$ cannot be isomorphic. Why? Because $\mathcal{M}$ contains an $\alpha-\beta-\gamma$ triangle with side lengths a, b, c. Therefore it does *not* contain an $\alpha-\beta-\gamma$ triangle with side lengths Ka, Kb, Kc by the AAA Congruence Theorem (since $K \neq 1$). But $\mathcal{M}'$ does contain such a triangle– therefore they are not isomorphic models.

Although hyperbolic geometry is not categorical, it turns out (perhaps surprisingly) that it is "almost categorical" in the sense that the only way to get non-isomorphic models is to scale the distance up or down by a factor of K as in the previous proof. In other words, there is a natural bijection between the set of models of hyperbolic geometry and the set of positive real numbers. In particular, if we re-scale the distances appropriately, we see that the Beltrami-Klein disk, the Poincaré half-plane, and the hyperboloid are all isomorphic to the Poincaré disk, and therefore to one another! So these models only look different on the surface– intrinsically they are the same.

21 Hyperbolic Geometry: The Angle of Parallelism

Our first theorem is actually neutral. For an informal explanation of what it says, first note that for any line ℓ and any point P not on ℓ, if you drop a perpendicular t from point P to ℓ, and draw a ray $\overrightarrow{r}$ emanating from P at a 90° angle relative to t, then $\overrightarrow{r}$ will not intersect ℓ by the Alternate Interior Angles Theorem 11.16. The theorem below says that there exists a unique "tipping point" angle $\theta \leq 90°$, such that if you draw a ray $\overrightarrow{r}$ from P making an angle of at least θ relative to t, then $\overrightarrow{r}$ will not intersect ℓ, but if $\overrightarrow{r}$ makes an angle strictly less than θ relative to t, then it will intersect ℓ.

Theorem 21.1 (The Angle of Parallelism Exists)**.** *Let ℓ be a line and P a point not on ℓ. Then there exists a positive real number $\theta \leq 90°$ (called the* ***angle of parallelism*** *for ℓ and P) which has the following property: If t is the line containing the perpendicular from P to ℓ, and m is a line passing through P, then*

(1) if all four angles formed by the intersection of m and t have measures greater than or equal to θ, then $m \parallel \ell$; and
(2) if one of the angles formed by the intersection of m and t has measure strictly less than θ, the $m \nparallel \ell$.

Proof. Let F denote the foot of the perpendicular from P to ℓ (so t and ℓ meet at F). Consider the following set

$$\mathcal{I} = \{\alpha \in [0, 90] : \text{there exists a point } Q \in \ell \text{ such that } \mathrm{m}\angle FPQ = \alpha\}.$$

Note that $\mathcal{I}$ is a nonempty set of real numbers, since $0 \in \mathcal{I}$ (as witnessed by the foot F and the zero angle $\angle FPF$). On the other hand, observe that $90 \notin \mathcal{I}$ by the Alternate Interior Angles Theorem 11.16.

Now we set

$$\theta = \sup \mathcal{I}.$$

(Recall that $\sup \mathcal{I}$ denotes the least upper bound of $\mathcal{I}$.) By definition $0 \le \theta \le 90$.

Now let $\vec{r}$ be any ray making an angle of at least θ with $\overrightarrow{PF}$ (including possibly an angle of exactly θ). We claim $\vec{r}$ does not meet ℓ. To see this, suppose for the sake of a contradiction that it does, and let Q denote the point of intersection. We have assumed $m\angle FPQ \ge \theta$. Then find a point S on ℓ so that $F * Q * S$. We have $\overrightarrow{FP} * \overrightarrow{FQ} * \overrightarrow{FS}$ by the Betweenness vs. Betweenness Theorem 10.29, and hence $m\angle FPQ < m\angle FPS$. But then $m\angle FPS > \theta$ and $m\angle FPS \in \mathcal{I}$, contradicting our choice of θ as the least upper bound of $\mathcal{I}$. This contradiction proves our claim.

Conversely, suppose $\vec{r}$ is any ray making an angle strictly less than θ with $\overrightarrow{PF}$. We claim $\vec{r}$ intersects ℓ. Let T be a point on $\vec{r}$ distinct from P, so $\vec{r} = \overrightarrow{PT}$, and we have assumed $m\angle FPT < \theta$. If $m\angle FPT = 0$ then it is obvious that $\vec{r}$ intersects ℓ, so assume $0 < m\angle FPT < \theta$. Since θ is the least upper bound of $\mathcal{I}$, there exists an angle $\alpha_0 \in \mathcal{I}$ which is strictly bigger than $m\angle FPT$. Since $\alpha_0 \in \mathcal{I}$, there exists a point $Q \in \ell$ such that $m\angle FPQ = \alpha$. Assume without loss of generality that T and Q are on the same side of t (by reconstructing T on the opposite side if necessary). Now $\triangle FPQ$ is a triangle, and $\overrightarrow{PF} * \overrightarrow{PT} * \overrightarrow{PQ}$. Then $\vec{r} = \overrightarrow{PT}$ intersects $\overline{FQ}$ by the Crossbar Theorem 11.2, i.e. $\vec{r}$ intersects ℓ as claimed.

Finally we are ready to prove claims (1) and (2) in the statement of the theorem. In case (1), if m makes four angles with t of angle at least θ, then both rays in m make an angle of at least θ with $\overrightarrow{PF}$, and therefore neither one intersects ℓ. In other words $m \parallel \ell$. In case (2), if one of the angles formed by m and t is strictly less than θ, then one of the rays in m makes an angle strictly less than θ with $\overrightarrow{PF}$, whence m intersects ℓ. □

Theorem 21.2. *Relative to neutral geometry, the following statements are equivalent:*

(I) Any version of the Euclidean Parallel Postulate. (See Theorem 19.1.)

(II) For every line ℓ and every point P not on ℓ, the angle of parallelism for ℓ and P is $90°$.

Proof. (I) $\Longrightarrow$ (II). Let ℓ be a line and P a point not on ℓ, and let F denote the foot of the perpendicular from P to ℓ. By Playfair's postulate there is a unique line m containing P parallel to ℓ, and it is perpendicular to $\overleftrightarrow{F}$ by Euclidean Theorem 13.1. Thus if n is any line making an angle less than $90°$ with $\overleftrightarrow{PF}$, it must be the case that $n \neq m$, whence must intersect ℓ. So the angle of parallelism is exactly $90°$.

(II) $\Longrightarrow$ (I). By contrapositive, assume Playfair's postulate fails and thus there are at least two distinct lines m, n passing through P parallel to ℓ. At least one of them makes an angle with $\overleftrightarrow{PF}$ strictly less than $90°$, and θ must be less than or equal to this angle by definition. □

Therefore if we assume the hyperbolic parallel postulate (which we now do), there must be angles of parallelism below $90°$.

Hyperbolic Geometry Corollary 21.3 (The Angle of Parallelism is Strictly Less Than $90°$)**.** *For every line ℓ and every point P, the angle of parallelism for ℓ and P is strictly less than $90°$.*

22 A Brief Real-Analytic Interlude: Continuity of Distance and Area

Theorem 22.1 (Distance is Continuous)**.** *Let ℓ be a line, P a point not on ℓ, and F the foot of the perpendicular from P to ℓ. Let $f : \ell \to \mathbb{R}$ be a coordinate function satisfying $f(F) = 0$. Define a function $D : \mathbb{R} \to [0, \infty)$ by the following rule: given $x \in \mathbb{R}$, set $X = f^{-1}(x)$, and define $D(x) = PX$. Then the function D is continuous, even, increasing on $[0, \infty)$, unbounded above, and obtains its absolute minimum at 0.*

Proof. D is even: Let x any positive real number; we must show $D(x) = D(-x)$. Set $X = f^{-1}(x)$ and $Y = f^{-1}(-x)$; then $\triangle PFX \cong \triangle PFY$ by SAS Postulate 9, whence $PX = PY$. So $D(x) = D(-x)$ as claimed.

D is increasing on $[0,\infty)$: Let x be an arbitrary positive real number and let $y > x$. Find $X = f^{-1}(x)$ and $Y = f^{-1}(y)$ on the line ℓ. Since $\triangle PFX$ is a triangle with right angle $\angle PFX$, $\angle FXP$ must be acute, and therefore $\angle PXY$ is obtuse. But since $\triangle PFY$ is a triangle with right angle $\angle PFY$, $\angle FYP$ is acute. So $\mathrm{m}\angle PXY > \mathrm{m}\angle FYP$, whence $PY > PX$ by the Scalene Inequality Theorem 11.11. Therefore $D(y) > D(x)$ and D is increasing on $[0,\infty)$.

D is unbounded above: Let d be an arbitrary positive real number. Let $x > d$ and find $X = f^{-1}(x)$ on ℓ. Then $\triangle PFX$ is a triangle; since $\angle F$ is right, it must be the case that $\angle X$ is acute by Corollary 11.14. Thus the Scalene Inequality Theorem 11.11 implies $D(x) = PX > FX > d$. Since d was arbitrary, this shows D takes arbitrarily large values.

D takes its minimum at 0: Let x be any non-zero real number. So $X = f^{-1}(x)$ is some point on ℓ distinct from F. Again $\angle F$ is right and $\angle X$ is acute, so $\mathrm{m}\angle F > \mathrm{m}\angle X$, whence $D(x) = PX > PF = D(0)$ by the Scalene Inequality Theorem 11.11. Therefore D obtains its absolute minimum at 0.

D is continuous: We must show that for every $x \in \mathbb{R}$, we have $\lim_{y\to x} D(y) = D(x)$. To see this, fix $x \in \mathbb{R}$ and let $\epsilon > 0$ be arbitrary. We must find a $\gamma > 0$ so that $|x - y| < \gamma$ implies $|D(x) - D(y)| < \epsilon$.

In fact we claim that $\gamma = \epsilon$ works. Because if $|x - y| < \gamma = \epsilon$, and $X = f^{-1}(x)$ and $Y = f^{-1}(y)$, we must have $XY < \epsilon$. Assume without loss of generality that $F * X * Y$ (so $PY > PX$ by our previous arguments). By the Triangle Inequality Theorem 11.9, we have $PY < PX + XY$. Therefore $|D(y) - D(x)| = PY - PX < XY < \epsilon$. This concludes the argument and shows that D is continuous as claimed. □

The main reason for the preceding digression was to invoke the following well-known theorem about continuous real-valued functions.

Theorem 22.2 (Intermediate Value Theorem)**.** *Let $F : [a,b] \to \mathbb{R}$ be a continuous function on some closed interval $[a,b] \subseteq \mathbb{R}$. Then for every C between $F(a)$ and $F(b)$, there exists $c \in [a,b]$ for which $F(c) = C$.*

Corollary 22.3. *Let ℓ be a line, P a point not on ℓ, and F the foot of the perpendicular from P to ℓ. Then for every positive distance $d > PF$, on either side of $\overleftrightarrow{PF}$, there exists a point $X \in \ell$ for which $PX = d$.*

Proof. Let D be the function defined in Theorem 22.1. Since D is unbounded above, there exists a real number u such that $D(u) > d$. Then since D is continuous on the closed interval $[0,u]$, and d lies between $D(0) = PF$ and $D(u)$, we know by the Intermediate Value Theorem that there exists a point $x \in [0,u]$ for which $D(x) = d$. Likewise $D(-x) = d$ since D is even. These numbers x and $-x$ correspond to points $X, Y \in \ell$ satisfying $PX = PY = d$, and clearly $X * F * Y$, so one such point is on each side of $\overleftrightarrow{PF}$ as claimed. □

We omit the proof of the following theorem, which we will use one time later when we characterize area functions in hyperbolic geometry.

Theorem 22.4 (Area is Continuous)**.** *Let α be any area function (see Definition 14.1). Let $\triangle ABC$ be a triangle, and define a function $\mathcal{A} : (0, \mathrm{m}\angle A] \to [0,\infty)$ as follows: for each $x \in (0, \mathrm{m}\angle A]$ find a ray $\overrightarrow{r_x}$ starting at A making an angle of x degrees with $\overrightarrow{AB}$, on the same side of $\overleftrightarrow{AB}$ as C. By the Crossbar Theorem 11.2, $\overrightarrow{r_x}$ meets $\overline{AC}$ at a point P_x. Let $\mathcal{A}(x) = \alpha(\triangle ABP_x)$. Then the function $\mathcal{A}$, so defined, is continuous.*

Lastly, we would like to remind the reader of the following theorem from real analysis, also well-known, which we will also use later on when studying hyperbolic area functions.

Theorem 22.5. *Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be continuous real-valued functions. Then the sum function $f + g$, the difference function $f - g$, and the product function fg are all continuous, and the quotient function $\frac{f}{g}$ is continuous provided g is nowhere zero.*

23 Hyperbolic Geometry: Cutting and Pasting Polygons

Definition 23.1. The **defect** of a convex polygon $\mathcal{P}$ with n vertices (where n is a positive integer) is

$$\delta(\mathcal{P}) = (n-2) \cdot 180^\circ - \sigma(\mathcal{P}),$$

where $\sigma(\mathcal{P})$ denotes the sum of the measures of all the interior angles of $\mathcal{P}$. The defect of a non-convex polygon can be defined very similarly; we just need to take special care when measuring those angles which are "inverted," because the inverted angles should be assigned measures greater than 180° instead of less than 180°– we omit the details of the definition, and leave it to the reader's intuition on the subject.

If $\mathcal{P}$ is a polygon, and $\mathcal{T}$ is a collection of triangles, we say that $\mathcal{P}$ is **triangulated** by $\mathcal{T}$ if all the triangles in $\mathcal{T}$ are non-overlapping, and the region determined by $\mathcal{P}$ is exactly the union of the triangular regions determined by the triangles in $\mathcal{T}$. We also call $\mathcal{T}$ a **triangulation** of $\mathcal{P}$.

The following fact has a lot of intuitive appeal (and is true in neutral geometry) but its proof is complicated and well beyond our scope, so we omit it.

Theorem 23.2. *Every polygon can be triangulated.*

Definition 23.3. Two polygons $\mathcal{P}_1$ and $\mathcal{P}_2$ are called **decomposition-equivalent** if there exists a triangulation $\mathcal{T}_1$ of $\mathcal{P}_1$ and a triangulation $\mathcal{T}_2$ of $\mathcal{P}_2$, and a one-to-one correspondence between the triangles in $\mathcal{T}_1$ and $\mathcal{T}_2$, such that all corresponding pairs of triangles are congruent. In this case we write $\mathcal{P}_1 \equiv \mathcal{P}_2$.

The following theorem is immediate from the definition.

Theorem 23.4. *Any two decomposition-equivalent polygons determine polygonal regions with the same area.*

Recall from Theorem 12.6 that if a convex quadrilateral is partitioned into two triangles, then the defect of the quadrilateral is equal to the sum of the defects of the two smaller triangles. In fact, a much more general fact is true, which we also state without proof.

Theorem 23.5 (Additivity of the Defect)**.** *If a polygon is triangulated in any manner, the defect of the polygon is equal to the sum of the defects of the component triangles.*

Corollary 23.6. *Any two decomposition-equivalent polygons have the same defect.*

Corollary 23.7. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are polygons so that the polygonal region determined by $\mathcal{P}_1$ is a subset of the polygonal region determined by $\mathcal{P}_2$, then $\delta(\mathcal{P}_1) \leq \delta(\mathcal{P}_2)$.*

Corollary 23.8 (Small Triangles Have Small Defect)**.** *For any real number $\epsilon > 0$, there exists a real number d such that whenever a triangle has all three side lengths less than d, we have $\delta(\triangle ABC) < \epsilon$.*

Proof. First construct an arbitrary right triangle $\triangle A_0B_0C$ with right angle $\angle C$. Let A_1 be the midpoint of $\overline{A_0C}$ and let B_1 be the midpoint of $\overline{B_0C}$. Let A_2 be the midpoint of $\overline{A_1C}$ and let B_2 be the midpoint of $\overline{B_2C}$. Continue this process, i.e. for each positive integer k find points A_k and B_k so that A_k is the midpoint of $\overline{A_{k-1}C}$ and B_k is the midpoint of $\overline{B_{k-1}C}$.

Note that for each positive integer k, the region determined by the quadrilateral ABB_kA_k is a subset of the triangular region determined by $\triangle ABC$. So $\delta(ABB_kA_k) \leq \delta(\triangle A_0B_0C)$, for all k. But by the additivity of the defect, $\delta(ABB_kA_k) = \sum_{i=1}^{k} \delta(A_{i-1}B_{i-1}B_iA_i)$. So the infinite series $\sum_{i=1}^{\infty} \delta(A_{i-1}B_{i-1}B_iA_i)$, which consists of all positive terms, is bounded above by $\delta(\triangle A_0B_0C)$, and therefore it converges!

In particular, $\lim_{i\to\infty} \delta(A_{i-1}B_{i-1}B_iA_i) = 0$. So given $\epsilon > 0$, we can find a fixed integer I for which $\delta(A_{I-1}B_{I-1}B_IA_I) < \epsilon$.

Relabel $A_{I-1}B_{I-1}B_IA_I$ for ease of notation: we call it $DEFG$ from now on. Let M be the midpoint of $\overline{DE}$. Drop perpendiculars from M to $\overline{DG}$, $\overline{FG}$, and $\overline{EF}$; these perpendiculars have lengths d_1, d_2, d_3 respectively. Let $d = \frac{1}{2}\min(d_1, d_2, d_3)$. We claim this is the d we want.

Now suppose $\triangle PQR$ is any triangle with all three side lengths less than d. Then we can construct a triangle $\triangle MQ'R'$ with a vertex at M, which is congruent to $\triangle PQR$, in such a way that the region determined by $\triangle MQ'R'$ is just a subset of the region determined by $DEFG$. (Try this on your own, or wait till we draw a picture in class.) It follows that $\delta(\triangle PQR) = \delta(\triangle MQ'R') \leq \delta(DEFG) < \epsilon$, as claimed. □

Remark 23.9 (Philosophical Remark – Which Geometry Reflects Reality?)**.** Some people object to the study of hyperbolic geometry on the basis that it doesn't reflect reality. Is it obvious that real physical space is Euclidean in nature and not hyperbolic?

There is a tale (almost certainly fictional) that Karl Friedrich Gauss, upon his private discovery of the consequences of the hyperbolic parallel postulate, set out to determine if the physical world is actually Euclidean or if it is hyperbolic. So he climbed to the tops of three distant mountains in Europe which were in sight of one another, and measured the three angles of the triangle formed by his three positions to determine whether the angle sum was exactly 180° or not. In the end he measured approximately 180°, but decided that his instruments were not sufficiently sensitive to know for sure.

To illustrate the point, suppose $\epsilon = 10^{-10}$, or some other very small number. The previous corollary implies there is some distance d such that whenever a triangle has side lengths less than d, its defect is no more than ϵ. What if the universe were truly hyperbolic in nature rather than Euclidean, and the value of this d were, say, a billion light years (roughly one forty-sixth of the radius of the observable universe)? Do you think it would be possible to detect physical triangle defects at this scale? What if ϵ were just $\frac{1}{100000}$ and d were the distance from the Earth to the sun– would we be able to take a measurement and determine if space is hyperbolic or Euclidean?

Lemma 23.10 (Triangles Into Saccheri Quadrilaterals)**.** *Every triangle is decomposition-equivalent to a Saccheri quadrilateral. Moreover this Saccheri quadrilateral can be chosen so the summit coincides with any given side of the triangle.*

Proof. Let $\triangle ABC$ be any triangle. Let M be the midpoint of $\overline{AB}$ and N the midpoint of $\overline{BC}$. Let P, Q, R be the feet of the perpendiculars from A, B, C to $\overleftrightarrow{MN}$ respectively. We claim that $APRC$ is a Saccheri quadrilateral decomposition-equivalent to $\triangle ABC$, and that $\delta(\triangle ABC) = \delta(APRC)$.

Let's note that there are several cases here: it could be that $Q = M$ or $Q = N$, or $Q * M * N$, or $M * Q * N$, or $M * N * Q$. This appears to be five cases. However, if $Q = N$, then we can swap labels for the points B, C and M, N and reduce to the case $Q = M$. Likewise if $M * N * Q$, we can swap labels and reduce to the case $Q * M * N$. So we need only check three cases.

Case 1 ($Q = M$): In this case, first note that P and R are on opposite sides of $\overleftrightarrow{BC}$. For if not, then either $R = N$ or R is on the same side of $\overleftrightarrow{BC}$ as P. If $R = N$ then $\angle PNC$ is right, and therefore $\angle BNP$ is right, contradicting Corollary 11.14 (since $\angle BPN$ is already right). Likewise if R is on the same side of $\overleftrightarrow{BC}$ as P, then $\angle RNC$ must be acute by Corollary 11.14, whence $\angle BNP$ is obtuse, another contradiction. So we must have $P * N * R$.

Now the fact that $\angle BMN$ is right implies that $\angle AMN$ is also right, in which case we have $P = Q = M$ as well. By SAS Postulate 9, $\triangle BNP \cong \triangle ANP$, and therefore $\overline{AN} \cong \overline{BN} \cong \overline{CN}$. Also $\angle BNM \cong \angle CNR$ by the Vertical Angles Theorem 10.12. So $\triangle BNM \cong \triangle CNR$ by the AAS Theorem 11.8. In particular $\overline{AP} \cong \overline{PB} \cong \overline{RC}$, and we conclude that $APRC$ is indeed a Saccheri quadrilateral as claimed.

Lastly, note that $APRC$ is triangulated by $\{\triangle ANC, \triangle APN, \triangle NRC\}$ and $\triangle ABC$ is triangulated by $\{\triangle ANC, \triangle APN, \triangle NPB\}$. Since $\triangle NRC \cong \triangle NPB$, this shows $APRC \equiv \triangle ABC$, and the claim is proved for Case 1.

Case 2 ($Q * M * N$): In this case Q and M are distinct, so $\triangle BMQ$ is a right triangle with right angle $\angle BQM$. Therefore $\angle BMQ$ is acute by Corollary 11.14, and $\angle QMA$ is obtuse. It follows that $P \neq M$ (since otherwise $\angle QMA = \angle QPA$ would be right), and that P is not on the same side of $\overleftrightarrow{AB}$ as Q (since otherwise $\angle QMA = \angle PMA$ would be an obtuse angle in the right triangle $\triangle APQ$, which is impossible). So P is on the opposite side of $\overleftrightarrow{AB}$ from Q. Therefore by the Vertical Angles Theorem 10.12, $\angle BMQ \cong \angle AMP$. So $\triangle BMQ \cong \triangle AMP$ by the AAS Theorem 11.8.

Very similar aguments will show that R is on the opposite side of $\overleftrightarrow{BC}$ from P. Then similarly, the Vertical Angles Theorem 10.12 and the AAS Theorem 11.8 together will imply that $\triangle BQN \cong \triangle CRN$. In particular, we get $\overline{AP} \cong \overline{BQ} \cong \overline{CR}$ and therefore $APRC$ is again a Saccheri quadrilateral.

Use the Segment Construction Lemma 9.9 to find a point S on $\overrightarrow{RN}$ satisfying $RS = QM$. So $\triangle CRS \cong \triangle BQM \cong \triangle APM \cong \triangle APN$, $\triangle NCS \cong \triangle NBM$, and $\triangle BMN \cong \triangle QSN$ by SAS Postulate 9. Since $APRC$ is triangulated by $\{\triangle APN, \triangle ANC, \triangle NCS, \triangle CRS\}$ and $\triangle ABC$ is triangulated by $\{\triangle APN, \triangle ANC, \triangle NBM, \triangle APM\}$, we see that $APRC \equiv \triangle ABC$, and the claim is proved for Case 2.

Case 3 ($M * Q * N$): Philosophically very similar to Cases 1 and 2. We leave it as an exercise. □

Lemma 23.11 (Saccheri Quadrilaterals Into Isosceles Triangles)**.** *Every Saccheri quadrilateral is decomposition-equivalent to an isosceles triangle.*

Exercise 23.12. Prove Lemma 23.11 above.

Corollary 23.13. *For every triangle $\triangle ABC$, there exists another triangle $\triangle DEF$ with exactly half the defect, i.e. $\delta(\triangle DEF) = \frac{1}{2}\delta(\triangle ABC)$. Moreover $\triangle ABC$ is decomposition-equivalent to two non-overlapping copies of $\triangle DEF$.*

Proof. By Lemmas 23.10 and 23.11, $\triangle ABC$ is decomposition-equivalent to an isosceles triangle $\triangle DEG$, where $\angle E \cong \angle G$. Let F be the midpoint of $\overline{EG}$. Then SAS Postulate 9 implies $\triangle DEF \cong \triangle DGF$ and hence $\delta(\triangle DEF) = \delta(\triangle DGF)$. But $\delta(\triangle ABC) = \delta(\triangle DEG) = \delta(\triangle DEF) + \delta(\triangle EFG)$, so it must be that $\delta(\triangle DEF) = \delta(\triangle DGF) = \frac{1}{2}\delta(\triangle ABC)$ as claimed. Obviously the region determined by $\triangle DEG$ is the union of the regions determined by the congruent triangles $\triangle DEF$ and $\triangle EFG$, so we see that $\triangle ABC$ is decomposition-equivalent to two non-overlapping copies of $\triangle DEF$. □

Hyperbolic Geometry Theorem 23.14 (Summit-Defect Congruence)**.** *If two Saccheri quadrilaterals have the same defect and congruent summits, then they are congruent.*

Proof. Suppose $ABCD$ is a Saccheri quadrilateral with summit $\overline{CD}$, and $EFGH$ is a Saccheri quadrilateral with summit $\overline{GH}$. Suppose $\overline{CD} \cong \overline{GH}$ and $\delta(ABCD) = \delta(EFGH)$. First of all, note that since their defects are equal, all of the summit angles in both quadrilaterals must be congruent. Then if $AD = EH$, it is easy to see that $ABCD \cong EFGH$, so suppose otherwise, and assume $AD > EH$.

Then we can find points $E' \in \overline{AD}$ and $F' \in \overline{BC}$ so that $E'D = EH$ and $F'C = FG$. But then $E'F'CD \cong EFGH$, and the region determined by $E'F'CD$ is a subset of the region determined by $ABCD$. So $\delta(ABF'E') = \delta(ABCD) - \delta(E'F'CD) = \delta(ABCD) - \delta(EFGH) = 0$, and thus $ABF'E'$ is a rectangle, whose existence contradicts the Hyperbolic Parallel Postulate. This contradiction ensures that $AD = EH$, and $ABCD \cong EFGH$ as claimed. □

Hyperbolic Geometry Theorem 23.15. *If two triangles have the same defect and a pair of congruent sides, then they are decomposition-equivalent.*

Proof. Let $\triangle ABC$ and $\triangle DEF$ be two triangles satisfying $\delta(\triangle ABC) = \delta(\triangle DEF)$ and $\overline{AC} \cong \overline{DF}$. By Lemma 23.10, there exist Saccheri quadrilaterals $APRC$ with summit $\overline{AC}$, and $DQSF$ with summit $\overline{DF}$, such that $\delta(APRC) \equiv \delta(\triangle ABC)$ and $\delta(\triangle DEF) \equiv \delta(DQSF)$. Since $APRC$ and $DQSF$ therefore have the same positive defect, they are congruent by Hyperbolic Theorem 23.14, and therefore certainly decomposition-equivalent. So $\triangle ABC \equiv APRC \equiv DQSF \equiv \triangle DEF$ as claimed. □

Hyperbolic Geometry Theorem 23.16 (Bolyai's Theorem for Triangles)**.** *If two triangles have the same defect, then they are decomposition-equivalent.*

Proof. Let $\triangle ABC$ and $\triangle DEF$ be two triangles satisfying $\delta(\triangle ABC) = \delta(\triangle DEF)$; we will show $\triangle ABC \equiv \triangle DEF$.

If the triangles have a pair of congruent sides, then we are done by Lemma 23.15. So assume they do not; without loss of generality we may assume $AB < DE$. Upon $\overline{AC}$, by Lemma 23.10 we may construct a Saccheri quadrilateral $APRC$ with summit $\overline{AC}$ which is decomposition-equivalent to $\triangle ABC$. Now by Corollary 22.3, there is a point $S \in \overrightarrow{PR}$ for which $AS = \frac{1}{2}DE$. Find a point T so that $A * S * T$ and $AS = ST$, and consider the triangle $\triangle ATC$.

Our first claim is that the Saccheri quadrilateral associated to $\triangle ATC$ is exactly $APRC$. To see this, let U denote the point where $\overline{CT}$ intersects $\overleftrightarrow{PR}$ (this line exists because T, C are on opposite sides); we already know that S is the midpoint of $\overline{AT}$, so we only need to check that U is the midpoint of $\overline{TC}$. So drop a perpendicular from T to $\overleftrightarrow{PR}$, and let Q denote the foot. We see that $\triangle APS \cong \triangle TQS$ by SAA Theorem 11.8, whence $\overline{TQ} \cong \overline{PA} \cong \overline{RC}$. But then $\triangle TQU \cong \triangle CRU$ by SAA Theorem 11.8, and we have $TU = UC$ as desired; this proves the claim.

Therefore we have $\delta(\triangle ATC) = \delta(APRC) = \delta(\triangle ABC) = \delta(\triangle DEF)$. But then $\triangle ATC$ and $\triangle DEF$ are two triangles with the same defect, and with congruent sides $\overline{AT} \cong \overline{DE}$– so they are decomposition-equivalent by Hyperbolic Theorem 23.15! Therefore $\triangle ABC \equiv \triangle ATC \equiv \triangle DEF$, and Bolyai's Theorem is proved. □

Hyperbolic Geometry Corollary 23.17. *Two triangles are decomposition-equivalent if and only if they have the same defect.*

Hyperbolic Geometry Corollary 23.18. *If two triangles have the same defect, then they have the same area.*

24 Hyperbolic Geometry: Continuity of the Defect

Theorem 24.1 (The Defect is Continuous)**.** *Let $\triangle ABC$ be a triangle, and define a function $\mathcal{D} : (0, \mathrm{m}\angle A] \to [0, 180)$ as follows: for each $x \in (0, \mathrm{m}\angle A]$ find a ray $\overrightarrow{r_x}$ starting at A making an angle of x degrees with $\overrightarrow{AB}$, on the same side of $\overleftrightarrow{AB}$ as C. By the Crossbar Theorem 11.2, $\overrightarrow{r_x}$ meets $\overline{AC}$ at a point P_x. Let $\mathcal{D}(x) = \delta(\triangle ABP_x)$. Then the function $\mathcal{D}$, so defined, is continuous.*

Proof. To make the proof go a bit easier, let us assume that $AB \leq AC$. In this case note that $AP_x \leq AC$ for any $x \in (0, \mathrm{m}\angle A]$. (Why?)

We need to show that for any input $x \in (0, \mathrm{m}\angle A]$, we have $\lim\limits_{y \to x} = \mathcal{D}(x)$. In other words, we need to show that for any $\epsilon > 0$, there exists a $\gamma > 0$ such that if $|x - y| < \gamma$, then $|\mathcal{D}(x) - \mathcal{D}(y)| < \epsilon$. So fix x and ϵ, and we proceed to try and find our desired γ.

First construct a sequence of points $C_0, C_1, C_2, C_3, ...$ as follows: let $C_0 = C$, and let C_1 be the unique point so that $B * C * C_1$ and $\overline{BC} \cong \overline{CC_1}$. If C_{k-2} and C_{k-1} have already been defined, let C_k be the unique point for which $C_{k-2} * C_{k-1} * C_k$ and $\overline{C_{k-2}C_{k-1}} = \overline{C_{k-1}C_k}$. Note that for each positive integer k,

$$\sum_{i=1}^{k} \delta(AC_{i-1}C_i) = \delta(AC_0C_k) < 180°.$$

So the infinite series $\sum_{i=1}^{k} \delta(AC_{i-1}C_i)$ is bounded above and hence converges. In particular, $\lim_{i\to\infty} \delta(AC_{i-1}C_i) = 0$. So there exists an integer I with $\delta(AC_{I-1}C_I) < \epsilon$. Let $\gamma = \mathrm{m}\angle C_{I-1}AC_I$. We claim this is our desired γ.

To see this, suppose $y \in (0, \mathrm{m}\angle A]$ is a number satisfying $|x - y| < \gamma$. Then $\mathrm{m}\angle P_xAP_y < \gamma = \mathrm{m}\angle C_{I-1}AC_I$, and $AP_x, AP_y \leq AC \leq AC_{I-1} \leq AC_I$. So we can construct a congruent copy of $\triangle AP_xP_y$ inside of $\triangle AC_{I-1}A_I$, from which we conclude that $\delta(AP_xP_y) < \delta(AC_{I-1}C_I) < \epsilon$. Since $|\mathcal{D}(x)-\mathcal{D}(y)| = |\delta(ABP_x) - \delta(ABP_y)| = \delta(AP_xP_y)$, this concludes the proof. □

25 Hyperbolic Geometry: Hyperbolic Area

All the tools are now in place for us to completely characterize the notion of area in the hyperbolic plane. The first thing to note is that we have had a reasonable hyperbolic area function floating around this entire time... check Definition 14.1 and compare with Theorem 23.5 to confirm that the defect function δ is in fact an example of an area function in the hyperbolic plane!

Hyperbolic Geometry Theorem 25.1. *The defect function δ is an area function. Consequently, any constant multiple $K \cdot \delta$ (for a positive constant K) is also an area function.* □

This fact on its own is quite surprising. What is perhaps more surprising, however, is that the *only possible* area functions in the hyperbolic plane are constant multiples of the defect. We prove this below.

To understand the proof, recall that a **dyadic rational** number is a number of the form $\frac{i}{2^j}$ for some integer i and some nonnegative integer j. The set of all dyadic rational numbers is *dense* in $\mathbb{R}$, i.e. for any two distinct real numbers $x < y$ in $\mathbb{R}$, there always exists a dyadic rational q satisfying $x < q < y$. Also recall the following general fact about continuous functions: if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are continuous functions on a domain $D \subseteq \mathbb{R}$, and $f(q) = g(q)$ for all members q of some dense subset of D, then $f = g$.

Hyperbolic Geometry Theorem 25.2. *If α is any area function, then $\alpha = K \cdot \delta$ for some positive constant K.*

Proof. Let α be any area function. By Postulate 2, a triangle $\triangle XYZ$ exists. Set

$$K = \frac{\alpha(\triangle XYZ)}{\delta(\triangle XYZ)},$$

so K is a positive constant. We claim that $\alpha = K \cdot \delta$, i.e. $\alpha(\mathcal{P}) = K\delta(\mathcal{P})$ for every polygonal region $\mathcal{P}$. Since every polygon can be triangulated, and both α and δ are additive, it suffices to check the equality for all triangles. In other words, to finish the proof, it suffices to check that $\alpha(\triangle ABC) = K\delta(\triangle ABC)$ for every triangle $\triangle ABC$. This is equivalent to checking that $K = \dfrac{\alpha(\triangle ABC)}{\delta(\triangle ABC)}$ for every triangle $\triangle ABC$.

First consider the case where $\triangle ABC$ is a triangle with the same defect as $\triangle XYZ$, i.e. $\delta(\triangle ABC) = \delta(\triangle XYZ)$. Then by Bolyai's Theorem 23.16, $\triangle ABC$ and $\triangle XYZ$ are decomposition equivalent, and it follows that $\alpha(\triangle ABC) = \alpha(\triangle XYZ)$. So $K = \frac{\alpha(\triangle XYZ)}{\delta(\triangle XYZ)} = \frac{\alpha(\triangle ABC)}{\delta(\triangle ABC)}$ as claimed.

Next consider the case where

$$\frac{\delta(\triangle ABC)}{\delta(\triangle XYZ)} = \frac{1}{2^j} \text{ for some nonnegative integer } j.$$

Now by Corollary 23.13, there exists a triangle $\triangle X_1Y_1Z_1$ such that $\delta(\triangle X_1Y_1Z_1) = \frac{1}{2}\delta(\triangle XYZ)$, and $\triangle XYZ$ is decomposition-equivalent to two copies of $\triangle X_1Y_1Z_1$. Then by Corollary 23.13 applied again, there exists a triangle $\triangle X_2Y_2Z_2$ such that $\delta(\triangle X_2Y_2Z_2) = \frac{1}{2}\delta(\triangle X_1Y_1Z_1) = \frac{1}{4}\delta(\triangle XYZ)$, and $\triangle XYZ$ is decomposition-equivalent to 4 copies of $\triangle X_2Y_2Z_2$. In fact, by applying Corollary 23.13 j-many times, we get that there exists a triangle $\triangle X_jY_jZ_j$ satisfying $\delta(\triangle X_jY_jZ_j) = \frac{1}{2^j}\delta(\triangle XYZ)$ and $\triangle XYZ$ is decomposition-equivalent to 2^j-many non-overlapping copies of $\triangle X_jY_jZ_j$.

But $\triangle X_jY_jZ_j$ and $\triangle ABC$ have the same defect, and are therefore decomposition-equivalent by Bolyai's Theorem 23.16. So we conclude that $\triangle XYZ$ is decomposition-equivalent to 2^j many non-overlapping copies of $\triangle ABC$. Therefore $\alpha(\triangle XYZ) = 2^j\alpha(\triangle ABC)$. So $K = \dfrac{\alpha(\triangle XYZ)}{\delta(\triangle XYZ)} = \dfrac{2^j\alpha(\triangle ABC)}{2^j\delta(\triangle ABC)} = \dfrac{\alpha(\triangle ABC)}{\delta(\triangle ABC)}$ as claimed.

Next consider the case where

$$\frac{\delta(\triangle ABC)}{\delta(\triangle XYZ)} = \frac{i}{2^j} \text{ for some nonnegative integer } j \text{ and some positive integer } i < 2^j.$$

Since defect is continuous (Theorem 24.1), by the intermediate value theorem we can find points $P_1, P_2, ..., P_i$ on $\overline{YZ}$ for which $\delta(\triangle XYP_\ell) = \dfrac{\ell}{2^j}\cdot\delta(\triangle XYZ)$ whenever $1 \leq \ell \leq i$. Note that $\triangle ABC$ and $\triangle XYP_i$ have the same defect and are thus decomposition-equivalent by Bolyai's Theorem 23.16. So $\alpha(\triangle ABC) = \alpha(\triangle XYP_i)$.

If we set $P_0 = Y$, then by the additivity of the defect (Theorem 23.5), $\delta(\triangle XP_{\ell-1}P_\ell) = \dfrac{1}{2^j}\delta(\triangle XYZ)$ for each $1 \leq \ell \leq i$. Then in particular, by our arguments in the previous case, $K = \dfrac{\alpha(\triangle XP_{\ell-1}P_\ell)}{\delta(\triangle XP_{\ell-1}P_\ell)}$ for each ℓ.

But the triangles $\triangle XP_{\ell-1}P_\ell$ $(1 \leq \ell \leq i)$ are all mutually decomposition-equivalent by Bolyai's Theorem 23.16! So in particular, they all have the same area. Consequently $\alpha(\triangle XYP_i) = i\cdot\alpha(\triangle XYP_1)$. So putting together all our equalities, we get $\dfrac{\alpha(\triangle ABC)}{\delta(\triangle ABC)} = \dfrac{\alpha(\triangle XYP_i)}{\delta(\triangle XYP_i)} = \dfrac{i\cdot\alpha(\triangle XYP_1)}{i\cdot\delta(\triangle XYP_1)} = \dfrac{i}{i}\cdot K = K$, as desired.

Lastly we are ready for the general case. Let $\triangle ABC$ be any triangle whatsoever– we wish to show $\dfrac{\alpha(\triangle ABC)}{\delta(\triangle ABC)} = K$. We may without loss of generality assume $\alpha(\triangle ABC) < \alpha(XYZ)$, and hence $r = \dfrac{\alpha(\triangle ABC)}{\alpha(\triangle XYZ)} < 1$. (Why?)

Since the dyadic rationals are dense in $\mathbb{R}$, we may find a non-decreasing sequence of dyadic rational numbers $(q_n)_{n=1}^{\infty}$ for which $\lim_{n\to\infty} q_n = r$. By the continuity of the defect and the intermediate value theorem, we may find points P_n and P on $\overline{YZ}$ such that $\delta(\triangle XYP_n) = q_n$ and $\delta(\triangle XYP) = r$. Since $\triangle ABC$ and $\triangle XYP$ have the same defect, they have the same area.

Now consider the functions $\mathcal{A}$ and $\mathcal{D}$ defined in Theorems 22.4 and 24.1, respectively. Find angle measures $(x_n)_{n=1}^{\infty}$ and x such that $\mathcal{D}(x_n) = q_n$ and $\mathcal{D}(x) = r$. The quotient function $\dfrac{\mathcal{A}}{\mathcal{D}}$ is continuous by Theorem 22.5. Note that for each n, our previous arguments imply $\dfrac{\mathcal{A}(x_n)}{\mathcal{D}(x_n)} = K$. Therefore $\dfrac{\alpha(\triangle XYP)}{\delta(\triangle XYP)} = \dfrac{\mathcal{A}(x)}{\mathcal{D}(x)} = \lim_{n\to\infty}\dfrac{\mathcal{A}(x_n)}{\mathcal{D}(x_n)} = \lim_{n\to\infty} K = K.$

So $\dfrac{\alpha(\triangle ABC)}{\delta(\triangle ABC)} = K$ as well. This concludes the proof. □

Hyperbolic Geometry Corollary 25.3. *There is a universal upper bound to the area of any triangle.*

Proof. Let α be any area function. Then $\alpha = K \cdot \delta$ for some positive constant K. Therefore if $\triangle ABC$ is any triangle, we must have $\alpha(\triangle ABC) \leq K \cdot 180$. □