Metric Space Topology (Spring 2016) Selected Homework Solutions

HW1 Q1.2. Suppose that $d$ is a metric on a set $X$. Prove that the inequality $|d(x, y) - d(z, w)| \leq d(x, z) + d(y, w)$ holds for all $w, x, y, z \in X$.

Proof. Let $w, x, y, z \in X$ be arbitrary. By the triangle inequality applied twice in succession,

$$d(x, y) \leq d(x, z) + d(z, y) \leq d(x, z) + d(z, w) + d(w, y).$$

Subtracting $d(z, w)$ from both sides yields

$$d(x, y) - d(z, w) \leq d(x, z) + d(y, w).$$

Now, the equality above holds for all points $w, x, y, z$. So swapping $x$ with $z$, and $y$ with $w$, and using the symmetry of the metric $d$, we obtain $d(z, w) - d(x, y) \leq d(x, z) + d(y, w)$. Flipping signs on this inequality yields

$$-d(x, z) - d(y, w) \leq d(x, y) - d(z, w).$$

So we have shown $-d(x, z) - d(y, w) \leq d(x, y) - d(z, w) \leq d(x, z) + d(y, w)$, which is equivalent to the inequality we hoped to show. □

HW1 Q1.8. Let $\mathcal{F}(S)$ be the set of all finite subsets of a set $S$. Define a function $d$ on $\mathcal{F}(S) \times \mathcal{F}(S)$ by the rule $d(A, B) = \#(\Delta(A, B))$ for all $A, B \in \mathcal{F}(S)$, where $\Delta$ denotes symmetric difference and $\#$ denotes cardinality. Is $d$ a metric?

We claim $d$ is a metric, and prove our assertion below.

Proof. Clearly $d(A, B) \geq 0$ for all $A, B \in \mathcal{F}(S)$, with $d(A, A) = \#(\emptyset) = 0$. If $d(A, B) = \#(\Delta(A, B)) = 0$, then $\Delta(A, B) = \emptyset$, from which we conclude $A = B$. Since $\Delta(A, B) = \Delta(B, A)$, we have $d(A, B) = d(B, A)$. So to finish the proof, we need only check the triangle inequality for $d$.

So let $A, B, C \in \mathcal{F}(S)$ be arbitrary. We know $\Delta(A, B) = (A \setminus B) \cup (B \setminus A)$; let us first consider $A \setminus B$. We note that

$$A \setminus B = [(A \setminus B) \cap C] \cup [(A \setminus B) \setminus C] \subseteq (C \setminus B) \cup (A \setminus C).$$

Similarly, $B \setminus A \subseteq (C \setminus A) \cup (B \setminus C)$. So we have
\[ \Delta(A, B) = (A \setminus B) \cup (B \setminus A) \]
\[ \subseteq (C \setminus B) \cup (A \setminus C) \cup (C \setminus A) \cup (B \setminus C) \]
\[ = (A \setminus C) \cup (C \setminus A) \cup (C \setminus B) \cup (B \setminus C) \]
\[ = \Delta(A, C) \cup \Delta(C, B). \]

It follows that
\[ d(A, B) = \#(\Delta(A, B)) \]
\[ \leq \#(\Delta(A, C) \cup \Delta(C, B)) \]
\[ \leq \#(\Delta(A, C)) + \#(\Delta(C, B)) \]
\[ = d(A, C) + d(C, B). \]

\(\square\)

**HW2 #1.** Suppose \((X, d)\) is a metric space and \(e(x, y) = \frac{d(x, y)}{1 + d(x, y)}\) for every \(x, y \in X\). Prove that \(e\) is a metric on \(X\).

**Proof.** Since \(d\) is a metric to begin with, the positivity and symmetry conditions for \(e\) obviously hold. So we need to check the triangle inequality for \(e\).

To that end, suppose \(x, y, z \geq 0\) are three nonnegative real numbers, and further suppose that \(x \leq y + z\). Since adding nonnegative terms can only make the right side bigger, it follows that \(x \leq y + z + 2yz + xyz\). Now, adding \((xy + xz + xyz)\) to both sides yields the inequality
\[ x + xy + xz + xyz \leq y + xy + yz + xyz + z + xz + yz + xyz, \]

in other words,
\[ x(1 + y)(1 + z) \leq y(1 + x)(1 + z) + z(1 + x)(1 + y). \]

Dividing through above by the positive number \((1 + x)(1 + y)(1 + z)\), we obtain
\[ \frac{x}{1 + x} \leq \frac{y}{1 + y} + \frac{z}{1 + z}. \]

To finish the proof, we simply observe that if \(a, b, c\) are any three points in \(X\), then we can set \(x = d(a, b), y = d(a, c), z = d(c, b)\). Since \(d\) is a metric, \(x \leq y + z\), and now the above line implies that the triangle inequality holds for \(e\).

\(\square\)

**HW2 #2.** For \((X, e)\) in the previous problem, prove that \(X\) has finite diameter.
Proof. Let \( x, y \in X \) be arbitrary. Then \( d(x, y) < 1 + d(x, y) \), and hence \( e(x, y) = \frac{d(x, y)}{1 + d(x, y)} < 1 \). So 1 is a lower bound for the set \( \{d(x, y) : x, y \in x\} \). Therefore \( \text{diam}(X) = \inf\{d(x, y) : x, y \in x\} \leq 1 \).

\[\text{HW2 } \#3. \text{ For the metric space } (\mathbb{N}, d) \text{ in Example 1.1.12, show that } \mathbb{N} \text{ has exactly one accumulation point.}\]

Proof. We will show \( \infty \) is the sole accumulation point. To see this, compute the distance from \( \infty \) to \( \mathbb{N}\{\infty\} = \mathbb{N} \):

\[
\text{dist}(\infty, \mathbb{N}) = \inf \{d(\infty, n) : n \in \mathbb{N}\} = \inf \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.
\]

Now the infimum in the last line above is at least 0, since \( 0 < \frac{1}{n} \) for every \( n \in \mathbb{N} \). On the other hand, notice that for every \( \epsilon > 0 \), \( \epsilon \) fails to be a lower bound: because there are always integers \( n \in \mathbb{N} \) for which \( n > \frac{1}{\epsilon} \), whence \( \frac{1}{n} < \epsilon \). So the infimum above is exactly 0. This shows \( \infty \) is an accumulation point.

On the other hand, if \( n \in \mathbb{N} \), we claim \( n \) is an isolated point of \( \mathbb{N} \) (and hence \( \infty \) is the only accumulation point). To see this, we must show \( \text{dist}(n, \mathbb{N}\{n\}) > 0 \). So, for our fixed \( n \), let us consider the following quantity:

\[
B = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}.
\]

Now consider an arbitrary \( m \in \mathbb{N}\{\infty\} \). Either \( m = \infty \), or \( m \in \mathbb{N} \) and \( m < n \), or \( m \in \mathbb{N} \) and \( m > n \). If \( m = \infty \), then \( d(n, m) = d(n, \infty) = \frac{1}{n} > \frac{1}{n} - \frac{1}{n+1} = B \).

If \( m \in \mathbb{N} \) and \( m < n \), then \( m \leq n-1 \). Therefore \( \frac{1}{m} \geq \frac{1}{n-1} \), whence \( d(n, m) = \frac{1}{m} - \frac{1}{n} \geq \frac{1}{n-1} - \frac{1}{n} = \frac{1}{n(n-1)} > \frac{1}{n(n+1)} = B \).

If \( m \in \mathbb{N} \) and \( m > n \), then \( n+1 \leq m \). Therefore \( \frac{1}{n+1} \geq \frac{1}{m} \), whence \( d(n, m) = \frac{1}{n} - \frac{1}{m} \geq \frac{1}{n+1} - \frac{1}{n} = B \).

By checking all cases, we have shown that \( B \) is a lower bound for the set \( \{d(n, m) : m \in \mathbb{N}\{n\}\} \). It follows that \( \text{dist}(n, \mathbb{N}\{n\}) \geq B > 0 \), and \( n \) is an isolated point as claimed. This completes the proof.

\[\text{HW2 } \#4. \text{ For the metric space } (\mathbb{R}^2, d) \text{ in Example 1.1.15, show that there exists exactly one isolated point of } \mathbb{R}^2.\]

Proof. First, we claim \((0, 0)\) is an isolated point. To see this, note that by definition of the metric, \( d((0, 0), x) \geq 1 \) for every \( x \in \mathbb{R}^2 \), \( x \neq (0, 0) \). It follows that \( \text{dist}((0, 0), \mathbb{R}^2\{(0, 0)\}) \geq 1 > 0 \), i.e. \((0, 0)\) is isolated.

On the other hand, we claim every other point \( x \in \mathbb{R}^2 \), \( x \neq (0, 0) \) is an accumulation point. To see this, let \( e \) denote the usual metric on \( \mathbb{R}^2 \), and observe that by the definition of \( d \), we have \( (\mathbb{R}^2\{(0, 0)\}, d) = (\mathbb{R}^2\{(0, 0)\}, e) \). That is to say, \( d \) and \( e \) agree everywhere
except at the origin.

Now let \( \epsilon > 0 \) be arbitrary. The set of points \( y \) for which \( e(x, y) < \epsilon \) is clearly infinite; so find such a \( y \) satisfying \( y \neq (0, 0) \). Then \( d(x, y) = e(x, y) < \epsilon \). It follows that \( \text{dist}(x, \mathbb{R}^2 \setminus \{x\}) = \inf\{d(x, y) : y \in \mathbb{R}^2, y \neq (0, 0)\} \) cannot be equal to any positive \( \epsilon \) (since no such \( \epsilon \) is a lower bound for \( \{d(x, y) : y \in \mathbb{R}^2, y \neq (0, 0)\} \)). So \( \text{dist}(x, \mathbb{R}^2 \setminus \{x\}) = 0 \), i.e. \( x \) is an accumulation point. \( \square \)

**HW3 #1.** Let \((X, d)\) be a metric space. Let \( x \in X \) and let \( r \) be a positive real number. Define the set

\[
B = \{ y \in X : d(y, x) < r \}.
\]

Show that \( B \) is open and \( \text{diam}(B) \leq 2r \).

**Proof.** Let \( b \in B \); we will show \( b \notin \partial B \). Since \( b \in B \), \( d(b, x) < r \). Set \( R = d(b, x) \), and set \( t = r - R \), so \( t > 0 \).

We claim that for every \( z \in B^c \), \( d(b, z) \geq t \). Indeed, if this were not the case, we would have \( d(b, z) < t \), and therefore by the triangle inequality we would have \( d(x, z) \leq d(x, b) + d(b, z) < R + t = r \), contradicting the fact that \( z \in B^c \).

So \( d(b, z) \geq t \) for all \( z \in B^c \). It follows that \( \text{dist}(b, B^c) = \inf\{d(b, z) : z \in B^c\} \geq t > 0 \), and hence \( b \notin \partial B \). Since \( B \) was arbitrary, we have shown \( B \cap \partial B = \emptyset \) and hence \( B \) is open. \( \square \)

**HW3 #2.** Let \((X, d)\) be a metric space. For any \( x \in X \), show that \( x \in \text{iso}(X) \) if and only if \( \{x\} \) is open.

**Proof.** For any \( x \in X \),

\[
x \in \text{iso}(X) \iff \text{dist}(x, X \setminus \{x\}) > 0
\]

\[
\iff \text{dist}(x, \{x\}^c) > 0
\]

\[
\iff x \notin \partial \{x\}
\]

\[
\iff \{x\} \cap \partial \{x\} = \emptyset
\]

\[
\iff \{x\} \text{ is open.}
\]

\( \square \)

**HW3 #3.** Let \( X \) be any set and let \( d \) be the discrete metric on \( X \) (see Example 1.1.7). Show that every subset of \( X \) is open. Conclude that every subset of \( X \) is closed.

**Proof.** First note that for each \( x \in X \), \( x \in \text{iso}(X) \) since

\[
\text{dist}(x, X \setminus \{x\}) = \inf\{d(x, y) : y \in X, y \neq x\} \geq \inf\{1\} = 1.
\]

Therefore \( \{x\} \) is open by the result in Problem 2 above.

Now let \( S \subseteq X \) be an arbitrary subset. Then \( S = \bigcup \{ \{x\} : x \in S \} \) is the union of its collection of singleton subsets. Hence, \( S \) is a union of open sets and therefore open. Since \( S \) was arbitrary, every subset of \( X \) is open.
Likewise, if \( S \subseteq X \) is an arbitrary subset, then the argument above shows that \( S^c \) is open. Therefore \( S \) is closed. Therefore every subset of \( X \) is closed. \( \square \)

**HW3 #4.** Consider the metric space \((\mathbb{N}, d)\) as defined in Example 1.1.12. Let \( A \subseteq \mathbb{N} \) (so \( \infty \notin A \)). Show that \( \infty \in A \) if and only if \( A \) is an infinite set.

**Proof.** First suppose \( A \) is an infinite subset of \( \mathbb{N} \). It means that \( A \) is not bounded above. So if \( \epsilon > 0 \) is arbitrary, we may find \( n \in A \) for which \( n > \frac{1}{\epsilon} \) and hence \( \frac{1}{n} < \epsilon \). Therefore \( \text{dist}(\infty, A) = \inf \{d(\infty, n) : n \in A\} = \inf \{\frac{1}{n} : n \in A\} = 0 \). So \( \infty \in \partial A \subseteq A \).

Conversely, suppose \( A \) is a finite subset of \( \mathbb{N} \). In this case, \( \text{dist}(\infty, A) = \inf \{d(\infty, n) : n \in A\} = \min \{d(\infty, n) : n \in A\} > 0 \), since the infimum of a finite set is just the minimum, and all distances \( d(\infty, n) \) for \( n \in A \) are strictly positive. So \( \infty \notin \partial A \). Since \( \infty \notin A \) by hypothesis, we conclude \( \infty \notin A \). \( \square \)

**HW3 #5.** Again consider \((\mathbb{N}, d)\) as in Example 1.1.12. Let \( A \subseteq \mathbb{N} \) and suppose \( \infty \in A \). Show that \( A \) is open if and only if \( A \) is cofinite (that is, \( \mathbb{N} \setminus A \) is a finite set).

**Proof.** First suppose \( A \) is cofinite. Then \( A^c \) is finite, and therefore closed. So \( A \) is open.

Conversely, suppose \( A \) is not cofinite. Then \( A^c \) is an infinite set, and \( \infty \notin A^c \) by hypothesis. Then by the result of Problem 4 above, \( \infty \in \partial A^c \) (whence \( A^c = A^c \) is open). It follows that \( A^c \) is not closed, and therefore \( A \) is not open. \( \square \)

**HW3 #6.** Consider \( \mathbb{Q} \) as a metric space with the usual metric. Let \( a \) and \( b \) be irrational numbers such that \( a < b \). Define the set \( E = \{q \in \mathbb{Q} : a < q < b\} \)

Is \( E \) open? Is \( E \) closed? Prove your answers.

We claim \( E \) is both open and closed, and prove it below.

**Proof.** To show \( E \) is clopen, it suffices to prove that \( \partial E = \emptyset \) (whence \( E = E^o = \overline{E} \)). So let \( x \in \mathbb{Q} \); we will show \( x \notin \partial E \). There are three cases: either \( x < a \), or \( x > b \), or \( a < x < b \).

First suppose \( x < a \). If \( q \in E \) is arbitrary, then \( a < q \), and hence \( d(x, q) = q - x > a - x \). It follows that \( \text{dist}(x, E) = \inf \{d(x, q) : q \in E\} \geq a - x > 0 \).

So \( x \notin \partial E \).

Next suppose \( x > b \). For every \( q \in E \), we have \( q < b \) and therefore \( d(x, q) = x - q > x - b \). So \( \text{dist}(x, E) = \inf \{d(x, q) : q \in E\} \geq x - b > 0 \).

So again \( x \notin \partial E \).

Lastly, suppose \( a < x < b \). Set \( R = \min\{x - a, b - x\} \). For every \( p \in E^c \), we have either \( p < a \) or \( p > b \). If \( p < a \), then \( d(x, p) = x - p > x - a \geq R \). If \( p > b \), then \( d(x, p) = p - x > b - x \geq R \). So we have shown that
\[ \text{dist}(x, E^c) = \inf \{ d(x, p) : p \in E^c \} \geq R > 0. \]

So \( x \notin \partial E \). Since we checked all cases, \( \partial E \) is empty and \( E \) is clopen. \( \square \)

**HW4 \#1.** Show that every subset of \( \tilde{\mathbb{N}} \) is open or closed, but not every subset is clopen.

**Proof.** Let \( A \subseteq \tilde{\mathbb{N}} \). Note that for each \( n \in \mathbb{N} \), \( n \) is an isolated point of \( \tilde{\mathbb{N}} \) by HW2 \#3. So each singleton \( \{n\} \) is open by HW3 \#2, for \( n \in \mathbb{N} \). Thus, if \( A \subseteq \mathbb{N} \), then \( A = \bigcup_{n \in A} \{n\} \) is a union of open singletons and is therefore an open set. On the other hand, if \( A \nsubseteq \mathbb{N} \), then \( \infty \in A \). It then follows that \( A^c \subseteq \mathbb{N} \), whence \( A^c \) is open by our previous remarks. So \( A \) is closed. Since \( A \) was arbitrary, we have shown every subset of \( \tilde{\mathbb{N}} \) is either open or closed.

On the other hand, \( \tilde{\mathbb{N}} \) has non-clopen subsets. One example is the subset \( \mathbb{N} \). \( \mathbb{N} \) is open in \( \tilde{\mathbb{N}} \), but the closure of \( \mathbb{N} \) contains \( \infty \) by HW3 \#4. So \( \mathbb{N} = \tilde{\mathbb{N}} \neq \mathbb{N} \), whence \( \mathbb{N} \) is not closed. It follows from these arguments that the complementary set \( \{\infty\} \) is a closed set which is not open. \( \square \)

**HW4 \#2.** Show that the open subsets of the subspace \( \mathbb{R} \times \{0\} \) of \( \mathbb{R}^2 \) with the usual metric are precisely those subsets of the form \( U \times \{0\} \) where \( U \) is open in \( \mathbb{R} \). Show also that none of those sets, except the empty set, is open in \( \mathbb{R}^2 \).

Let us start by proving the following lemma.

**Lemma.** Suppose \((X, d)\) and \((Y, e)\) are metric spaces and \( f : X \to Y \) is an isometry of \( X \) onto \( Y \). Then \( U \subseteq X \) is open in \( X \) if and only if \( f(U) \subseteq Y \) is open in \( Y \).

**Proof.** Let \( U \subseteq X \) be an arbitrary subset. Since \( f \) is a bijection, for any point \( x \in U \) there corresponds some point \( y \in f(U) \), such that \( y = f(x) \). Now since \( f \) is an isometry, i.e. a distance-preserving bijection, we compute the following equality for all \( x \in U \) and \( y = f(x) \in f(U) \):

\[
\text{dist}(y, Y\setminus f(U)) = \inf \{ e(y, w) : w \in Y\setminus f(U) \} \\
= \inf \{ e(f(x), f(z)) : f(z) \in Y\setminus f(U) \} \\
= \inf \{ e(f(x), f(z)) : z \in X\setminus U \} \\
= \inf \{ d(x, z) : z \in X\setminus U \} \\
= \text{dist}(x, X\setminus U).
\]

Given the equality above, it is easy to finish proving the lemma:

\[
U \text{ is open in } X \iff \forall x \in U \text{ dist}(x, X\setminus U) > 0 \\
\iff \forall y \in f(U) \text{ dist}(y, Y\setminus f(U)) > 0 \\
\iff f(U) \text{ is open in } Y.
\]

\( \square \)
Proof of HW4 #2. Note that $\mathbb{R}$ is isometric to $\mathbb{R} \times \{0\}$ via the isometry $f$ defined by $f(x) = (x, 0)$, for all $x \in \mathbb{R}$. (The fact that $f$ is an isometry is extremely easy to check.) So by the lemma we just proved, a subset $W$ of $\mathbb{R} \times \{0\}$ is open if and only if $U = f^{-1}(W)$ is open in $\mathbb{R}$, if and only if $W = U \times \{0\}$ for some open $U \subseteq \mathbb{R}$.

On the other hand, no such set is open in $\mathbb{R}^2$, unless $U = \emptyset$. If $U$ is nonempty, then there exists $(x, 0) \in U \times \{0\}$. Every open ball about $(x, 0)$ will intersect the complement of $\mathbb{R} \times \{0\}$, so $(x, 0) \in \partial(U \times \{0\})$. This shows $U \times \{0\}$ is not open. \hfill $\square$

HW4 #3. Suppose $X$ is a metric space and $S \subseteq X$. Show that $S$ is dense in $X$ if and only if $S$ has nonempty intersection with every open ball of $X$.

Proof. First suppose $S$ is dense and let $b[x; r)$ be some open ball in $X$ (for some $x \in X$, $r \in \mathbb{R}^+)$). Since $S$ is dense, $x \in \overline{S}$. It follows then that $S \cap b[x; r) \neq \emptyset$.

Conversely, suppose $S$ has nonempty intersection with every open ball of $X$. Let $x \in X$ be arbitrary. Then for every $r \in \mathbb{R}^+$, $S \cap b[x; r) \neq \emptyset$ by hypothesis. Therefore $x \in \overline{S}$. Since $x$ was arbitrary, we have shown $X \subseteq \overline{S}$, which implies $S$ is dense. \hfill $\square$

HW4 #4. Find all dense subsets of $\mathbb{N}$. Prove you have found them all.

We claim $\mathbb{N}$ and $\mathbb{N}$ are the only two dense subsets of $\mathbb{N}$.

Proof. It is obvious that $\mathbb{N}$ is dense in $\mathbb{N}$. To see that $\mathbb{N}$ is dense, simply note that $\mathbb{N}$ is an infinite set and hence $\infty \in \overline{\mathbb{N}}$ by HW3 #4; so $\mathbb{N} \subseteq \mathbb{N}$.

Now if $A$ is any other subset of $\mathbb{N}$ other than $\mathbb{N}$ or $\mathbb{N}$, then $A$ must be missing some integer, i.e. there exists $n \in \mathbb{N}$ with $n \notin A$. But we have shown in previous problems that $\{n\}$ is open; hence $\{n\}$ is an open ball in $\mathbb{N}$ that has empty intersection with $A$. By the previous problem, $A$ is not dense. \hfill $\square$

HW5 #1. Prove the Bounded Monotone Convergence Theorem: If $(x_n)$ is a bounded monotone sequence in $\mathbb{R}$ (with the usual metric), then $(x_n)$ converges to some point in $\mathbb{R}$.

Proof. Either $(x_n)$ is nondecreasing or nonincreasing; first suppose nondecreasing. Set $z = \sup\{x_n : n \in \mathbb{N}\}$. Note that since $(x_n)$ is bounded above by some number $M$ by hypothesis, we have $z \leq M < \infty$ and hence $z \in \mathbb{R}$. We claim that $x_n \to z$.

To see this, let $r \in \mathbb{R}^+$ be arbitrary. By our definition of $z$, and our bonus homework characterizing suprema, we know that there exists $m \in \mathbb{N}$ for which $x_m > z - r$. But then since $(x_n)$ is a nondecreasing sequence, we have $z - r < x_m \leq x_n \leq z$ for all $n \geq m$. In other words, we see that $\operatorname{tail}_m(x_n) \subseteq (z - r, z] \subseteq b[z; r)$. Since $r$ was arbitrary, we have shown $x_n \to z$.

The nonincreasing case is very similar and we leave it to the reader. \hfill $\square$

HW5 #2. Prove the Squeeze Theorem: Suppose $(a_n)$, $(b_n)$, and $(c_n)$ are sequences in $\mathbb{R}$ (with the usual metric), and suppose there exists an $M \in \mathbb{N}$ such that

$$
\text{for all } n \geq M, \quad a_n \leq b_n \leq c_n.
$$

If $a_n \to L$ and $c_n \to L$ for some $L \in \mathbb{R}$, then $b_n \to L$. 
**Proof.** Let \( r \in \mathbb{R}^+ \) be arbitrary. Since \( a_n \to L \), it means there exists \( m_1 \in \mathbb{N} \) such that
\[
\text{tail}_{m_1}(x_n) \subseteq b[L; r] = (L - r, L + r).
\]

Likewise since \( c_n \to L \), there exists \( m_2 \in \mathbb{N} \) for which
\[
\text{tail}_{m_2}(x_n) \subseteq b[L; r] = (L - r, L + r).
\]

Now set \( N = \max(M, m_1, m_2) \). Then for every \( n \geq N \), the term \( b_n \) satisfies \( a_n \leq b_n \leq c_n \) by hypothesis, and also \( b_n \) lies in both tails mentioned above. So we have
\[
L - r < a_n \leq b_n \leq c_n < L + r,
\]
so \( b_n \in (L - r, L + r) \). Therefore \( \text{tail}_N(x_n) \subseteq b[z; r] \). Since \( r \) was arbitrary, \( b_n \to L \) and the theorem is proved. \( \square \)

**HW6 #1.** Let \( X \) and \( Y \) be sets, let \( A \subseteq X \) and \( B \subseteq Y \), and let \( f : X \to Y \) be a function. Prove that \( f^{-1}(f(A)) \supseteq A \) and \( f(f^{-1}(B)) \subseteq B \), but it is not necessarily the case that \( f^{-1}(f(A)) = A \), nor that \( f(f^{-1}(B)) = B \).

**Proof.** Let \( a \in A \). Then \( f(a) \in f(A) \), and therefore \( a \in f^{-1}(f(A)) \). Since \( a \in A \) was taken arbitrarily, we have shown \( A \subseteq f^{-1}(f(A)) \). However, it is not necessarily the case that \( f^{-1}(f(A)) = A \) for instance one could take the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^2 \) for all \( x \in \mathbb{R} \), and observe that \( f^{-1}(f([0, 2])) = f^{-1}([0, 4]) = [-2, 2] \neq [0, 2] \).

Next let \( y \in f(f^{-1}(B)) \). It means there exists an \( x \in f^{-1}(B) \) so that \( f(x) = y \). Since \( x \in f^{-1}(B) \), \( f(x) \in B \). Therefore \( y \in B \). So we have shown \( f(f^{-1}(B)) \subseteq B \). But equality need not hold: for instance take \( f \) as in the previous paragraph, and observe that \( f(f^{-1}([-1, 0])) = f(\{0\}) = \{0\} \neq [-1, 0] \).

**HW6 #2.** Let \( X, Y \), and \( f \) be as in the previous problem. Prove that \( f^{-1}(f(A)) = A \) for all \( A \subseteq X \) if and only if \( f \) is an injection.

**Proof.** (\( \Rightarrow \)) First assume \( f^{-1}(f(A)) = A \) for all \( A \subseteq X \). Let \( x, y \in X \) be arbitrary and assume \( f(x) = f(y) \). Then \( f(\{x\}) = f(\{y\}) \), and therefore by hypothesis \( \{x\} = f^{-1}(f(\{x\})) = f^{-1}(f(\{y\})) = \{y\} \). So \( x = y \). This shows \( f \) is injective.

(\( \Leftarrow \)) On the other hand assume \( f \) is injective, and let \( A \subseteq X \) be arbitrary. By the previous problem, \( f^{-1}(f(A)) \supseteq A \), so we need only show that \( f^{-1}(f(A)) \subseteq A \). So let \( x \in f^{-1}(f(A)) \) be arbitrary. It means \( f(x) \in f(A) \). I.e. there exists an \( a \in A \) so that \( f(a) = f(x) \). But since \( f \) is injective, this implies \( a = x \). Therefore \( x \in A \) in the first place. So \( f^{-1}(f(A)) \subseteq A \) as claimed. \( \square \)

**HW6 #4.** Prove that every affine map \( f \) is a continuous bijection. Deduce as an immediate corollary that the inverse mapping \( f^{-1} \) is also continuous.

**Proof.** Let \( f : \mathbb{R} \to \mathbb{R} \) be an affine map, so \( f(x) = mx + b \) for all \( x \in \mathbb{R} \), for some \( m, b \in \mathbb{R} \) with \( m \neq 0 \).

\( f \) is injective.) Let \( x, y \in \mathbb{R} \) be arbitrary and assume \( f(x) = f(y) \). Clearly \( mx + b = my + b \) implies \( x = y \).
(f is surjective.) Let \( z \in \mathbb{R} \) be arbitrary. Setting \( x = \frac{z - b}{m} \) (which makes sense because \( m \neq 0 \)), we see that \( f(x) = z \).

(f is continuous.) Fix any \( x \in \mathbb{R} \) and any \( \epsilon \in \mathbb{R}^+ \). Set \( \delta = \frac{\epsilon}{|m|} \). Assume \( z \in \mathbb{R} \) satisfies \( |x - z| < \delta \). Then \( |f(x) - f(z)| = |mx + b - mz - b| = m|x - z| < |m|\delta = \epsilon \). So \( f \) is continuous at \( x \), for each \( x \in \mathbb{R} \).

\( (f^{-1} \) is continuous.\( ) The inverse mapping is given by \( f^{-1}(y) = \frac{1}{m}y - \frac{b}{m} \), which is also an affine map, hence continuous by our earlier remarks.

 HW6 #6. Prove that the map \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = x^3 \) for all \( x \in \mathbb{R} \) is continuous.

Proof. Fix any \( x \in \mathbb{R} \) and any \( \epsilon \in \mathbb{R}^+ \). Let \( \delta = \min\{1, \frac{\epsilon}{3x^2 + 3|x| + 1}\} \), and let \( z \in \mathbb{R} \) be such that \( |x - z| < \delta \). Then we have

\[
|f(x) - f(z)| = |x^3 - z^3| = |x - z||x^2 + xz + z^2|.
\]

Since \( |x - z| \leq 1 \), we have \( |z| \leq |x| + 1 \) by the triangle inequality, and hence the above implies:

\[
|f(x) - f(z)| \leq |x - z|[x^2 + |x||z| + z^2] \\
\leq |x - z|[x^2 + |x|(|x| + 1) + (|x| + 1)^2] \\
= |x - z|[3x^2 + 3|x| + 1] \\
< \delta[3x^2 + 3|x| + 1] \\
\leq \epsilon.
\]

Thus \( f \) is continuous at \( x \), for every \( x \in \mathbb{R} \).

 HW6 #7. Prove that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 0 \) iff \( x \in \mathbb{Q} \), and \( f(x) = 1 \) iff \( x \in \mathbb{R}\setminus\mathbb{Q} \), is not continuous at any point in \( \mathbb{R} \).

Proof. Let \( x \in \mathbb{R} \); we will show \( f \) is not continuous at \( x \). Set \( \epsilon = 1 \), and let \( \delta \in \mathbb{R}^+ \) be arbitrary. Now either \( x \in \mathbb{Q} \) or \( x \in \mathbb{R}\setminus\mathbb{Q} \). Both \( \mathbb{Q} \) and \( \mathbb{R}\setminus\mathbb{Q} \) are dense in the real line. Hence, we can find \( z \in (x - \delta, x + \delta) \) so that \( x \) and \( z \) are not both rational. Then by the definition of the map \( f \), we get \( |f(x) - f(z)| = 1 \geq \epsilon \). So \( f \) fails continuity at \( x \) as claimed.

 HW6 #8. Prove that the function \( f : \mathbb{R} \to \mathbb{R} \) defined by \( f(x) = 0 \) iff \( x \in \mathbb{Q} \), and \( f(x) = x \) iff \( x \in \mathbb{R}\setminus\mathbb{Q} \), is continuous at 0.

Proof. Let \( \epsilon \in \mathbb{R}^+ \). Set \( \delta = \epsilon \). Let \( z \in \mathbb{R} \) be such that \( |z| = |z - 0| < \delta = \epsilon \). Either \( z \in \mathbb{Q} \) or \( z \not\in \mathbb{Q} \). If \( z \in \mathbb{Q} \), then \( |f(0) - f(z)| = |0 - 0| = 0 < \epsilon \). If \( z \not\in \mathbb{Q} \), then \( |f(0) - f(z)| = |0 - z| = |z| < \epsilon \). So \( f \) is continuous at 0.