1 The basics of Ordinary Differential Equations

You start learning MATH 266: Introduction to Ordinary Differential Equations. I will often abbreviate Ordinary Differential Equation (or Equations) as ODE, irrespective of whether I am speaking about one or several equations, you should read this abbreviation depending on the context. There will be other abbreviations in this course, but this one is by far the most frequent one, so please memorize it. As in any other course, it is of paramount importance to understand the meaning of every word that I am using. Can you actually explain to yourself what every word in the title of the course means?

So let us start with "Introduction." This obviously means that we will not go very deep into the mathematical theory of ODE, however, you should not deceive this word with "simple"; we are going to cover quite a number of topics at a reasonably fast pace, so start working from the first lecture. The word "Equation" means "equality with unknowns," or even more formally, "a formula of the form $A = B$, where A are B are expressions containing one or several variables called unknowns." The word "Differential" means that we are dealing with derivatives; from your Calculus course you should remember that we find derivatives of *functions*, and it implies that our unknowns or variables in the equations will be functions. Finally, to decipher the word "Ordinary" you should recall that in Calculus you studied ordinary derivatives and partial derivatives; the distinction is that in the former case the function in the question depends on a single variable, whereas in the latter, the function is multivariable.

Therefore, the conclusion is that the main object of our study in this course is the equations which contain unknown functions, depending on a single variable, and derivatives of these functions. (*Q:* Can you now think of an example of an ODE? I will use the letter "*Q:*" to ask questions throughout the notes, and a student should spend a minute or two contemplating on these questions.)

1.1 Main definitions

So, here is a general definition.

Definition 1. *An explicit ODE of the n-th order is the expression of the form*

$$y^{(n)}(x) = f\big(x, y(x), y'(x), \ldots, y^{(n-1)}(x)\big). \tag{1}$$

A few words of explanation are necessary.

- First, the function f of n variables is given.
- In the definition there is the word "explicit." This means that the highest derivative $y^{(n)}(x)$ can be expressed as the function of the independent variable x, unknown function or dependent variable $y(x)$, and its derivatives $y'(x), \ldots, y^{(n-1)}(x)$ up to the order $n-1$. I remind that the notation $y^{(n)}(x)$ means the n-th derivative (so we use $y'(x), y''(x), y'''(x)$ but $y^{(4)}(x)$, $y^{(5)}(x)$, etc). Only such ODE will be studied in this course. Therefore, the word "explicit" will be usually dropped. (*Q*: Can you come up with an example of an *implicit* ODE?)
- *The order* of an ODE is the order of the highest derivative in the equation (the most important definitions will be usually given in the formal form as above; however, for a number of terms I

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024

will use the italic font to give a definition, so pay attention). For instance, an ODE of the first order (note I dropped the word *explicit*) is defined as the equation of the form

$$y'(x) = f(x, y(x)).$$

- Very often the dependence on the independent variable x of the unknown function is suppressed. I.e., the definition is very often stated as

  $$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

  for the ODE of the n-th order and

  $$y' = f(x, y)$$

  for the ODE of the first order.

- Is is very often that y denotes the dependent variable (the unknown function) and x denotes the independent variable (the variable with respect to which the derivative is taken), however, nothing prohibits considering the letter x as the dependent variable and y as an independent variable, or choose different letters. For instance, in many occasions instead of x I will use t since this variable will be playing the role of time. Therefore, it is the first task for a person studying a particular differential equation to determine what is what, and which letter actually denotes the unknown function. For example, in the equation

  $$\rho' = \rho(1 - \rho),$$

  ρ is the dependent variable (unknown function) and we actually do not know how the independent variable is denoted, hence we free to choose our own, e.g., θ (ρ and θ are the usual notations for the *polar coordinates* on the plane, often letter ρ denoting *radial distance* is replaced with r).

- There are several different notations to denote derivatives. We will use the following: the usual prime notation $y'(x), y''(x), \ldots$ (this notation is due to Joseph-Louis Lagrange, 1736–1813, a French mathematician, physicist and astronomer); Isaac Newton's (1642–1725, English polymath) notation is when the derivatives are denoted by dots:

  $$\dot{y} := y', \quad \ddot{y} := y'', \quad \dddot{y} := y''',$$

  this notation is still very popular in classical mechanics and other fields when the independent variable physically is *time*; and finally the differential notation due to Gottfried Wilhelm Leibnitz (1646–1716, German polymath and philosopher, together with Newton is generally considered the founding father of *Calculus*) is given by

  $$\frac{\mathrm{d}y}{\mathrm{d}x} := y', \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} := y'', \quad \ldots, \quad \frac{\mathrm{d}^ny}{\mathrm{d}x^n} := y^{(n)}.$$

  The sign ":=" means "the expression on the left is defined to be equal to the expression on the right" (think about the assignment operator, if you have some programming experience). Of course, to use this sign we need to know already what the right-hand side means.

- Finally, f in the definitions above is different for different definitions, which is hopefully understood.

Now it is time to switch from the general abstract definitions to some concrete examples.

Example 2. Consider the following ODE of the second order

$$y'' + \lambda y = 0, \quad \lambda \in \mathbf{R}.$$

Convince yourself that this ODE is explicit. This ODE includes the parameter λ, which is assumed to be a real number. The notation $\lambda \in \mathbf{R}$ means that λ is an element of the set of real numbers, which is usually denoted as $\mathbf{R}$ or $\mathbb{R}$ (I prefer the former). y in this equation is obviously the dependent variable or our unknown, and we do not know how the independent variable is denoted, so we can pick any letter, e.g., x; thence $y = y(x)$. A significant part of the course will be devoted to solving such ordinary differential equations. It turns out that the form of the solution depends on the sign of λ. The solution to this ODE is given by (we will learn how to actually find this solution in due course)

$$y(x) = \begin{cases} Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}, & \text{if } \lambda < 0, \\ A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x), & \text{if } \lambda > 0. \end{cases}$$

(Q: Can you find a solution if $\lambda = 0$?) Here A and B are arbitrary real constants, i.e., $A, B \in \mathbf{R}$.

How to check that the given function is actually a solution to the given differential equation? The same as for the usual algebraic equations — just plug the formula into the equation (make this exercise for the example above!)

Example 3. Actually, during your Calculus courses you already solved a lot of differential equations (Isaac Newton invented Calculus to solve ODE). Indeed, consider a very special case of the first order ODE

$$y'(x) = f(x).$$

Here the right-hand side depends only on the independent variable. If the derivative of y is given by $f(x)$, what is y? As you were taught in Calculus this is an antiderivative of $f(x)$, i.e., such function $F(x)$ that $F'(x) = f(x)$, and to find it we need to take the indefinite integral

$$y(x) = F(x) = \int f(x)\,\mathrm{d}x.$$

Here is a concrete example. Solve

$$y'(x) = 2x.$$

Obviously, one solution is x^2. But we also have more: e.g., $x^2 + 5$ or $x^2 - \pi$. In general,

$$y(x) = \int 2x\,\mathrm{d}x = x^2 + C,$$

where C is an arbitrary constant. Hence, a very important conclusion follows: *any ODE has infinitely many solutions* (well, to be precise, *most* of the ODE have infinitely many solutions since it is possible to build pathological examples for which this is not true. We will not bother about such examples. *Q:* Can you give an example of an ODE without any (real) solutions?).

Example 4. Consider a physical example. Assume that we drop a stone from the height of l meters. How long will it take for it to hit the ground? The movement of the stone is governed by the second Newton's law, which says that for the stone

$$ma = F,$$

where F is the net force applied to the stone. In our case, if we disregard the air resistance, this is only gravitation, which is given by the Newton gravitational law:

$$F = -G\frac{Mm}{r^2},$$

where G is the gravitational constant, M is the mass of the Earth, and r is the distance from the center of mass of the Earth to the stone, which simply can be taken as the Earth radius R (this is a very good approximation since R is *way* bigger than l). The minus sign here is necessary since we suppose that the axis along which the force is applied has the direction opposite to the force of gravity. So we obtain

$$a = -g, \quad g := G\frac{M}{R^2},$$

where g is the acceleration of the free falling body (*Q:* Do you remember the actual numerical value of this constant from your Physics classes?). So where is an ODE here? Recall that the acceleration a is the derivative of the velocity v, and velocity is the derivative of the displacement x, hence, denoting our independent variable here as t for "time," we obtain an ODE of our problem:

$$x'' = -g.$$

Again, here $x = x(t)$ is the dependent variable. Since we can write $(x')' = -g$, we find that

$$x'(t) = \int (-g)\,\mathrm{d}t = -gt + C_1,$$

and finally,

$$x(t) = \int x'\,\mathrm{d}t = \frac{-gt^2}{2} + C_1 t + C_2,$$

which is the *general solution* to our ODE. C_1 and C_2 are arbitrary constants.

There are several conclusions from this example. First, the number of arbitrary constants in the solution to ODE is equal to the order of this equation (look at the examples above). Second, for our physical problem, we actually would like to answer a simple and concrete question: How long will it take for the stone to hit the Earth? We cannot answer this question having infinitely many solutions. Something is missing. A few seconds thinking helps realize that we also need to know the initial position of the stone (in our case $x(0) = l$) and the initial velocity ($x'(0) = 0$). This is enough to determine uniquely what the values of C_1 and C_2 are. The initial position condition implies that

$$C_2 = l,$$

and the initial velocity condition implies that $C_1 = 0$, hence, finally, our unique solution is

$$x(t) = \frac{-gt^2}{2} + l.$$

(*Q:* So, if l is 20 meters, how long will it take for the stone to hit the ground?)

There is a lot of important information in the last examples. We summarize it in a formal way.

Definition 5. *A function $\phi(x)$ is a solution to ODE* (1) *on the interval $I = (a, b)$ if, first, this function is n times continuously differentiable with respect to x on I and, second,*

$$\phi^{(n)}(x) \equiv f\big(x, \phi(x), \phi'(x), \dots, \phi^{(n-1)}(x)\big), \quad x \in I.$$

In this definition $\equiv$ means "identically equal," i.e., for any $x \in I$. In the following lectures I will use the notation $\phi \in C^{(p)}(U; V)$ for the function $\phi \colon U \longrightarrow V$ to be p times continuously differentiable. Hence in the definition above I could have used the notation $\phi \in C^{(n)}(I; \mathbf{R})$. It is often abbreviated $C(U; V) := C^{(0)}(U; V)$ for the set of *continuous* functions.

We already learned that usually there are infinitely many solutions to ODE (1). Here is the way to present them all at the same time — the concept of the general solution to an ODE.

Definition 6. *The function $\phi(x) = \phi(x, C_1, ..., C_n)$ is called the general solution to ODE* (1) *if any solution to* (1) *can be obtained from $\phi(x)$ for a particular choice of constants $C_1, \dots, C_n$.*

The general solution is a formula that depends on arbitrary constants and any particular solution to ODE can be represented using this formula, we only need to choose correctly the values of $C_1, \dots, C_n$.

To find *a particular solution* to an ODE, it is usually necessary to supplement the ODE with *some initial conditions*; the number of the initial conditions coincides with the number of the arbitrary constants in the general solutions and hence with the order of the ODE. For example, for the first order ODE $y' = f(x, y)$ one needs one initial condition $y(x_0) = y_0$, where x_0, y_0 are given numbers.

Definition 7. *An ordinary differential equation plus the initial conditions are called the initial value problem* (*or Cauchy's problem*)*.*

Initial value problem is usually abbreviated as IVP, and this is the second abbreviation in our course that has to be memorized.

To sum up, if we are given an ODE then it is a good guess that we are looking for the general solution, i.e., for a formula, or for several formulas, that depend on arbitrary constants and describe all possible solutions to this ODE. If, instead, we consider IVP, i.e., equation plus initial conditions, then we are looking for a particular solution that 1) solves the equation, 2) satisfies these initial conditions.

1.2 Where do ODE come from?

Formally, ODE describe a deterministic process with a finite dimensional state space. The process is *deterministic* if its law does not change "spontaneously," and *finite dimensional* if a finite number of values is enough to describe the process at any time moment. A great deal of different real-world processes can be considered as deterministic and finite dimensional, at least at the first approximation. This is especially true for the physics and in particular for the classical mechanics.

Here are a few examples.

- *Physics.* As we already discussed in one of the examples, the second Newton's law is actually an ODE. To make things more realistic, recall that displacement, velocity, and acceleration generally are *vectors*, not *scalars*. In we consider our usual 3D space $\mathbf{R}^3$, then the second Newton's law has the form

$$m\boldsymbol{a} = \sum_k \boldsymbol{F}_k,$$

where the bold font means vectors, in particular, in $\mathbf{R}^3$, $\boldsymbol{a} = (a_1, a_2, a_3)$ in coordinates, and the same is true for the forces applied to the body. Now consider two bodies in $\mathbf{R}^3$. What differential equations govern the movement of these bodies if the only force that they experience is the gravitational force (i.e., these two bodies attract to each other with the force proportional to the product of their masses and inversely proportional to the square of the distance between them)? Let $\boldsymbol{x}_1(t)$, $\boldsymbol{x}_2(t)$ be the displacements of the two bodies respectively. Then the second Newton's law together with the Newton law of gravity say that for the first body

$$m_1\ddot{\boldsymbol{x}}_1 = G\frac{m_1m_2}{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^2}\frac{(\boldsymbol{x}_2 - \boldsymbol{x}_1)}{\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|},$$

where $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|$ is the usual Euclidian distance between the two bodies, and $\frac{(\boldsymbol{x}_2-\boldsymbol{x}_1)}{\|\boldsymbol{x}_2-\boldsymbol{x}_1\|}$ is the unit vector in the direction from the first to the second body. A similar equation holds for the second body. After some simplifications we obtain a system of two second order ODE

$$\ddot{\boldsymbol{x}}_1 = G\frac{m_2(\boldsymbol{x}_2 - \boldsymbol{x}_1)}{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^3},$$
$$\ddot{\boldsymbol{x}}_2 = G\frac{m_1(\boldsymbol{x}_1 - \boldsymbol{x}_2)}{\|\boldsymbol{x}_1 - \boldsymbol{x}_2\|^3}.$$

Recall that $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are vectors, each having three coordinates, therefore we actually have six second order ordinary differential equations, which should be solved if we would like to find out the behavior of the bodies. To get a unique solution one needs twelve initial conditions — three initial coordinates of the first body, three initial components of the velocity of the first body, and the same for the second body. *Q:* Can you guess what are possible solutions of this system (think about the system the Sun and the Earth)?

- *Biology.* Consider the change with time of the population number $N(t)$. In a small period of time Δt the change can be decomposed into births with the rate b, deaths with the rate d, and immigration with the rate i. Hence,

$$\underbrace{N(t+\Delta t)}_{\text{the size at } t+\Delta t} = \underbrace{N(t)}_{\text{the size at } t} + \underbrace{bN(t)\Delta t}_{\text{births}} - \underbrace{dN(t)\Delta t}_{\text{deaths}} + \underbrace{i\Delta t}_{\text{immigration}}.$$

Rearranging as

$$N(t+\Delta t) - N(t) = \big((b-d)N(t) + i\big)\Delta t,$$

dividing by Δt and using the limit $\Delta t \to 0$, we find

$$\dot{N} = (b-d)N + i.$$

Assuming that the population is closed (i.e., no immigration, $i = 0$) and denoting $m := b - d$, we finally obtain a most important ODE, which is called in biology *the Malthus equation* (after Thomas Robert Malthus, 1766–1834, an English economist and demographer),

$$\dot{N} = mN.$$

Q: Can you guess its solution (first think about the case $m = 1$)? In general, of course, both b and d should depend on the population size $N(t)$. Here we need only one initial condition: The initial population size $N(t_0) = N_0$.

- *Chemistry.* Consider a theoretical chemical reaction when compound A reacts with compound B to produce, at the rate k, compound C, schematically

  $$A + B \xrightarrow{k} C.$$

  The *law of mass action* states that the reaction rate is proportional to the reactant concentrations. This means in our case, denoting by $[X]$ the concentration of X,

  $$\frac{\mathrm{d}[A]}{\mathrm{d}t} = -k[A][B],$$
  $$\frac{\mathrm{d}[B]}{\mathrm{d}t} = -k[A][B],$$
  $$\frac{\mathrm{d}[C]}{\mathrm{d}t} = k[A][B].$$

  This is a system of three nonlinear first order ODE.

The list of various application of ODE in natural sciences can be easily continued, as we will see in our course. In particular, an ODE can be stumbled upon in *Psychology*, *Economics*, *Social Sciences*, *Geometry*, etc; in short, you can find ODE in almost any field of study.