10 Complex numbers. Solving second order linear ODE with constant coefficients

This lecture opens the second part of our course. From now on the main object of the study will be the linear ODE. And even not simply linear, but linear ODE with constant coefficients. In these lectures I will use t to denote the independent variable (it is often useful to think about it as a time variable) and $y = y(t)$ for the dependent variable, the unknown function which we will need to determine.

But before embarking on dealing with ODE, let me briefly review the necessary material about the complex numbers, which will be of great help to us during the rest of the course.

10.1 Complex numbers

Complex numbers are defined as

$$z = x + \mathrm{i}\, y,$$

where x and y are real numbers, and i is the *imaginary unit* that has the characteristic property $\mathrm{i}^2 = -1$ (engineers usually use for the same purpose the letter "j" but we will stick to the mathematical notation). In other words, a complex number is defined if there is a pair of real numbers (x, y). x is called the *real part* of the complex number, $x = \operatorname{Re} z$, and y is called the *imaginary part* of z, $y = \operatorname{Im} z$. We have hence $z = \operatorname{Re} z + \mathrm{i} \operatorname{Im} z$. The set of complex numbers is denoted as $\mathbf{C}$. Note that $\mathbf{R} \subset \mathbf{C}$, since any real number $x \in \mathbf{R}$ can be written as $x = x + \mathrm{i} \cdot 0$ (for a slight confusion: this sentence is not exactly true, but these subtleties will not concern us here).

Two complex numbers z_1 and z_2 are the same if $\operatorname{Re} z_1 = \operatorname{Re} z_2$ and $\operatorname{Im} z_1 = \operatorname{Im} z_2$:

$$z_1 = z_2 \Leftrightarrow \operatorname{Re} z_1 = \operatorname{Re} z_2 \text{ and } \operatorname{Im} z_1 = \operatorname{Im} z_2.$$

Since any complex number is actually a pair of real numbers, it is convenient to represent complex numbers as points on the plane or vectors on the plane, with the beginning at the origin (see Fig. 1). According to the general terminology in this case $\mathbf{R}^2$ is called the *complex plane*, x-axis is called the real axis and y-axis is called the imaginary axis.

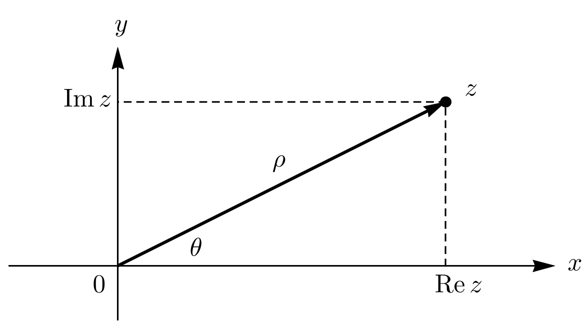

Figure 1: Geometric interpretation of complex numbers. θ is the angle between vector z and x-axis; ρ is the length of the same vector: $\rho = |z|$.

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I said that a complex number is the same as a pair of two real numbers. This is not exactly so, because we additionally need the rules for the arithmetic operations. The four arithmetic operations are defined for complex numbers similarly to the usual arithmetic operations for real numbers; in these definitions below we use i as a number that is subject to the same arithmetic rules as usual numbers modulo the key property that $\mathrm{i}^2 = -1$.

Let $z_1 = x_1 + \mathrm{i}y_1$ and $z_2 = x_2 + \mathrm{i}y_2$. Then

$$\begin{aligned} z_1 + z_2 &= x_1 + \mathrm{i}y_1 + x_2 + \mathrm{i}y_2 = (x_1 + x_2) + \mathrm{i}(y_1 + y_2), \\ z_1 - z_2 &= x_1 + \mathrm{i}y_1 - x_2 - \mathrm{i}y_2 = (x_1 - x_2) + \mathrm{i}(y_1 - y_2), \\ z_1 \cdot z_2 &= (x_1 + \mathrm{i}y_1)(x_2 + \mathrm{i}y_2) = (x_1x_2 - y_1y_2) + \mathrm{i}(x_1y_2 + x_2y_1). \end{aligned}$$

Before defining the division of two complex numbers we define the complex conjugate. For any complex number $z = x + \mathrm{i}y$ its *complex conjugate* (or simply conjugate) is the number $\overline{z} := x - \mathrm{i}y$. Using the definition for the product of two complex numbers, we have

$$z\overline{z} = x^2 + y^2 \geq 0 \text{ and } z\overline{z} = 0 \Leftrightarrow z = 0.$$

If you look at Fig. 1, you will see that actually $x^2 + y^2$ gives the square of the length of the vector z, hence we can define the modulus of a complex number z as the length of the corresponding vector on the plane:

$$|z|^2 = x^2 + y^2 = z\overline{z}.$$

Now, if we need to divide z_2 by $z_1 \neq 0 + \mathrm{i} \cdot 0$, we can do the following

$$\frac{z_2}{z_1} = \frac{z_2\overline{z}_1}{z_1\overline{z}_1} = \frac{z_1\overline{z}_1}{|z_1|^2} = \frac{x_1x_2 + y_1y_2}{x_1^2 + y_1^2} + \mathrm{i}\frac{x_1y_2 - x_2y_1}{x_1^2 + y_1^2}\,.$$

Thus defined the four arithmetic operations possess all the usual properties (commutativity, distributivity, associativity) that we so get used to while dealing with the usual real numbers. In mathematical terms, it means that the set of complex numbers **C** forms a *field*. Here are a few exercises to practice working with the complex numbers:

- Show that

$$\overline{z_1 \pm z_2} = \overline{z}_1 \pm \overline{z}_2; \quad \overline{z_1z_2} = \overline{z}_1\overline{z}_2; \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}\,.$$

- Find all complex numbers that solve the equation

$$\overline{z} = z^2.$$

Trigonometric form of the complex number. In Fig. 1 you can see that the point z can be defined as either $z = x + \mathrm{i}y$, where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$ are its Cartesian coordinates, or using the polar coordinates θ (polar angle) and ρ (distance from the origin). We have

$$\rho\cos\theta = x, \quad \rho\sin\theta = y,$$

from where

$$\rho = |z| = \sqrt{x^2 + y^2}\,,$$

and
$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}\,, \quad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}\,.$$
Hence any complex number can be written as
$$z = x + \mathrm{i}y = \rho(\cos\theta + \mathrm{i}\sin\theta).$$
$\rho = |z|$ is the *modulus* of the complex number and θ (defined up to $2\pi k$ additive constant, $k \in \mathbf{Z}$) is the *argument* of z:
$$\rho = |z|, \quad \theta = \arg z.$$
Argument of a complex number is not defined uniquely, and it is convenient also to have the principal value of the polar angle, which is denoted $\operatorname{Arg} z$ and satisfies $0 \le \operatorname{Arg} z < 2\pi$ (or, sometimes, $-\pi < \operatorname{Arg} z \le \pi$).

Convince yourself that
$$|\mathrm{i}| = 1,\ \operatorname{Arg}\mathrm{i} = \frac{\pi}{2}\,, \quad |1+\mathrm{i}| = \sqrt{2},\ \operatorname{Arg}(1+\mathrm{i}) = \frac{\pi}{4}\,.$$

The expression
$$z = \rho(\cos\theta + \mathrm{i}\sin\theta)$$
is the *trigonometric form* of the complex number z. Using this form it is possible (do it!) to show that if we are given two complex numbers z_1 and z_2 then their product has the modulus equal to $|z_1||z_2|$ and argument $\theta_1 + \theta_2$.

Euler's formula. For any complex z it is true that
$$e^{\mathrm{i}z} = \cos z + \mathrm{i}\sin z.$$
The last equality is called *Euler's formula* (we already mentioned this name in Lecture 9), but to fully appreciate it we would need to discuss what the function of the complex argument is, and this is beyond the scope of the course. Instead, we will talk about a particular case, which is true for any $x \in \mathbf{R}$:
$$e^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x.$$
To get an idea where this remarkable identity is coming from, recall that functions e^x, $\cos x$, $\sin x$ have Taylor's series around zero absolutely convergent for any $x \in \mathbf{R}$:
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}\,,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}\,,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{2k-1}}{(2k-1)!}\,.$$
Now plug $\mathrm{i}x$ instead of x into the series for the exponent function, use the property that $\imath^2 = -1$, $\mathrm{i}^3 = -\mathrm{i}$, $\mathrm{i}^4 = 1, \ldots$, rearrange the series and obtain Euler's formula.

Using Euler's formula we can write any complex number in the *exponential form* as

$$z = x + \mathrm{i}y = \rho(\cos\theta + \mathrm{i}\sin\theta) = \rho e^{\mathrm{i}\theta}.$$

Using Euler's formula we also can express usual cos and sin functions through the exponent:

$$\cos x = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2}, \quad \sin x = \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{2\mathrm{i}}.$$

The exponential form of the complex number lets us obtain a number of results. For instance,

$$z^n = (\rho e^{\mathrm{i}\theta})^n = \rho^n e^{\mathrm{i}n\theta} = \rho^n(\cos n\theta + \mathrm{i}\sin n\theta),$$

which is called de Moivre's formula (after Abraham de Moivre, 1667–1754, French mathematician) if $\rho = 1$:

$$(\cos\theta + \mathrm{i}\sin\theta)^n = (\cos n\theta + \mathrm{i}\sin n\theta).$$

Example 1. What is $(1+\mathrm{i})^6$? We can, of course, multiply the factor $(1+\mathrm{i})$ six times (or use the binomial theorem). However, a better approach would be to consider

$$1 + \mathrm{i} = \sqrt{2}e^{\mathrm{i}\frac{\pi}{4}} \implies (1+\mathrm{i})^6 = (\sqrt{2})^6 e^{\mathrm{i}\frac{3\pi}{2}} = -8\mathrm{i}.$$

For an arbitrary $z = x + \mathrm{i}y$ we have

$$e^{x+\mathrm{i}y} = e^x e^{\mathrm{i}y} = e^x(\cos y + \mathrm{i}\sin y).$$

Hence we have

$$\operatorname{Re} e^z = e^x \cos y, \quad \operatorname{Im} e^z = e^x \sin y.$$

Conversely, the sine and cosine functions can be expressed through the complex exponentials. For example, for sine we have

$$\sin x = \operatorname{Im} e^{\mathrm{i}x},$$

or, sometimes more convenient,

$$\sin x = \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{2\mathrm{i}}.$$

You should write the corresponding formulas for cosine.

Example 2. Find the formula for $\cos^3 x$. We can use

$$\cos x = \frac{e^{\mathrm{i}x} + e^{-\mathrm{i}x}}{2},$$

therefore

$$\cos^3 x = \frac{1}{8}(e^{\mathrm{i}x} + e^{-\mathrm{i}x})^3 = \frac{1}{8}(e^{3\mathrm{i}x} + 3e^{\mathrm{i}x} + 3e^{-\mathrm{i}x} + e^{-3\mathrm{i}x}),$$

which can be rewritten as

$$\cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.$$

You should do the same calculations for $\sin^3 x$.

The complex numbers appeared while solving polynomial equations. Consider a polynomial $P_n(z)$ of complex variable z with complex coefficients:

$$P_n(z) = z^n + c_1 z^{n-1} + c_2 z^{n-2} + \ldots + c_{n-1} z + c_n,$$

where $c_j \in \mathbf{C}$. Any number $\hat{z} \in \mathbf{C}$ such that $P_n(\hat{z}) = 0$ is called a root of $P_n(z)$. The most important fact here is called the *fundamental theorem of algebra*:

Theorem 3. *Any nonconstant complex polynomial of degree n has exactly n complex roots if the roots are counted according to their multiplicities.*

We will discuss what "multiplicity" of a root means later, for now let me illustrate this theorem with an example.

The theorem implies, e.g., that the quadratic equation

$$x^2 + 1 = 0,$$

which is usually said not to possess real roots (indeed, there are no such $x \in \mathbf{R}$ that $x^2 + 1 = 0$), has perfectly fine two complex roots:

$$\hat{x}_{1,2} = \pm \mathrm{i}.$$

Example 4. Find the roots of

$$x^3 + 1 = 0.$$

Clearly, one obvious (and real) root is $\hat{x}_1 = -1$, but the fundamental theorem of algebra says that there should be two more (potentially complex) roots. To find them let me rewrite constant -1 in the exponential form:

$$-1 = 1 \cdot e^{\mathrm{i}\pi} = e^{\mathrm{i}(\pi + 2\pi k)}, \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

I look for a solution in the form $x = \rho e^{\mathrm{i}\theta}$, and hence

$$(\rho e^{\mathrm{i}\theta})^3 = \rho^3 e^{3\mathrm{i}\theta} = e^{\mathrm{i}(\pi + 2\pi k)},$$

which implies that

$$\rho^3 = 1 \implies \rho = 1,$$

and

$$\theta = \frac{\pi}{3} + \frac{2\pi k}{3}, \quad k = 0, \pm 1, \pm 2, \ldots,$$

so technically I found infinitely many roots! Note, however, that the argument of complex number is not defined uniquely, only up to $2\pi k$ constant, and if I consider the cases $k = 0, 1, 2, 3$ I get

$$\theta_0 = \frac{\pi}{3}, \quad \theta_1 = \pi, \quad \theta_2 = \frac{5\pi}{3}, \quad \theta_3 = \frac{7\pi}{3} = \frac{\pi}{3} + 2\pi,$$

but the last argument is the same as $\pi/3$ hence my formula gives only 3 distinct roots, as was expected (check some other values of k). Finally I conclude that my roots are

$$\hat{x}_0 = e^{\mathrm{i}\frac{\pi}{3}} = \frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}, \quad \hat{x}_1 = e^{\mathrm{i}\pi} = -1, \quad \hat{x}_2 = e^{\mathrm{i}\frac{5\pi}{3}} = \frac{1}{2} - \mathrm{i}\frac{\sqrt{3}}{2}.$$

You can check that these are indeed the roots by plugging them into the equation and simplifying.

Q: Can you now solve $x^4 + 1 = 0$?

10.2 Solving homogeneous second order linear ordinary differential equation with constant coefficients

I start with the general definition.

Definition 5. *Linear ordinary differential equation of the n-th order is*

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_2(t)y'' + a_1(t)y' + a_0y = f(t), \tag{1}$$

where

$$a_{n-1}(t),\ a_{n-2}(t), \ldots, a_1(t),\ a_0(t),\ f(t)$$

are given functions.

The general solution to (1) depends on n arbitrary constants, and if we need to identify a unique particular solution, we will need n initial conditions of the form

$$y(t_0) = y_0,\ y'(t_0) = y_1, \ldots, y^{(n-1)}(t_0) = y_{n-1}.$$

Equation (1) is called *homogeneous* if $f(t) \equiv 0$, otherwise it is *non-homogeneous* (or *inhomogeneous*). If all $a_i(t)$, $i = 0, \ldots, n-1$ are constants, then equation (1) is called *linear ODE with constant coefficients*. This is one of the few classes of ODE for which an exhaustive theory exists. Now we switch from the general definitions to the most important particular case.

We study here ODE of the form (and I do not want to write again the full title of this equation)

$$y'' + a_1y' + a_0y = 0,$$

or, choosing other letters for a_1, a_0,

$$y'' + py' + qy = 0, \quad p, q \in \mathbf{R}. \tag{2}$$

Before solving (2), consider the first order homogeneous linear ODE with constant coefficient:

$$y' - \lambda y = 0, \quad \lambda \in \mathbf{R}.$$

(I choose "$-$" for the consistence with the following discussion). We know that its solution is given by

$$y(t) = Ce^{\lambda t}.$$

It turns out that exponents play an extremely important role in solving general linear equations with constant coefficients of an arbitrary order. To see this let me rewrite equation (2) in the form

$$(D^2 + pD + q)y = 0,$$

where I denote by D the derivative with respect to t. The expression $D^2 + pD + q$ is a quadratic polynomial with respect to variable D and hence can be factored as $D^2 + pD + q = (D - \lambda_1)(D - \lambda_2)$, where $\lambda_{1,2}$ are the roots. Each factor corresponds to the first order ODE $y' - \lambda_{1,2}y = 0$, and hence it is reasonable to predict that my equation will have exponential solutions.

For the moment let me assume that the general solution to (2) has the form (this of course must be justified)

$$y(t) = C_1y_1(t) + C_2y_2(t),$$

where $y_1(t)$ and $y_2(t)$ are such that one is not a multiple of another one (the exact mathematical term is that $y_1(t)$ and $y_2(t)$ are *linearly independent*), and look for a solution in the form $y(t) = e^{\lambda t}$, where λ is some constant to be determined. By plugging this expression into our equation:

$$\lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = 0 \implies \lambda^2 + p\lambda + q = 0$$

we find the equation for the unknown λ. This equation is called the *characteristic equation*.

How to solve quadratic equations. The characteristic equation for (2) is a quadratic equation. I remind you that it usually has two roots that can be found as

$$\lambda_{1,2} = \frac{-p \pm \sqrt{\Delta}}{2}, \quad \Delta = p^2 - 4q,$$

where Δ is called the discriminant. In general, if $\Delta > 0$, then we have two real roots $\lambda_1 \in \mathbf{R} \neq \lambda_2 \in \mathbf{R}$. If $\Delta = 0$ then there is one real root $\lambda \in \mathbf{R}$ of multiplicity 2, finally, if $\Delta < 0$ then we obtain two complex solutions $\lambda_1 \in \mathbf{C} \neq \lambda_2 \in \mathbf{C}$ such that $\lambda_1 = \overline{\lambda}_2$, where the bar means complex conjugate (this is only true for the quadratic equations with real coefficients $p \in \mathbf{R}$, $q \in \mathbf{R}$). Very often Vieta's formulas are useful, that say that

$$\lambda_1 + \lambda_2 = -p, \quad \lambda_1\lambda_2 = q.$$

(*Q:* Can you prove these formulas?) And final note here is that if the quadratic equation $\lambda^2+p\lambda+q = 0$ has roots λ_1 and λ_2 (note that I include the case $\lambda_1 = \lambda_2$) then $\lambda^2 + p\lambda + q = (\lambda - \lambda_1)(\lambda - \lambda_2)$.

Now consider some examples.

Example 6. Solve

$$y'' - 4y' - 12y = 0.$$

The characteristic equation is

$$\lambda^2 - 4\lambda - 12 = 0,$$

which has the roots $\lambda_1 = 6$, $\lambda = -2$, hence the general solution is

$$y(t) = C_1 e^{6t} + C_2 e^{-2t}.$$

Note that the function $y(t) = 0$ is a solution to our equation (as well as to any other linear homogeneous equation). This solution is called the *trivial solution* and can be obtained from the general solution for $C_1 = C_2 = 0$. Similar to the equilibria for the autonomous equations we can talk about the stability of the trivial solution. More specifically, we say that this solution is *stable* if any solution that starts close to it stays close to it for all the future times; *asymptotically stable* is any solution that starts close to it tends to it as $t \to \infty$; and finally *unstable* if there is a solution that starts close to the trivial one and moves away from it with time.

Since $\lambda_1 = 6 > 0$ in this example then $e^{6t} \to \infty$ when $t \to \infty$, hence the trivial solution in this particular case is unstable.

Q: Can you think of a sufficient condition that guarantees the asymptotic stability of the trivial solution?

Example 7. Solve
$$y'' + 2y' + 8y = 0.$$
The characteristic equation is
$$\lambda^2 + 2\lambda + 8 = 0,$$
and has the roots
$$\lambda_{1,2} = -1 \pm \mathrm{i}\sqrt{7},$$
where i is the imaginary unit with the characteristic property $\mathrm{i}^2 = -1$.

Hence the general solution is
$$y(t) = C_1 e^{(-1+\mathrm{i}\sqrt{7})t} + C_2 e^{(-1-\mathrm{i}\sqrt{7})t}.$$
This solution is perfectly fine, and actually C_1 and C_2 are arbitrary *complex* constants here. However, since our original equation has the real coefficients then it would be important to try to obtain real valued solutions. For this Euler's formula is very important: For any $x \in \mathbf{R}$
$$e^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x.$$
The second ingredient to figure out the real valued solutions is to note that if $z(t) \in \mathbf{C}$ for a fixed t is a complex valued function of the real argument t that solves our equation, then its real and imaginary parts are also solutions (in this case real valued). Recall that if $z(t) = u(t) + \mathrm{i}v(t)$, then $u(t)$ is called the real part of $z(t)$ and denoted $u(t) = \operatorname{Re} z(t)$, and $v(t)$ is called the imaginary part of $z(t)$, $v(t) = \operatorname{Im} z(t)$. To see that $u(t)$ and $v(t)$ are the solutions, plug $z(t) = u(t) + \mathrm{i}v(t)$ into the equation and remember that differentiating complex valued functions satisfies the same rules as differentiating real valued functions.

So, one of our solutions is $z(t) = e^{(-1+\mathrm{i}\sqrt{7})t} = e^{-t}e^{\mathrm{i}\sqrt{7}t} = e^{-t}(\cos\sqrt{7}t + \mathrm{i}\sin\sqrt{7}t)$. We therefore have
$$\operatorname{Re} z(t) = e^{-t}\cos\sqrt{7}t = y_1(t), \quad \operatorname{Im} z(t) = e^{-t}\sin\sqrt{7}t = y_2(t).$$
Now both $y_1(t)$ and $y_2(t)$ are real valued and not multiple of one another, therefore, we can present the general solution to our equation as
$$y(t) = C_1 e^{-t}\cos\sqrt{7}t + C_2 e^{-t}\sin\sqrt{7}t = e^{-t}(C_1\cos\sqrt{7}t + C_2\sin\sqrt{7}t).$$
I hope it is clear that the trivial solution in this example is asymptotically stable.

In general, if two roots of the characteristic equation are complex conjugate:
$$\lambda_{1,2} = \alpha \pm \mathrm{i}\beta,$$
then the general real-valued solution can be written as
$$y(t) = e^{\alpha t}(C_1\cos\beta t + C_2\sin\beta t).$$
Q: What are the necessary and sufficient conditions for α and β so that the trivial solution would be asymptotically stable?

Example 8. Solve

$$y'' + 4y' + 4y = 0.$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

and has the root $\lambda = -2$ multiplicity two. This means that we have $y_1(t) = e^{-2t}$ but missing the second linearly independent solution. To determine it, let us look for a solution to our equation in the form $y_2(t) = u(t)y_1(t) = u(t)e^{-2t}$ where $u(t)$ is an unknown function. We get

$$y_2'(t) = u'(t)e^{-2t} - 2u(t)e^{-2t}, \quad y_2''(t) = u''(t)e^{-2t} - 4u'(t)e^{-2t} + 4u(t)e^{-2t}.$$

After plugging this into the equation and canceling we obtain

$$u''(t) - 4u'(t) + 4u(t) + 4u'(t) - 8u(t) + 4u(t) = u''(t) = 0 \implies u(t) = At + B,$$

where A, B are arbitrary constants. This means that any function of the form $(At + B)e^{-2t}$ is also a solution to our ODE. To guarantee the linear independence, we can choose $y_2(t) = te^{-2t}$. Therefore, the general solution is

$$y(t) = C_1e^{-2t} + C_2te^{-2t} = e^{-2t}(C_1 + C_2t).$$

Despite that fact that in this example one of the solutions has the form te^{-2t}, it is still asymptotically stable since, using, e.g., l'Hôpital's law (fill in the missed details),

$$\lim_{t\to\infty} te^{-2t} = 0.$$

In general, if the characteristic equation has the root of multiplicity two, $\lambda = \lambda_1$, then the general solution to the ODE is

$$y(t) = e^{\lambda_1 t}(C_1 + C_2t).$$