11 Elements of the general theory of the linear ODE

In the last lecture we looked for a solution to the second order linear homogeneous ODE with constant coefficients in the form 
\[ y(t) = C_1 y_1(t) + C_2 y_2(t), \]
where \( C_1, C_2 \) are arbitrary constants and \( y_1(t) \) and \( y_2(t) \) are some particular solutions to our problem, the only condition is that one of them is not a multiple of another (this prohibits, e.g., using \( y_1(t) = 0 \), which, as we know, is a solution). Using the ansatz \( y(t) = e^{\lambda t} \), we actually were able to produce these two solutions \( y_1(t) \) and \( y_2(t) \) (ansatz is an educated guess). However, the questions remain: Did our assumption actually lead to a legitimate solution? And how do we know that all possible solutions to our problem can be represented by our formula? In this lecture I will try to answer these questions. Since no advantage is gained by restricting the attention to the case of constant coefficients, I will be talking here about arbitrary second order linear ODE.

11.1 General theory. Linear operators

**Definition 1.** The equation of the form
\[ y'' + p(t)y' + q(t)y = f(t), \]
where \( p(t), q(t), f(t) \) are given functions, is called second order linear ODE. This ODE is homogeneous if \( f(t) \equiv 0 \) and nonhomogeneous in the opposite case. If \( p(t), q(t) \) are constant, then this equation is called linear ODE with constant coefficients.

For equation (1) the following theorem of uniqueness and existence of the solution to the IVP holds. Here is the statement without proof.

**Theorem 2.** Consider the initial values problem
\[ y'' + p(t)y' + q(t)y = f(t) \]
and
\[ y(t_0) = y_0, \quad y'(t_0) = y_1. \]
Assume that \( t_0 \in I = [a, b] \) and \( p(t), q(t), f(t) \in C(I; \mathbb{R}) \), i.e., these are continuous functions on the interval \( I = [a, b] \). Then the solution to (1)–(2) exists and unique on the interval \( I \).

Note that for the linear equation the major difference of this statement from the statement of the uniqueness and existence theorem in Lecture 3 is its global character: The theorem guarantees that the unique solution exists on the whole interval \( I = [a, b] \) where our functions \( p(t), q(t), f(t) \) behave nicely. This implies in particular that for the second order linear homogeneous ODE with constant coefficients the unique solution exists for the whole \( \mathbb{R} \). For nonlinear equations the only thing we can expect is a local uniqueness and existence.

The word “linear” appears so frequently in different contexts in mathematics, that it is worth pausing for a second and discuss what linear means in general. For this let me rewrite (1) in the following form:
\[ \frac{d^2 y}{dt^2} + p(t) \frac{dy}{dt} + q(t)y = f(t). \]
Now I denote
\[ L = \frac{d^2}{dt^2} + p(t) \frac{d}{dt} + q(t), \]
hence my equation can be rewritten in the form (I suppress the dependence on \( t \))
\[ Ly = f. \]

In the last expression \( L \) is a \textit{differential operator}. Think about generalizing the notion of a function: Functions take some numerical values as input and produce other numerical values as output. Now consider a function that takes other functions as input and produces functions as output. Such functions are commonly called \textit{operators} (to distinguish from the usual functions). Using the notion of the operator, we can give

**Definition 3.** Operator \( L \) is \textit{linear} if
\[ L(\alpha y_1 + \beta y_2) = \alpha Ly_1 + \beta Ly_2, \]
where \( \alpha \) and \( \beta \) are constants.

\[ Q: \] Can you show that operator \( \frac{d^2}{dt^2} \) is linear?

Now we can give a general definition of a \textit{linear equation}, which works equally well for algebraic equations, matrix equations, differential equations, integral equations (can you think of an example?), etc.

**Definition 4.** The equation
\[ Ly = f \]
for the unknown quantity \( y \) is \textit{linear} if the operator \( L \) is linear. This equation is \textit{homogeneous} if \( f \equiv 0 \) and nonhomogeneous otherwise.

Let us check that our equation (1) is linear using this new definition of the linear operator. We need to show that for any two functions \( y_1 \) and \( y_2 \) and constants \( \alpha \) and \( \beta \), our differential operator
\[ L = \frac{d^2}{dt^2} + p(t) \frac{d}{dt} + q(t) \]
is linear. This is indeed the case since the differentiation is a linear operation (the derivative of the sum equals to the sum of derivatives and constant can be factored our of the derivative):
\[
L(\alpha y_1 + \beta y_2) = \left( \frac{d^2}{dt^2} + p(t) \frac{d}{dt} + q(t) \right) (\alpha y_1 + \beta y_2) = \\
= \frac{d^2}{dt^2} (\alpha y_1 + \beta y_2) + p(t) \frac{d}{dt} (\alpha y_1 + \beta y_2) + q(t)(\alpha y_1 + \beta y_2) = \\
= \alpha \frac{d^2}{dt^2} y_1 + \beta \frac{d^2}{dt^2} y_2 + \alpha p(t) \frac{d}{dt} y_1 + \beta p(t) \frac{d}{dt} y_2 + \alpha q(t) y_1 + \beta q(t) y_2 = \\
= \alpha \frac{d^2}{dt^2} y_1 + \alpha p(t) \frac{d}{dt} y_1 + \alpha q(t) y_1 + \beta \frac{d^2}{dt^2} y_2 + p(t) \beta \frac{d}{dt} y_2 + \beta q(t) y_2 = \\
= \alpha \left( \frac{d^2}{dt^2} + p(t) \frac{d}{dt} + q(t) \right) y_1 + \beta \left( \frac{d^2}{dt^2} + p(t) \frac{d}{dt} + q(t) \right) y_2 = \\
= \alpha Ly_1 + \beta Ly_2
\]
Therefore, our two definitions do not contradict each other. Moreover, using the notion of the operator, the general definition of the linear ODE follows:

**Definition 5. The n-th order ODE**

\[ Ly = f \] \hspace{1cm} (3)

is linear if \( L \) is a linear differential operator of the n-th order.

Now, equipped with the powerful notations of the linear operator, we can list important properties of the n-th order linear ODE (3) and its homogeneous counterpart

\[ Ly = 0. \] \hspace{1cm} (4)

Properties:

- If \( y_1, y_2 \) are solutions to the homogeneous linear equation \( Ly = 0 \) then their linear combination \( \alpha y_1 + \beta y_2 \) is also a solution to (4). This follows right from the fact that \( L \) is a linear differential operator. This property sometimes called the superposition principle.

- If \( y_1, y_2 \) are solutions to the nonhomogeneous equation (3) then their difference \( y_1 - y_2 \) solves homogeneous equation (4). Indeed, from the fact that \( Ly_1 = f, Ly_2 = f \), and linearity of \( L \) it follows
  \[ Ly_1 - Ly_2 = f - f \implies L(y_1 - y_2) = 0. \]

- Any solution to (1) can be represented as \( y = y_h + y_p \), where \( y_h \) is the general solution to the homogeneous equation \( Ly = 0 \) and \( y_p \) is a particular solution to the nonhomogeneous equation \( Ly = f \). To prove this fact note that if \( y(t) \) is any solution to (1) and \( y_p(t) \) is some fixed solution to (1), then their difference, due to the previous, has to solve the homogeneous equation, i.e., \( y - y_p = y_h \), from which the conclusion follows.

- Assume that we need to solve
  \[ Ly = f_1 + f_2. \]
  Then the general solution to this equation can be represented as
  \[ y = y_h + y_p^1 + y_p^2, \]
  where \( y_h \) is the general solution to the homogeneous equation, \( y_p^1 \) solves \( Ly = f_1 \), and \( y_p^2 \) solves \( Ly = f_2 \). A proof is left as an exercise. I hope at this point it is clear how we can generalize this property.

### 11.2 The structure of the general solution to the homogeneous second order linear ODE

Now we are ready to prove that the general solution to the homogeneous equation

\[ Ly = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0, \] \hspace{1cm} (5)

where \( L \) is the second order differential operator, can be represented as \( y_h = C_1y_1 + C_2y_2 \), where \( y_1 \) and \( y_2 \) are not multiple of each other. We used this fact without proof in the previous lecture, and here is a justification. I start with some definitions.
Definition 6. Two functions \( y_1 \) and \( y_2 \) defined on the same interval \( I = [a, b] \) are called linearly independent if none of them is multiple of another. In the opposite case they are called linearly dependent.

You probably saw the definition of linear independence in your Linear Algebra class. Formally, two vectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are linearly independent if there are no constants \( \alpha_1 \) and \( \alpha_2 \), which do not equal to zero simultaneously, that the linear combination

\[
\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2
\]

is equal to zero. Spend a few second to realize that this exactly means that none of the vectors is a multiple of another. First I provide a criterion to check whether two functions are linearly dependent.

Proposition 7. If the expression

\[
W(t) := y_1(t)y_2'(t) - y_2(t)y_1'(t)
\]

is not zero at some point \( t_0 \in I \), then \( y_1, y_2 \) are linearly independent.

Proof. To prove I show that if \( y_1, y_2 \) are linearly dependent then there is \( t_0 \) such that \( W(t_0) = 0 \). If \( y_1, y_2 \) are linearly dependent then there are constants \( \alpha_1, \alpha_2 \) such that

\[
\begin{align*}
\alpha_1 y_1 + \alpha_2 y_2 &= 0, \\
\alpha_1 y_1' + \alpha_2 y_2' &= 0,
\end{align*}
\]

for any \( t \in I \), and hence for some \( t_0 \in I \). This means that the system above with respect to constants \( \alpha_1, \alpha_2 \) has nontrivial solution, hence its determinant, which is exactly \( W(t_0) \), has to be equal to zero for any \( t \) and hence for \( t_0 \).

By the way, we also proved that

Proposition 8. If \( y_1, y_2 \) are linearly dependent then

\[
W(t) \equiv 0, \quad t \in I.
\]

Finally,

Proposition 9. If \( y_1, y_2 \) solve (5) on \( I = [a, b] \) then \( W(t) \) is either identically zero or not equal to zero at any point of \( I \).

Proof. First,

\[
W'(t) = y_1 y_2'' - y_1'' y_2,
\]

and since \( y_1, y_2 \) solve (5), then

\[
y_i'' = -p(t)y_i' - q(t)y_i, \quad i = 1, 2.
\]

Therefore,

\[
W'(t) = -p(t)(y_1 y_2' - y_1' y_2) = -p(t)W(t).
\]

The last ODE has the solution

\[
W(t) = W(t_0)e^{-\int_{t_0}^{t} p(\xi)d\xi},
\]

which is either identically zero, or never zero, depending on \( W(t_0) \).
Putting everything together, we showed that if \( y_1, y_2 \) solve (5) and linearly independent then \( W(t) \neq 0 \) for any \( t \). Finally, the big theorem

**Theorem 10.** Consider the homogeneous equation (5)

\[
Ly = y'' + p(t)y' + q(t)y = 0.
\]

Assume that \( y_1 \) and \( y_2 \) solve this equation and linearly independent. Then

\[
y_h = C_1y_1 + C_2y_2
\]

is the general solution to our equation.

**Proof.** We already know that \( y_h = C_1y_1 + C_2y_2 \), as a linear combination of solutions, is a solution. To prove that it is the general solution, we must, for any existing solution \( \tilde{y} \), guarantee that it is possible to pick \( C_1 \) and \( C_2 \) such that

\[
\tilde{y} = C_1y_1 + C_2y_2.
\]

Let \( \tilde{y} \) be an arbitrary solution to our equation that satisfies the initial conditions \( \tilde{y}(t_0) = \tilde{y}_0, \tilde{y}'(t_0) = \tilde{y}_1 \).

Consider the system

\[
\begin{align*}
C_1y_1(t_0) + C_2y_2(t_0) &= \tilde{y}_0, \\
C_1y'_1(t_0) + C_2y'_2(t_0) &= \tilde{y}_1.
\end{align*}
\]

Here \( C_1 \) and \( C_2 \) are unknowns. Due to our assumption \( y_1'y_2' - y_2'y_1' \neq 0 \) at \( t_0 \), thence this system of two linear algebraic equations has the unique solution, which I denote \( \hat{C}_1, \hat{C}_2 \). Consider now the solution

\[
y_h = \hat{C}_1y_1 + \hat{C}_2y_2.
\]

It has the required form, i.e., it is a linear combination of \( y_1 \) and \( y_2 \). And it coincides with \( \tilde{y} \) due to the existence and uniqueness theorem, because \( \hat{C}_1, \hat{C}_2 \) were chosen such that \( y_h(t_0) = \tilde{y}(t_0) \) and \( y'_h(t_0) = \tilde{y}'(t_0) \).

It is convenient to introduce more terminology.

**Definition 11.** For any two functions \( y_1 \) and \( y_2 \) the expression

\[
W(t) = y_1y'_2 - y'_1y_2 = \det \begin{bmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{bmatrix}
\]

is called the Wronskian.

Finally,

**Definition 12.** Two linearly independent functions \( \{y_1, y_2\} \) that solve the homogeneous second order ODE (5) are called a fundamental set of solutions.
11.3 Summary

That was a lot of information in this lecture. Let me go quickly through the main points.

We consider the equation

$$y'' + p(t)y' + q(t)y = f(t).$$

(1)

We proved that the general solution to this equation has the form

$$y = y_h + y_p.$$

Here $y_h$ is the general solution to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

and $y_p$ is any particular solution to (1).

To find the general solution to the homogeneous equation, we will need

$$y_h = C_1 y_1 + C_2 y_2,$$

where $C_1$ and $C_2$ are arbitrary constants, and $y_1$ and $y_2$ are a fundamental set of solutions, i.e., two linearly independent solutions to the homogeneous equation. To check that two solutions are linearly independent we need to evaluate the Wronskian

$$W(t) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

at any point $t_0$ in the interval of existence of solutions (it is advisable to pick such a point $t_0$ so that to simplify the subsequent calculations). If $W(t_0) \neq 0$ then $y_1$ and $y_2$ are linearly independent. It is a good exercise to go back to the previous lecture and check that the solutions we found for the equation with constant coefficients are linearly independent, now you have a tool for this. In particular, our fundamental solution sets were

- $\{e^{\lambda_1 t}, e^{\lambda_2 t}\}$ in case when the characteristic equation has two real distinct roots;
- $\{e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t\}$ in case when the characteristic equation has complex conjugate roots;
- $\{e^{\lambda t}, te^{\lambda t}\}$ in case when the characteristic equation has a real root multiplicity two.

And all these wonderful facts were the direct consequences of the linearity of the equation (1), i.e., of the fact that it can be written as

$$Ly = f,$$

where $L$ is a linear differential operator.

11.4 Generalization. Solving linear $n$-th order ODE with constant coefficients

Using the theory outlined above, we can generalize the procedure from the previous lecture to the case when our ODE has an arbitrary order.

Let us consider the $n$-th order homogeneous linear ODE with constant coefficients

$$y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0, \quad a_i \in \mathbb{R}, \ i = 0, \ldots, n - 1.$$

(6)
The general solution to the equation (6) has the form

\[ y(t) = C_1 y_1 + \ldots + C_n y_n, \]

where \( \{ y_i \}_{i=1}^n \) is a linearly independent set of solutions (which is also called a fundamental set of solutions), and \( C_i \) are arbitrary constants.

**Definition 13.** A set of functions \( \{ y_i \}_{i=1}^n \) is called linearly dependent on the interval \( I = [a, b] \) if there exist constants \( \alpha_1, \ldots, \alpha_n \) not equal zero simultaneously such that

\[ \alpha_1 y_1 + \ldots + \alpha_n y_n \equiv 0 \]

for any \( t \in I \). Otherwise the set of functions is called linearly independent.

With a slight abuse of language, I also call the functions \( y_i \) linearly independent if they belong to a linearly independent set.

A criterion when \( n \) solutions to (6) are linearly independent is given in terms of the Wronskian of the set of functions. Consider the Wronskian of the set \( \{ y_1, \ldots, y_n \} \):

\[
W(t) = \begin{vmatrix}
y_1 & y_2 & y_3 & \cdots & y_n \\
y_1' & y_2' & y_3' & \cdots & y_n' \\
y_1'' & y_2'' & y_3'' & \cdots & y_n'' \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}
\]

**Criterion:** If \( y_1, \ldots, y_n \) are solutions to (6) and \( W(t_0) \neq 0 \) for any \( t_0 \) (pick any!) then the set \( \{ y_1, \ldots, y_n \} \) is linearly independent and forms a fundamental set of solutions.

So the question boils down to the problem to find \( y_1, \ldots, y_n \). For this we write down the characteristic equation

\[
\lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0 \quad (7)
\]

and then solve it. This is a polynomial equation, and therefore, counting multiplicities, any such equation has exactly \( n \) roots (see below). The following cases are possible:

- \( \lambda \) is a real root of (7) multiplicity 1, then the solution to (6) corresponding to this root is \( e^{\lambda t} \).
- Two complex conjugate roots \( \lambda = \alpha + i\beta \) and \( \bar{\lambda} = \alpha - i\beta \) correspond to the two solutions \( e^{\alpha t} \cos \beta t, \ e^{\alpha t} \sin \beta t \).
- Real root \( \lambda \in \mathbb{R} \) multiplicity \( k \) produces \( k \) solutions to (6):
  \[ e^{\lambda t}, te^{\lambda t}, \ldots, t^{k-1}e^{\lambda t} \].
• Two complex conjugate roots $\lambda = \alpha + i\beta$ and $\overline{\lambda} = \alpha - i\beta$ of multiplicity $k$ correspond to $2k$ solutions

$$e^{\alpha t} \cos \beta t, e^{\alpha t} \sin \beta t, te^{\alpha t} \cos \beta t, te^{\alpha t} \sin \beta t, \ldots, k^{k-1} e^{\alpha t} \cos \beta t, k^{k-1} e^{\alpha t} \sin \beta t.$$  

It can be proved that obtained in this manner $n$ solutions are linearly independent. This means that forming a fundamental set of solutions actually amounts to solving the algebraic equation (7). Here is some useful information in this respect.

**Solving polynomial equations.** Consider complex polynomial of degree $n$

$$P_n(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0, \quad a_i \in \mathbb{C}.$$  

A complex number $\hat{z} \in \mathbb{C}$ is called a root of $P_n(z)$ if $P_n(\hat{z}) = 0$.

The fundamental theorem of algebra says that any non-constant complex polynomial has at least one complex root. This means that if $\hat{z}_1$ is this root, our polynomial can be represented as

$$P_n(z) = (z - \hat{z}_1)Q_{n-1}(z),$$  

where $Q_{n-1}(z)$ is a complex polynomial of degree $n-1$, for which, if not constant, the same fundamental theorem of algebra holds. $Q_{n-1}(z)$ can be found using, e.g., the long division procedure. If we continue to apply the fundamental theorem of algebra, we will eventually arrive at

$$P_n(z) = (z - \hat{z}_1)^{l_1} \ldots (z - \hat{z}_k)^{l_k},$$  

where $\hat{z}_1, \ldots, \hat{z}_k$ are the roots of $P_n(z)$ and $l_1, \ldots, l_k$ are, by definition, the corresponding multiplicities. Thence, finding the roots of a polynomial is equivalent to factoring it.

We usually deal with polynomials with real coefficients $a_i \in \mathbb{R}$. In this case it is true that if a complex number $\hat{z}$ is a root of $P_n(z)$ then $\overline{\hat{z}}$ is also a root (can you prove this?).

How to actually find the roots of a polynomial? If it is quadratic $P_2(z)$ then it is easy: We know the formula. What about polynomials degree $n \geq 3$? Actually, for $n = 3$ and $n = 4$ there are also formulas. It is a remarkable results that for $n = 5$ there is no such formula.

It is useful to remember the following fact: Consider the polynomial with integer coefficients:

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$  

and assume that $a_0 \neq 0$ and $a_n \neq 0$. It is possible to prove that all the rational roots of this polynomial have the form $p/q$, where $p$ divides $a_0$ and $q$ divides $a_n$. Hence the usual first step is to write down all potential rational solutions and check them one by one by plugging them into $P_n(x)$.

To conclude this short review, note that if a polynomial has real coefficients then it is always possible to factor it in the form

$$P_n(x) = (x - \hat{x}_1)^{l_1} \ldots (x - \hat{x}_k)^{l_k}(x^2 + p_1x + q_1)^{n_1} \ldots (x^2 + p_nx + q_m)^{n_m},$$  

where $\hat{x}_1, \ldots, \hat{x}_k$ are real roots with multiplicities $l_1, \ldots, l_k$, $p_i, q_i \in \mathbb{R}$ and such that $p_i^2 - 4q_i < 0$ for any $i = 1, \ldots, m$, and $l_1 + \ldots + l_k + n_1 + \ldots + n_m = n.$
Example 14. Find the general solution to
\[ y^{(5)} - 2y^{(4)} + 2y''' - 4y'' + y' - 2y = 0. \]

The characteristic polynomial is
\[ P_5(\lambda) = \lambda^5 - 2\lambda^4 + 2\lambda^3 - 4\lambda^2 + \lambda - 2. \]

Note that we have \( a_5 = 1 \) and \( a_0 = -2 \), hence \( p = \pm 1, \pm 2 \), \( q = \pm 1 \). Therefore the potential rational (integer in our case) roots are
\[ -1, +1, -2, +2. \]

By plugging them into \( P_5(\lambda) \) we will find that \( \hat{\lambda} = 2 \) is actually a root. Therefore,
\[ P_5(\lambda) = (\lambda - 2)Q_4(\lambda), \]

where \( Q_4(\lambda) \) can be found using the long division:
\[
\begin{array}{c|ccccc}
\lambda^5 & -2\lambda^4 & +2\lambda^3 & -4\lambda^2 & +\lambda & -2 \\
\hline
\lambda^5 & -2\lambda^4 & & & & \\
\hline
& & +2\lambda^3 & -4\lambda^2 & +\lambda & -2 \\
& & +2\lambda^3 & -4\lambda^2 & & \\
& & & +\lambda & -2 & \\
& & & +\lambda & -2 & \\
& & & & 0 & \\
\end{array}
\]

Hence,
\[ P_5(\lambda) = \lambda^5 - 2\lambda^4 + 2\lambda^3 - 4\lambda^2 + \lambda - 2 = (\lambda - 2)(\lambda^4 + 2\lambda^2 + 1) = (\lambda - 2)(\lambda^2 + 1)^2, \]

where in the last equality I used the fact that
\[ a^2 + 2ab + b^2 = (a + b)^2. \]

This is the factoring of our polynomial if we are allowed only to use real numbers. Using complex numbers, we finally get
\[ P_5(\lambda) = (\lambda - 2)(\lambda + i)^2(\lambda - i)^2, \]

hence our polynomial has a real root 2 multiplicity 1, and a pair of complex conjugate roots \( \pm i \) multiplicity 2. Using the general strategy, we have that a fundamental set of solutions is
\[ \{e^{2t}, \cos t, \sin t, t \cos t, t \sin t\}. \]

The fact that these functions are linearly independent can be proved using the Wronskian. Finally, the general solution is
\[ y(t) = C_1e^{2t} + C_2 \cos t + C_3 \sin t + t(C_4 \cos t + C_5 \sin t). \]