12 Solving nonhomogeneous equations: Method of an educated guess

Consider n-th order linear ODE with constant coefficients

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(t). \tag{1}$$

We know by now that the general solution to this equation can be represented in the following form:

$$y = y_h + y_p,$$

where $y_h = C_1 y_1 + \ldots + C_n y_n$ is the general solution to the homogeneous equation (i.e., (1) with $f(t) = 0$), $\{y_1, \ldots, y_n\}$ is the fundamental set of solutions, and y_p is a particular solution to the nonhomogeneous equation. "Particular solution" in this context means *any* solution, the only requirement is that it satisfies the equation. The last two lectures were devoted to the procedure of finding $y_h(t)$ (we need to write down the characteristic polynomial, find its roots together with multiplicities, and write down the linearly independent solutions). In this lecture we'll learn how to find y_p in the cases when $f(t)$ has a special form. In particular, we'll discuss the cases when

$$f(t) = P_m(t)e^{\alpha t}, \quad \text{or } f(t) = P_m(t)e^{\alpha t}\cos\beta t, \quad \text{or } f(t) = P_m(t)e^{\alpha t}\sin\beta t,$$

where $P_m(t)$ is a polynomial of degree m. The expressions of this form are sometimes called *quasipolynomials*. I start with some examples, and after that I will present a general scheme.

12.1 An example and the general rule

Example 1. Solve

$$y'' + y = 4e^t.$$

We start with solving the homogeneous equation

$$y'' + y = 0.$$

The characteristic polynomial is

$$\lambda^2 + 1 = 0 \implies \lambda_{1,2} = \pm \mathrm{i}.$$

Therefore, the real valued solutions are $y_1 = \cos t$ and $y_2 = \sin t$. Hence, the general solution to the homogenous equation is

$$y_h(t) = C_1 \cos t + C_2 \sin t.$$

To find a particular solution to the nonhomogeneous equation we *guess* that it has to have the form

$$y_p(t) = Ae^t,$$

where A is a constant to be determined. That is why sometimes this method is called the *method of undetermined coefficients*. We plug our guess into the equation, cancel exponents, and get

$$A + A = 4 \implies A = 2.$$

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Therefore, the solution

$$y_p(t) = 2e^t$$

solves nonhomogeneous equation, and hence final general solution to the nonhomogeneous equation is given by

$$y(t) = C_1 \cos t + C_2 \sin t + 2e^t.$$

Too easy? Let's proceed to another example.

Example 2. Solve

$$y'' - y = 4e^t.$$

As before, we find the characteristic polynomial

$$\lambda^2 - 1 = 0 \implies \lambda_1 = 1,\, \lambda_2 = -1,$$

which means that

$$y_h(t) = C_1 e^t + C_2 e^{-t}.$$

Now we can try to look for a particular solution to the nonhomogeneous equation as before: $y_p = Ae^t$. And it will not work. Actually, try it! What is the problem then? The reason this form for a particular solution does not work is that the coefficient at t in the exponent in $f(t)$ coincides with one of the roots (namely, λ_1). To bypass this obstacle we can try

$$y_p(t) = Ate^t.$$

We'll find

$$y_p'(t) = Ae^t + Ate^t$$
$$y_p''(t) = 2Ae^t + Ate^t,$$

and after plugging into our equation

$$2Ae^t + Ate^t - Ate^t = 4e^t \implies A = 2.$$

Hence the general solution is

$$y(t) = C_1 e^t + C_2 e^{-t} + 2te^t.$$

Example 3. Solve

$$y'' - 2y' + y = 4e^t.$$

The roots of the characteristic polynomial are $\lambda = 1$ of multiplicity 2. The particular solution $y_p(t) = Ate^t$ will not work here (try it!). Instead, one needs to try

$$y_p = At^2e^t.$$

We have

$$y_p'(t) = At^2e^t + 2Ate^t$$
$$y_p''(t) = At^2e^t + 4Ate^t + 2Ae^t.$$

Plugging these into the equation we obtain

$$2A = 4 \implies A = 2.$$

(And it is a mere coincidence that in all these examples I got $A = 2$.) Therefore, the general solution is

$$y(t) = C_1 e^t + C_2 t e^t + 2t^2 e^t.$$

Now we are ready to make a first generalization. It goes like this: If linear ODE with constant coefficients has the right hand side of the form $Ce^{\alpha t}$, where C is some constant, then a particular solution to the nonhomogeneous equations has to be sought in the form

$$y_p(t) = At^r e^{\alpha t},$$

where A is a constant to be determined, r is the multiplicity of α as a root of a characteristic polynomial ($r = 0$ is α is not a root, $r = 1$ if α is a simple root, $r = 2$ if α is a root multiplicity two and so on).

Example 4. Solve

$$y'' - 5y' + 4y = 4t^2 e^{2t}.$$

Now our $f(t)$ can be represented as $f(t) = P_2(t)e^{2t}$, where $P_2(t) = 4t^2$ is a polynomial of degree 2.

We start again with the characteristic equation

$$\lambda^2 - 5\lambda + 4 = (\lambda - 4)(\lambda - 1),$$

therefore

$$y_h(t) = C_1 e^{4t} + C_2 e^t.$$

Note that $4 \neq 2$ and $1 \neq 2$, where 2 is the coefficient of our exponent e^{2t}. Hence, it would be a good idea to look for a solution in the form

$$y_p(t) = Q_2(t)e^{2t} = (At^2 + Bt + C)e^{2t},$$

where $Q_2(t)$ is a polynomial of degree 2 with undetermined coefficients. Our task is to find A, B, C. For this we need to plug our $y_p(t)$ into the equation:

$$\begin{aligned} y_p'(t) &= (2At + B)e^{2t} + 2(At^2 + Bt + C)e^{2t} \\ y_p''(t) &= 2Ae^{2t} + 4(2At + B)e^{2t} + 4(At^2 + Bt + C)e^{2t}. \end{aligned}$$

Plugging this into the equation and canceling all the exponents implies

$$2A - 2At - B - 2At^2 - 2Bt - 2C = 4t^2,$$

or, after some rearrangement:

$$(-2A)t^2 + (-2A - 2B)t + (2A - B - 2C) = (4)t^2 + (0)t + (0).$$

Now we need to recall that two polynomials equal if and only if they have equal coefficients at t in the same power. Hence, we get

$$\begin{cases} -2A = 4, \\ -2A - 2B = 0, \\ 2A - B - 2C = 0, \end{cases}$$

which is a system of three linear algebraic equations with three unknowns. The solution is

$$A = -2,\ B = 2,\ C = -3.$$

Therefore,

$$y(t) = C_1 e^{4t} + C_2 e^{t} + (-2t^2 + 2t - 3)e^{2t}$$

is our general solution.

Finally, we state the general rule:

Rule: If $f(t) = P_m(t)e^{\alpha t}$ is the right-hand side of (1), where $P_m(t)$ is a polynomial of degree m, then a particular solution to the nonhomogeneous equation can be chosen in the form

$$y_p(t) = Q_m(t)t^r e^{\alpha t},$$

where $Q_m(t)$ is a polynomial of degree m with undetermined coefficients, and r is the multiplicity of α as a root of the characteristic equation of the homogeneous ODE.

Example 5. Consider

$$y'' - 5y' + 4y = f(t).$$

The following table lists, according to the general rule, the form of the particular solutions depending on $f(t)$. Make sure that you can fill the right column on your own:

$f(t)$	$y_p(t)$
$5e^{2t}$	Ae^{2t}
te^{2t}	$(At+B)e^{2t}$
te^{t}	$(At+B)te^{t}$
t^2	At^2+Bt+C

Consider one more example.

Example 6. Solve

$$y'' + y = 4\sin t.$$

The roots of the characteristic polynomial are

$$\lambda_{1,2} = \pm \mathrm{i},$$

and

$$y_h(t) = C_1\cos t + C_2 \sin t.$$

What about $y_p(t)$? It seems that we did not discuss this case yet. However, this case also can be covered by the general rule above! For this we need to note that if a complex valued function of real argument, let us call it $z(t) = x(t) + \mathrm{i}y(t)$, solves a linear ODE

$$Lz = f,$$

then its real and imaginary parts solve the equations

$$Lx = \operatorname{Re} f$$

and

$$Ly = \operatorname{Im} f$$

respectively (here we assume that L has only real valued coefficients). To prove this note that

$$L(x + \mathrm{i}y) = \operatorname{Re} f + \mathrm{i} \operatorname{Im} f \implies Lx + \mathrm{i}Ly = \operatorname{Re} f + \mathrm{i} \operatorname{Im} f,$$

which completes the proof. Therefore, here is a strategy: Instead of solving, e.g., $Ly = \operatorname{Im} f$, which is a real ODE, we can consider $Lz = f$. Solve this equation, and find $y = \operatorname{Im} z$. Similarly, we can work with x.

Coming back to our example, we can note that, by Euler's formula,

$$\sin t = \operatorname{Im} e^{\mathrm{i}t}.$$

Therefore, instead of solving the original equation, let us solve

$$z'' + z = 4e^{\mathrm{i}t},$$

and after that we can find $y = \operatorname{Im} z$. We are only interested in a particular solution, which, according to the general rule above, can be looked in the form

$$z_p(t) = Ate^{\mathrm{i}t},$$

since i is a root of the characteristic polynomial. We have (the derivatives of the exponent with imaginary unit are taken exactly in the same fashion as it was taught in Calculus, e.g., $(e^{\mathrm{i}t})' = \mathrm{i}e^{\mathrm{i}t}$)

$$z_p'(t) = Ae^{\mathrm{i}t} + \mathrm{i}Ate^{\mathrm{i}t}, \quad z_p''(t) = 2\mathrm{i}Ae^{\mathrm{i}t} - Ate^{\mathrm{i}t},$$

hence

$$2\mathrm{i}A = 4 \implies A = -2\mathrm{i},$$

and

$$z_p = -2\mathrm{i}te^{\mathrm{i}t}.$$

To get y_p, we have

$$y_p(t) = \operatorname{Im} z_p(t) = \operatorname{Im}(-2\mathrm{i}te^{\mathrm{i}t}) = \operatorname{Im}(-2\mathrm{i}t \cos t + 2t \sin t) = -2t \cos t.$$

Finally, the general solution to the original problem is

$$y(t) = C_1 \cos t + C_2 \sin t - 2t \cos t.$$

It usually takes some time for a student to feel comfortable with this approach, so here are some other examples, with some details left out.

Example 7. Solve

$$y'' + y = t \cos 2t.$$

The general solution to the homogeneous equation is

$$y_h(t) = C_1 \cos t + C_2 \sin t.$$

Note that $t\cos 2t = \operatorname{Re}(te^{2\mathrm{i}t})$. Therefore, instead of the original equation we consider

$$z'' + z = te^{2\mathrm{i}t}.$$

Since $2\mathrm{i}$ is not a root of the characteristic equation, then

$$z_p(t) = (At + B)e^{2\mathrm{i}t}.$$

Plugging this into the equation, we find

$$4\mathrm{i}A - 3At - 3B = t,$$

which implies

$$-3A = 1,\ 4\mathrm{i}A - 3B = 0 \implies A = -\frac{1}{3},\ B = -\frac{4\mathrm{i}}{9}.$$

Finally, to find y_p:

$$\begin{aligned} y_p(t) = \operatorname{Re} z_p(t) &= \operatorname{Re}\left(\left(-\frac{t}{3} - \frac{4\mathrm{i}}{9}\right)e^{2\mathrm{i}t}\right) = \\ &= \operatorname{Re}\left(\left(-\frac{t}{3} - \frac{4\mathrm{i}}{9}\right)(\cos 2t + \mathrm{i}\sin 2t)\right) = \\ &= \operatorname{Re}\left(-\frac{t}{3}\cos 2t - \frac{\mathrm{i}t}{3}\sin 2t - \frac{4\mathrm{i}}{9}\cos 2t + \frac{4}{9}\sin 2t\right) = \\ &= -\frac{t}{3}\cos 2t + \frac{4}{9}\sin 2t. \end{aligned}$$

Therefore, the general solution to the original equation is

$$y(t) = C_1\cos t + C_2\sin t - \frac{t}{3}\cos 2t + \frac{4}{9}\sin 2t.$$

Example 8. Find the general solution to

$$y'' - 9y = e^{3t}\cos t.$$

The general solution to the homogeneous equation is

$$y_h(t) = C_1e^{3t} + C_2e^{-3t}.$$

Consider instead the equation

$$z'' - 9z = e^{(3+\mathrm{i})t},$$

since

$$e^{3t}\cos t = \operatorname{Re} e^{(3+\mathrm{i})t}.$$

Look for a solution in the form

$$z_p(t) = Ae^{(3+\mathrm{i})t} \implies A = \frac{1}{6\mathrm{i} - 1}.$$

Hence

$$y_p = \operatorname{Re}\left(\frac{1}{-1+6\mathrm{i}}e^{(3+\mathrm{i})t}\right) = \operatorname{Re}\left(\frac{-1-6\mathrm{i}}{37}(e^{3t}\cos t + \mathrm{i}e^{3t}\sin t)\right) = -\frac{e^{3t}}{37}\cos t + \frac{6e^{3t}}{37}\sin t,$$

and the final answer is

$$y(t) = C_1e^{3t} + C_2e^{-3t} - \frac{e^{3t}}{37}\cos t + \frac{6e^{3t}}{37}\sin t.$$

12.2 *For a mathematically inclined student

The whole idea to guess a solution in mathematics should not sound very satisfactory. Here I give a justification of the method employed above to solve nonhomogeneous ODE.

12.2.1 Operator rules for differential operators with constant coefficients

I wrote

$$L = \frac{\mathrm{d}^n}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}}{\mathrm{d}t} + a_0$$

for a linear n-th order differential operator. I will use here the notation

$$D^n = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\,,$$

and therefore

$$L = P_n(D) = D^n + a_{n-1}D^{n-1} + \ldots + a_1 D + a_0$$

is a polynomial of the symbol D. $P_n(D)$ or simply $P(D)$ is a polynomial differential operator. Here are some rules for dealing with such operators:

- *Sum rule.* For any sufficiently differentiable function u and polynomial operators $P(D)$ and $Q(D)$

$$\big(P(D) + Q(D)\big)u = P(D)u + Q(D)u.$$

- *Linearity rule.* For any sufficiently differentiable functions u_1, u_2 and polynomial operator $P(D)$

$$P(D)(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 P(D)u_1 + \alpha_2 P(D)u_2.$$

- *Multiplication rule.* If $P(D) = Q(D)H(D)$ as polynomials in D then

$$P(D)u = Q(D)\big(H(D)u\big) = H(D)\big(Q(D)u\big).$$

  Note that this only true for polynomial differential operators (with constant coefficients).

  For example,

$$D^m(\alpha D^k u) = \alpha D^{m+k} u.$$

- *Substitution rule.*

$$P(D)e^{at} = P(a)e^{at},$$

  for any $a \in \mathbf{C}$. The proof follows from the fact that $D^k e^{at} = a^k e^{at}$.

  This rule allows to prove a very useful theorem.

  Theorem 9. *The ODE*

$$P(D)y = Ae^{at}, \quad A, a \in \mathbf{C}$$

  has a particular solution

$$y_p(t) = \frac{Ae^{at}}{P(a)}\,,$$

  if $P(a) \neq 0$.

Proof. For the proof note that

$$P(D)y_p = P(D)\frac{Ae^{at}}{P(a)} = Ae^{at}.$$

■

What if $P(a) = 0$? Then

$$y_p(t) = \frac{At^r e^{at}}{P^{(r)}(a)}\,,$$

if a is a root of $P(D)$ multiplicity r. The proof of this last formula is left as an exercise (*Hint:* First show that if a is a root of multiplicity r, then $P(D) = Q(D)(D-a)^r$, where $Q(a) \neq 0$).

- *The exponential shift rule.*

  $$P(D)e^{at}u = e^{at}P(D+a)u.$$

  The proof follows from the fact that

  $$De^{at}u = e^{at}Du + ae^{at}u = e^{at}(D+a)u.$$

12.2.2 Justification of the method of an educated guess

What is so special about the right hand sides for which the method works? They are actually solutions of some homogeneous linear ODE with constant coefficients. In other words, there exists a polynomial differential operator $H(D)$ for any of such function f that $H(D)f = 0$ (it is said that this polynomial operator *annihilates* f). Therefore, if we are looking for a particular solution of

$$P(D)y = f,$$

and f satisfies

$$H(D)f = 0,$$

then the form of a particular solution can be found from the form of the general solution to the homogeneous equation

$$H(D)P(D)y = 0.$$

Example 10. What is the form of a particular solution to

$$y'' - 2y' + y = e^t?$$

Let us note that the polynomial operator $H(D) = D - 1$ annihilates e^t:

$$(D-1)e^t = 0.$$

Therefore,

$$y'' - 2y' + 1 = (D-1)^2 y = e^t \implies (D-1)^3 y = 0.$$

We know that the general solution to the last homogeneous ODE is

$$y_h(t) = C_1e^t + C_2te^t + C_3t^2e^t.$$

But the first two terms can be dropped because they are the part of the general solution to $(D-1)^2y = 0$. Hence my particular solution has the form

$$y_p(t) = C_3t^2e^t,$$

exactly as we *guessed* in the first part of this section.