14 Mass on a spring and other systems described by linear ODE

Now we are in a position to consider applications of our mathematical technique to certain physical systems.

14.1 Mass on a spring

Consider a mass hanging on a spring (see Fig. 1). The position of the mass in uniquely defined by one coordinate $x(t)$ along the x-axis, whose direction is chosen to be along the direction of the force of gravity. Note that if this mass is in an equilibrium, then the string is stretched, say, by amount of s units. The origin of the x-axis is chosen such that the position of the mass at rest coincides with the zero coordinate.

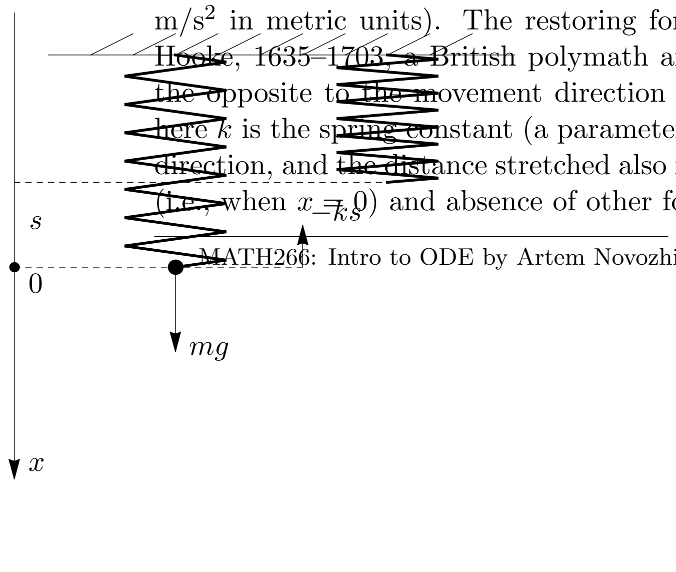

Figure 1: Mechanical system "mass on a spring."

We know that the movement of the mass is determined by the second Newton's law, that can be stated (for our particular one-dimensional case) as

$$ma = \sum_i F_i,$$

where m is the mass of the object, a is the acceleration, which, as we know from Calculus, is the second derivative of the displacement $x(t)$, $a = \ddot{x}$, and $\sum_i F_i$ is the net force applied (the sum of all the forces applied to the body along the chosen axis). What we know about the net force? This has to include the gravity, of course: $F_1 = mg$, where g is the acceleration due to gravity ($g \approx 9.8$ m/s^2 in metric units). The restoring force of the spring is governed by *Hooke's law* (after Robert Hooke, 1635–1703, a British polymath and natural scientist), which says that the restoring force in the opposite to the movement direction is proportional to the distance stretched: $F_2 = -k(s + x)$, here k is the spring constant (a parameter) and minus signifies that the force is acting in the opposite direction, and the distance stretched also includes the s units from above. Note that at the equilibrium (i.e., when $x = 0$) and absence of other forces we have $F_1 + F_2 = 0$ and hence $mg - ks = 0$ (this last

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024 expression often allows to find the spring constant k given the amount of s the spring is stretched by the mass m). When the mass is not at rest, we can also have damping, which is acting in the direction opposite to the direction of velocity. Observations say that it is reasonable to assume that the damping is proportional to the speed when x is small enough, hence $F_3 = -c\dot{x}$, where c is the constant of proportionality. Finally, we might have that an external force $F_4 = F(t)$ is applied to the mass. Summing,

$$m\ddot{x} = F_1 + F_2 + F_3 + F_4 \implies m\ddot{x} = mg - k(s + x) - c\dot{x} + F(t),$$

and finally, after some simplifications and rearrangements:

$$m\ddot{x} + c\dot{x} + kx = F(t), \tag{1}$$

which is a second order linear nonhomogeneous ODE with constant coefficients: Exactly the one we were studying in the last 3 lectures.

The parameters of the problem are all nonnegative: $m, c, k \geq 0$ (*Q:* What would be the physical meaning if $c < 0$ or $k < 0$?). To solve the problem of the movement of a mass on a spring it is necessary to set the initial conditions — the initial position and initial velocity:

$$x(0) = x_0, \quad \dot{x}(0) = v_0. \tag{2}$$

Remark about units: You should always remember about the units of the parameters and variables in the mathematical model and cannot use different units for different terms in the equation. Eventually, all the units in the model must agree. For the metric system the units are newtons (N) for the force, kilograms (kg) for mass, meters (m) for length and seconds (s) for time, therefore the velocity is measured in m/s, the acceleration is measured in m/s^2, the spring constant has units N/m, and the damping coefficient is measured in Ns/m= kg/s, finally newtons can be expressed through the basic measurements as N = kg $\cdot$ m/s^2, i.e., force in 1 N is the force required to accelerate a mass of 1 kg to the rate of 1 m per second squared.

For the English measurement system the force is measured in pounds (lbs), mass is in slugs (lbs $\cdot$ s^2/ft), length is in feet (ft), and time is in seconds (s) as well, you should express the units of the parameters in this system. In different measurement systems different numerical values for the same constants occur. For example, in metric system the acceleration of the free falling body $g \approx 9.8\,\mathrm{m/s^2}$, whereas in the English units $g \approx 32.2\,\mathrm{ft/s^2}$.

After the equation is written down we can forget about the original physical system and study the mathematical model (1)–(2), whose solutions can be given later a physical interpretation.

We start consider cases one by one, starting with the simplest one.

Harmonic oscillations

Here we assume that $c = 0$ and $F(t) \equiv 0$. Hence we have

$$m\ddot{x} + kx = 0,$$

or, after using the new notation $w_0^2 = k/m$ (note that both k and m are positive),

$$\ddot{x} + \omega_0^2 x = 0.$$

This equation has the general solution

$$x_h(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t,$$

where C_1, C_2 are arbitrary constants that are determined by the initial conditions (2).

For the following it will be convenient to rewrite the last expression in a different form. For this recall that if we have a and b such that $a^2 + b^2 = 1$ then it is always possible to find φ such that

$$\sin \varphi = a, \quad \cos \varphi = b.$$

We will also need the formula

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

Now, assuming that at least one of the arbitrary constants is not zero, we have

$$\begin{aligned} x_h(t) &= C_1 \cos \omega_0 t + C_2 \sin \omega_0 t = \\ &= \sqrt{C_1^2 + C_2^2} \left(\frac{C_1}{\sqrt{C_1^2 + C_2^2}} \cos \omega_0 t + \frac{C_2}{\sqrt{C_1^2 + C_2^2}} \sin \omega_0 t \right) = \\ &= A(\cos \omega_0 t \cos \varphi + \sin \omega_0 t \sin \varphi) = \\ &= A \cos(\omega_0 t - \varphi), \end{aligned}$$

where instead of old constants C_1, C_2 we have new constants A and φ, which can be determined by the initial conditions (2) and related to the old constants as

$$A = \sqrt{C_1^2 + C_2^2}, \quad \cos \varphi = \frac{C_1}{\sqrt{C_1^2 + C_2^2}}, \quad \sin \varphi = \frac{C_2}{\sqrt{C_1^2 + C_2^2}}.$$

The solution formula

$$x_h(t) = A \cos(\omega_0 t - \varphi)$$

gives a simple way to analyze the displacement $x_h(t)$ in this particular case (see Fig. 2).

We know that $\cos t$ is a periodic function with period 2π, and bounded between 1 and -1 (recall that function g is T-periodic if $g(t) = g(t + T)$ for all t and T is the smallest such number). Trigonometric functions cos and sin describe periodic oscillations that are called *simple harmonic motion*. Therefore the original system $\ddot{x} + \omega_0^2 x = 0$ is often called the *simple harmonic oscillator*. Function $\cos \omega_0 t$ has period (*Q:* Can you see why?)

$$T = \frac{2\pi}{\omega_0}.$$

The *frequency* f (number of complete oscillations per time unit, measured usually in hertz (Hz $= 1/\text{s}$)) is defined as the reciprocal of the period:

$$f = \frac{1}{T} = \frac{\omega_0}{2\pi},$$

and ω_0 is called the *angular frequency* ($\omega_0 = 2\pi f$, measured in radians per seconds).

Hence we have that the harmonic oscillator produces periodic motion with the angular frequency ω_0. By subtracting φ we simply shift the graph of our function, and this constant is called the *phase*.

Finally, the harmonic oscillations are bounded now by A and $-A$, and this constant is called the *amplitude* of oscillations. Therefore, if we are given a simple harmonic oscillator, then its behavior is defined by the angular frequency

$$\omega_0=\sqrt{\frac{k}{m}}\,,$$

which is the intrinsic property of the system, that is why it is sometimes called the *natural frequency of the system*, and by the amplitude and phase, which can be found given the initial conditions x_0, v_0. Note that the period of oscillations

$$T=2\pi\sqrt{\frac{m}{k}}$$

does not depend on the initial conditions and hence on the amplitude, which is the property of linear systems. For nonlinear system this usually does not hold.

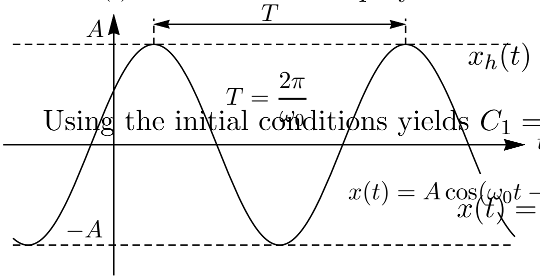

Figure 2: Simple harmonic oscillations.

Example 1. To consider a specific example, assume that a mass weighting 24 pounds attached to the end of the string stretches it four inches. Initially, the mass is released from a point 3 inches above the equilibrium position with the upward velocity 1 foot per second. Assuming that there is no damping find the equation of motion, the amplitude, and the period of oscillations.

First we need to convert inches into feet, which means that the mass stretches the spring $s=4/12=1/3$ foot. Since the weight of the mass is 24 pounds, this means that the actual mass (not force) satisfies $mg=24$, and since $g\approx 32$ foot per second squared, $m=24/32=3/4$ slug. Finally, from Hook's law I have $mg=ks$ or $k=24\cdot 3=72$. Hence my IVP takes the form

$$\frac{3}{4}\ddot{x}+72x=0,\quad x(0)=-\frac{1}{4},\,\dot{x}(0)=-1.$$

I have the characteristic polynomial $\lambda^2+96=0$ or $\lambda_{1,2}=\pm 4\sqrt{6}\mathrm{i}$. Therefore the general solution is

$$x_h(t)=C_1\cos 4\sqrt{6}t+C_2\sin 4\sqrt{6}t.$$

Using the initial conditions yields $C_1=-\frac{1}{4}, C_2=-\frac{1}{4\sqrt{6}}$ and hence I found the equation of motion

$$x(t)=-\frac{1}{4}\cos 4\sqrt{6}t-\frac{1}{4\sqrt{6}}\sin 4\sqrt{6}t.$$

The period of oscillations is simply

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} = \frac{\pi}{2\sqrt{6}} \approx 0.64 \text{ s}.$$

The amplitude is computed as

$$A = \sqrt{C_1^2 + C_2^2} = \frac{1}{4}\sqrt{\frac{7}{6}} \approx 0.27 \text{ ft}.$$

To present an equivalent formula for the equation of motion, let me also find the phase in the given example. I have that

$$\cos\varphi = \frac{C_1}{A} \approx -0.93\,, \quad \sin\varphi = \frac{C_2}{A} = -0.38,$$

and hence

$$\tan\varphi = \frac{C_2}{C_1} \approx 0.41.$$

Since both cosine and sine are negative, my angle must be in the third quadrant, i.e. between π and $3\pi/2$ radians, which finally gives the value

$$\varphi = \pi + \arctan 0.41 \approx 3.53 \text{ radians},$$

which allows me to write my equation of motion in an equivalent form

$$x(t) = 0.27\cos\left(4\sqrt{6}t - 3.53\right).$$

This last equation is more convenient for the analysis. For instance if I was asked to find the time moments at which the mass passes the equilibrium point, all I need to do is to solve the equation

$$0.27\cos\left(4\sqrt{6}t - 3.53\right) = 0$$

for the unknown time moments t (note that there will be infinitely many of them).

Simple harmonic oscillator predicts that the oscillations continue forever, which is not true for the real systems. The reason for this is that we assumed that there is no damping. Now consider the case when $c \neq 0$.

Damping

Let now $F(t) \equiv 0$ and $c > 0$. Hence,

$$m\ddot{x} + c\dot{x} + kx = 0.$$

To solve it we write down the characteristic equation

$$m\lambda^2 + c\lambda + k = 0,$$

which can be solved as

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}.$$

Here we need to consider 3 cases:

- *Overdamped motion.* Assume that $c^2 - 4mk > 0$, therefore, the characteristic equation has two negative real roots (*Q:* Do you see why they both are negative?) λ_1, λ_2, and the general solution is given by
$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$
Depending on the values of C_1 and C_2 we see that this solution will either never cross zero (the equilibrium point), or cross it only once. Moreover, since both lambdas are negative, the solution approaches zero: $x(t) \to 0$ as $t \to \infty$, which physically means that if the damping is really large, the mass on a spring will return to its equilibrium position either without complete oscillations or with just one oscillation.

- *Critically damped motion.* Let $c^2 - 4mk = 0$, then $\lambda = -c/(2m)$ is the only root of the characteristic polynomial with multiplicity 2. Therefore,
$$x(t) = C_1 e^{\lambda t} + C_2 t e^{\lambda t}.$$
Here the situation is very similar to the previous case. Since λ is negative, $x(t) \to 0$ as $t \to \infty$ without oscillations.

- *Damped oscillations.* Let $c^2 - 4mk < 0$, therefore we have two complex conjugate roots $\lambda_1 = \overline{\lambda_2} = \alpha + \mathrm{i}\beta$, where
$$\alpha = -\frac{c}{2m}, \quad \beta = \frac{\sqrt{4mk - c^2}}{2m} = \sqrt{\omega_0^2 - \left(\frac{c}{2m}\right)^2}.$$
We have
$$x(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t),$$
or, using the same manipulations as in the case of the simple harmonic oscillator,
$$x(t) = A e^{\alpha t} \cos(\beta t - \varphi),$$
where A and φ are new arbitrary constants. Note that if we consider $A(t) = Ae^{\alpha t}$ as our "amplitude," then, since $\alpha < 0$, $A(t) \to 0$ as should be expected for damped oscillations (Fig. 3). The solution in this case is not periodic, but sometimes called quasiperiodic, because we observe oscillations with decreasing amplitude and the quasiperiod is given by
$$T = \frac{2\pi}{\beta} = \frac{2\pi}{\sqrt{\omega_0^2 - \left(\frac{c}{2m}\right)^2}},$$
which is larger than the period of simple harmonic oscillations with the angular velocity ω_0, as also should be intuitively expected.

Harmonic oscillator with an external force

Now assume that $c = 0$ and $F(t) = F_0 \cos \omega t$, i.e., the external force is a periodic function with amplitude F_0 and angular frequency ω. We have, using the same notation as before,
$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t.$$

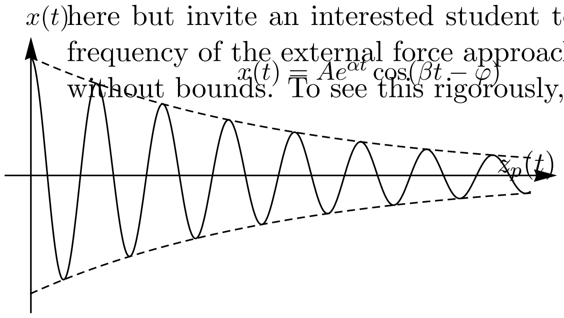

Figure 3: Damped oscillations.

The solution to this equation is, as we know,

$$x(t) = x_h(t) + x_p(t),$$

where $x_h(t)$ is the general solution to the homogeneous equation and $x_p(t)$ is a particular solution to the nonhomogeneous equation. $x_h(t)$ was already found in this lecture:

$$x_h(t) = A\cos(\omega_0 t - \varphi).$$

Now, since $\cos\omega t = \operatorname{Re} e^{\mathrm{i}\omega t}$, consider instead the equation

$$\ddot{z} + \omega_0^2 z = \frac{F_0}{m} e^{\mathrm{i}\omega t}.$$

Assume first that $\mathrm{i}\omega$ is not a root of the characteristic polynomial, i.e., $\omega \neq \omega_0$. Then

$$z_p(t) = Ce^{\mathrm{i}\omega t} \implies C = \frac{F_0}{m(\omega_0^2 - \omega^2)}\,.$$

Therefore,

$$x_p(t) = \operatorname{Re} z_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t,$$

and the general solution is

$$x(t) = A\cos(\omega_0 t - \varphi) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos\omega t,$$

where A and φ are determined by the initial conditions.

Here we need to note two things: First, the general solution is the sum of two periodic functions with different periods. Will the solution be also periodic? The answer is generally "no," for the general solution to be periodic we have to ask that ω_0/ω is a rational number (I will not go into further details here but invite an interested student to contemplate on the last statement). Second, if the angular frequency of the external force approaches the natural frequency of the system, then $|x_p(t)|$ will grow without bounds. To see this rigorously, now let $\omega = \omega_0$. In this case,

$$z_p(t) = Cte^{\mathrm{i}\omega_0 t} \implies C = \frac{F_0}{2m\omega_0 \mathrm{i}}\,,$$

and hence

$$x_p(t) = \operatorname{Re} z_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t,$$

which satisfies $|x_p(t)| \to \infty$ as $t \to \infty$ (see Fig. 4).

In physics the phenomenon when the amplitude grows without bounds if the natural frequency of the system equals the angular frequency of the external force is called *resonance* and is responsible for some very unfortunate events including the collapse of Tacoma bridge[1] (watch it on YouTube).

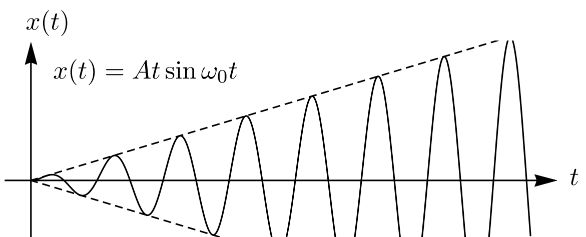

Figure 4: Resonance in the system without damping.

Ok, you might say that the resonance was actually the consequence of the assumption that $c = 0$, i.e., that we did not include damping, which is obviously not true for any mechanical system on our planet.

Full equation

Consider now the full equations with all the forces included:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t.$$

The general solution is given by the sum

$$x(t) = x_h(t) + x_p(t),$$

where $x_h(t)$ was already found above (I assume the damped oscillations occur in the system without external force):

$$x_h(t) = Ae^{\alpha t} \cos(\beta t - \varphi).$$

A particular solution can be found using the same approach as in the case $c = 0$ and is given by (fill in the details)

$$x_p(t) = \frac{F_0}{m\big((\omega_0^2 - \omega^2)^2 + (c\,\omega/m)^2\big)^{1/2}} \cos(\omega t - \varphi),$$

where

$$\tan \varphi = \frac{\omega c}{k - m\omega^2}\,.$$

Well, the reality is actually more complicated, if you want to see the details, read Billah, K. Y., & Scanlan, R. H. (1991). Resonance, Tacoma Narrows bridge failure, and undergraduate physics textbooks. American Journal of Physics, 59(2), 118–124.

Since $x_h(t) \to 0$ as $t \to \infty$ (this is the *transient* part of the solution), then $x(t) \to x_p(t)$, which is called the *stationary* part of the solution. Hence we conclude that the mass on the spring, when the damping and external periodic force are taken into account, will eventually produce oscillations with the frequency equal to the frequency of the external force, and with the amplitude given by

$$A(\omega) = \frac{F_0}{m\big((\omega_0^2 - \omega^2)^2 + (c\,\omega/m)^2\big)^{1/2}}\,,$$

which is maximal when (check it)

$$\omega^2 = \omega_0^2 - \frac{c^2}{2m^2}\,,$$

provided that $\omega_0^2 - c^2/(2m^2) > 0$ (in the opposite case the highest amplitude happens at $\omega = 0$). This value of the angular frequency of the external force is defined to be resonant (see Fig. 5).

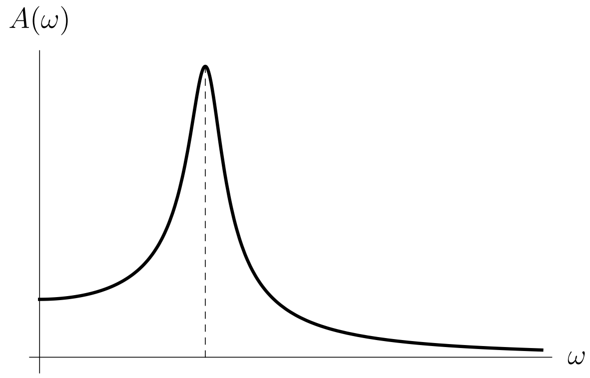

Figure 5: Resonance in the system with damping. The parameters are chosen $F_0 = k = m = 1, c = 0.2$. The resonance occurs at $\omega = \sqrt{0.98}$, which is shown by the dashed line.

Since it is easy to get lost in all the different parameters of the system, let me consider a specific example.

Example 2. A mass of 1 slug is attached to a spring whose constant is 5 lb/ft. Initially the mass is released 1 foot below the equilibrium position with downward velocity of 5 ft/s, and the motion takes place in the medium with damping coefficient $c = 2$. Find the equation of motion if the mass is driven by an external force $F(t) = 3\sin 2t$. Find the transient and steady state solution. Determine the eventual amplitude of oscillations.

My mathematical model takes the form

$$\ddot{x} + 2\dot{x} + 5x = 3\sin 2t, \quad x(0) = 1,\, \dot{x}(0) = 5.$$

Using the standard technique and the method of an educated guess I find (fill in the omitted steps) that the general solution is given by

$$x(t) = -\frac{3}{17}(4\cos 2t - \sin 2t) + C_1 e^{-t}\cos 2t + C_2 e^{-t}\sin 2t.$$

Using the initial conditions yields $C_1 = 29/17, C_2 = 54/17$, hence the transient solution is given by

$$x_{\text{transient}}(t) = \frac{29}{17}e^{-t}\cos(2t) + \frac{54}{17}e^{-t}\sin(2t),$$

which approaches zero as $t \to \infty$, and the steady state solution

$$x_{\text{steady-state}}(t) = -\frac{12}{17}\cos 2t + \frac{3}{17}\sin 2t.$$

Finally, to find the eventual amplitude I will use the same approach as in the discussion of the harmonic oscillator and find that

$$A_{\text{eventual}} = \sqrt{\left(\frac{12}{17}\right)^2 + \left(\frac{3}{17}\right)^2} = \frac{3}{\sqrt{17}}\,.$$

You can see the comparison of the full solution with its transient and steady state parts in Fig. 6.

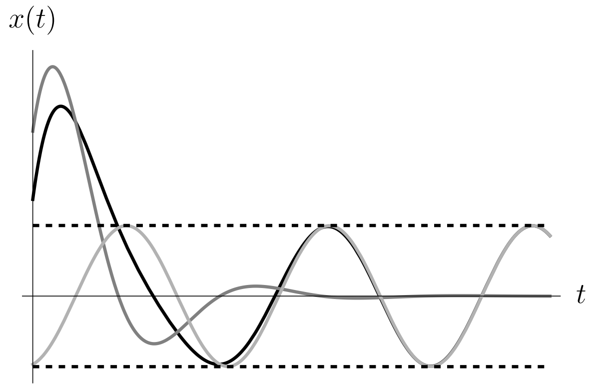

Figure 6: Comparison of the full solution (black) with its transient (dark grey) and steady-state (light grey) parts. The dotted lines show the eventual amplitude of oscillations.

14.2 Other physical systems

The power of mathematical modeling lies in the fact that literally the same models appear in quite unrelated physical systems. Here I give two more examples of physical systems, for which the analysis in the previous section provides (full or approximate) description of the dynamic behaviors.

14.2.1 *LRC*-circuit

Consider now the LRC-circuit as presented in Fig. 7. This circuit consists of three elements: resistor with resistance R measured in ohms (Ω), capacitor with capacitance C measured in farads (F), and an inductor with inductance L measured in henrys (H). There is also a source of electricity $E(t)$ measured in volts (V). We need to find the current $I(t)$ in the circuit measured in amperes (A).

> *Kirchhoff's law* states that the sum of the voltage drops across the circuit elements has to be equal $E(t)$.

To be able to apply this law to our situation, we need to know that the voltage drop on the resistor is given by

$$IR \quad \text{(Ohms' law)},$$

on the inductor is given by

$$L\frac{\mathrm{d}I}{\mathrm{d}t}\,,$$

and on the capacitor is

$$\frac{Q}{C}\,,$$

where $Q(t)$ is the charge of the capacitor (measured in coulombs (C)) that is related to the current as

$$I(t)=\frac{\mathrm{d}Q}{\mathrm{d}t}(t)\,.$$

Putting everything together we obtain

$$IR+L\frac{\mathrm{d}I}{\mathrm{d}t}+\frac{Q}{C}=E(t),$$

or

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2}+R\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{Q}{C}=E(t),$$

which is a second order linear nonhomogeneous equation with constant coefficients. By analogy with the mass on the spring we see that the inductance plays the role of the mass, the resistance is analogous to damping, and capacitance is inversely related to the spring constant. Assuming that $R=0$ we can find the natural frequency of the system

$$\omega_0=\sqrt{\frac{1}{LC}}\,,$$

and hence the harmonic oscillations of the charge in LRC-circuite will occur with the period

$$T=2\pi\sqrt{LC}\,.$$

Note that the resonance in an electric circuit may be a desirable phenomenon (signal amplification).

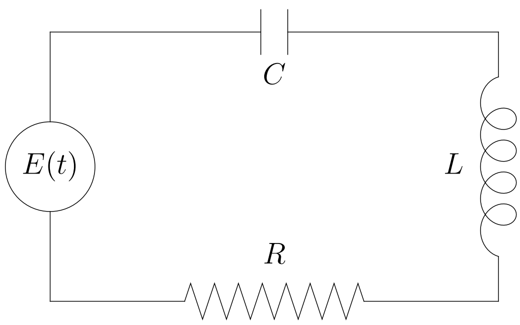

Figure 7: LRC-circuit.

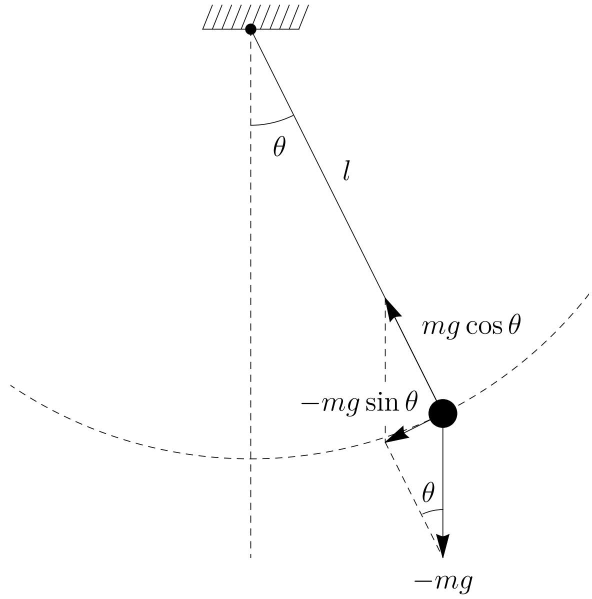

Figure 8: Pendulum.

14.2.2 Pendulum

Consider now the pendulum (see Fig. 8). We will use again the second Newton's law to find the equation of motion. Let the length of the pendulum be l and the mass be m. Now we need to consider the projections of the forces onto the two axes. One is along the pendulum itself, and there the force of tension is equal to $mg\cos\theta$, they cancel each other. The only projection on the axis perpendicular to the pendulum is $-mg\sin\theta$ (the minus is because the force is restoring).

Hence we have

$$m\ddot{s} = -mg\sin\theta,$$

where $s(t)$ is the displacement at time t. In our case $s(t) = l\theta(t)$, therefore, finally

$$\ddot{\theta} = -\frac{g}{l}\sin\theta,$$

or, using the notation $\omega_0^2 = g/l$:

$$\ddot{\theta} + \omega_0^2\sin\theta = 0.$$

This is a *nonlinear* equation of the second order, for which only an implicit solution can be written down. To make things easier, recall that if the angle, measured in radians, is small enough, then

$$\theta \approx \sin\theta.$$

Therefore, the nonlinear equation for the pendulum can be replaced, as a first approximation, with the equation of small oscillations:

$$\ddot{\theta} + \omega_0^2\theta = 0,$$

which produces harmonic oscillations with angular frequency ω_0 and period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}\,.$$

(I note, however, that in the full *nonlinear* equation the period of oscillations *depends* on the amplitude, but analysis of this phenomenon belongs to a next level course.)

Additionally, damping and an external force can be considered. In this case we end up with the equation

$$l\ddot{\theta} + c\,\dot{\theta} + g\theta = F(t),$$

where the resonance can occur (think about the swings).