## 16 Laplace transform. Solving linear ODE

I this lecture I will explain how to use the Laplace transform to solve an ODE with constant coefficients. The main tool we will need is the following property from the last lecture:

5° *Differentiation.* Let $\mathscr{L}\{f(t)\} = F(s)$. Then

$$\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0), \quad \mathscr{L}\left\{f''(t)\right\} = s^2F(s) - sf(0) - f'(0).$$

Now consider the second order IVP

$$y'' + py' + qy = f(t), \quad y(0) = y_0,\, y'(0) = y_1. \tag{1}$$

I consider the second order equation here, but it should be clear that similar considerations will lead to a solution of any order linear differential equation with constant coefficients.

Apply the Laplace transform to the left and right hand sides of ODE (1):

$$\begin{aligned}
\mathscr{L}\left\{y'' + py' + qy\right\} = \mathscr{L}\{f\} &\implies \\
\mathscr{L}\left\{y''\right\} + p\,\mathscr{L}\left\{y'\right\} + q\,\mathscr{L}\{y\} = \mathscr{L}\{f\} &\implies \\
s^2\,\mathscr{L}\{y\} - sy(0) - y'(0) + ps\,\mathscr{L}\{y\} - py(0) = \mathscr{L}\{f\} &\implies \\
(s^2 + ps + q)\,\mathscr{L}\{y\} - sy_0 - y_1 - py_0 = \mathscr{L}\{f\} &\implies \\
(s^2 + ps + q)Y(s) - sy_0 - y_1 - py_0 = F(s) &\implies \\
Y(s) = \frac{F(s) + sy_0 + y_1 + py_0}{s^2 + ps + q}\,,
\end{aligned}$$

where I used the notation $Y(s) = \mathscr{L}\{y\}$ and $F(s) = \mathscr{L}\{f\}$. The property of linearity was used, and also I used Property 5° to simplify $\mathscr{L}\{y''\}$ and $\mathscr{L}\{y'\}$. Since, due to Property 5°, the Laplace transform turns the operation of differentiation into the algebraic operation multiplication by $s$, then, instead of the initial *differential* equation, I end up with a simple *algebraic* equation for $Y(s)$. The final step (and usually the most complex one) is to restore $y(t)$ from $Y(s)$, i.e.,

$$y(t) = \mathscr{L}^{-1}\{Y(s)\}\,.$$

Note that to apply this approach, we need the initial conditions specified at the point zero. In general, if the initial conditions are not given, we can always put two parameters $y(0) = y_0$ and $y'(0) = y_1$ such that the solution will be a two parameter family.

**Example 1.** Solve using the Laplace transform

$$y' - y = e^{3t}, \quad y(0) = 2.$$

Application of the Laplace transform leads to

$$sY(s) - y(0) - Y(s) = \frac{1}{s-3}\,,$$

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therefore,

$$Y(s) = \frac{2}{s-1} + \frac{1}{(s-1)(s-3)} = \frac{2}{s-1} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s-3}\right).$$

Using the table to find the inverse Laplace transform, we obtain

$$y(t) = \mathscr{L}^{-1}\{Y\} = 2e^t - \frac{1}{2}(e^t - e^{3t}) = \frac{3}{2}e^t + \frac{1}{2}e^{3t}.$$

**Example 2.** Solve

$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = 1, \, y'(0) = 0.$$

Applying the Laplace transform, I find

$$(s^2 - 3s + 2)Y - s + 3 = \frac{1}{s-3} \implies Y(s) = \frac{s^2 - 6s + 10}{(s-3)(s-2)(s-1)}.$$

To find the inverse Laplace transform we will need first simplify the expression for $Y(s)$ using the partial fraction decomposition. Note that $Y(s)$ is given by a rational function, i.e, a polynomial divided by polynomial, moreover, the highest degree of the numerator is strictly less than the highest degree of the denominator. Hence, as we all learnt in Calculus,

$$Y(s) = \frac{s^2 - 6s + 10}{(s-3)(s-2)(s-1)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s-1}.$$

To find $A$, $B$ and $C$ here is especially simple. For example, for $A$ multiply both sides by $s-3$ and plug $s = 3$ into the expressions to obtain $A = \frac{1}{2}$. In a similar way $B = -2$ and $C = \frac{5}{2}$.

Therefore, using the linearity of the inverse Laplace transform, we find

$$y(t) = \mathscr{L}^{-1}\{Y\} = \frac{5}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t}.$$

**Example 3.** Solve

$$y'' - 10y' + 9y = 5t, \quad y(0) = -1, \, y'(0) = 2.$$

Applying the Laplace transform to both side, we find

$$(s^2 - 10s + 9)Y + s - 2 - 10 = \frac{5}{s^2} \implies Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}.$$

To find the inverse Laplace transform we will need first simplify the expression for $Y(s)$ using the partial fraction decomposition:

$$\frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}.$$

We find

$$B = \frac{5}{9}, D = -2, C = \frac{31}{81}, A = \frac{50}{81}.$$

Therefore, using the linearity of the inverse Laplace transform,

$$y(t) = \frac{50}{81} + \frac{5t}{9} + \frac{31}{81}e^{9t} - 2e^t.$$

**Example 4.** Solve

$$y'' - 6y' + 15y = 2\sin 3t, \quad y(0) = -1,\, y'(0) = -4.$$

We have

$$(s^2 - 6s + 15)Y + s - 2 = \frac{6}{s^2+9} \implies Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15}.$$

To find the constants, we need to simplify the expression on the right (find the common denominator) and equate the coefficients at the equal powers:

$$\begin{aligned} &s^3\colon A + C = -1 \\ &s^2\colon -6A + B + D = 2 \\ &s^1\colon 15A - 6B + 9C = -9 \\ &s^0\colon 15B + 9D = 24 \end{aligned}$$

The solution is

$$A = \frac{1}{10},\, B = \frac{1}{10},\, C = -\frac{11}{10},\, D = \frac{5}{2}.$$

Hence, we got

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right).$$

Now we need to find the inverse Laplace transform. Let us start with the first term:

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9} + \frac{1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\} = \cos 3t + \frac{1}{3}\sin 3t.$$

The second term is slightly more involved. Rearrange the expression in the following way (remember that we can always add and subtract the same expression and multiply and divide by the same expression different from zero):

$$\begin{aligned} \frac{-11s+25}{s^2-6s+15} &= \frac{-11s+25}{(s-3)^2+6} \\ &= \frac{-11(s-3)-8}{(s-3)^2+6} \\ &= -11\frac{(s-3)}{(s-3)^2+6} - \frac{8}{\sqrt{6}}\frac{\sqrt{6}}{(s-3)^2+6}. \end{aligned}$$

Now

$$\mathscr{L}^{-1}\left\{\frac{-11s+25}{s^2-6s+15}\right\} = -11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t.$$

The final answer hence is

$$y(t) = \mathscr{L}^{-1}\{Y\} = \frac{1}{10}\left(\cos 3t + \frac{1}{3}\sin 3t - 11e^{3t}\cos\sqrt{6}t - \frac{8}{\sqrt{6}}e^{3t}\sin\sqrt{6}t\right).$$