17 Laplace transform. Solving linear ODE with piecewise continuous right hand sides. Delta-function

In this lecture I will show how to apply the Laplace transform to the ODE $Ly = f$ with piecewise continuous f. I will also talk briefly about an incredibly interesting mathematical object — so-called delta-function.

17.1 Solving linear ODE with piecewise continuous right hand side

Definition 1. *A function f is piecewise continuous on the interval $I = [a, b]$ if it is defined and continuous on this interval except, probably, a finite number of points, $t_1, t_2, \ldots, t_k$, at each of which the left and right limits of this function exist (i.e., all the discontinuities of the first type, or "jump" discontinuities).*

Example 2. Consider the function

$$f(t) = \begin{cases} -1, & t \leq 4, \\ 1, & t > 4, \end{cases}$$

the graph of which is given in Fig. 1.

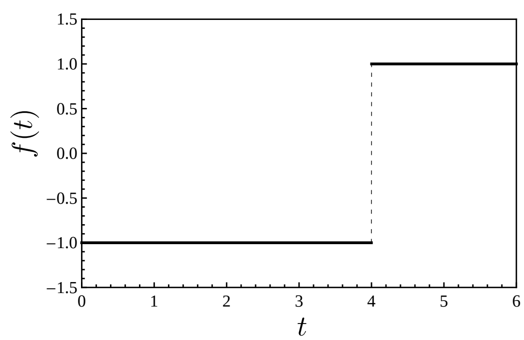

Figure 1: Piecewise continuous function $f(t)$ in Example 2.

You can see that there is one point of discontinuity, namely $t = 4$, at which $f(t)$ has the discontinuity of the first type, or jump discontinuity. We have

$$\lim_{t \to 4^-} f(t) = -1, \quad \lim_{t \to 4^+} f(t) = 1,$$

they both exist and are not equal to each other. The value of the jump is 2 (note that the value of the jump is the right limit minus the left limit, that is, it is possible in general to have both positive and negative jumps).

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024

What is the Laplace transform of $f(t)$? We can find it using the definition:

$$\begin{aligned}
\mathscr{L}\{f\} &= \int_0^\infty f(t)e^{-st}\,\mathrm{d}t \\
&= \int_0^4 (-1)e^{-st}\,\mathrm{d}t + \int_4^\infty 1 \cdot e^{-st}\,\mathrm{d}t \\
&= \frac{1}{s}e^{-st}|_0^4 - \frac{1}{s}e^{-st}|_4^\infty \\
&= \frac{1}{s}e^{-4s} - \frac{1}{s} + \frac{1}{s}e^{-4s} \\
&= \frac{1}{s}(2e^{-4s} - 1).
\end{aligned}$$

Example 3 (Heaviside function)**.** The most important example for us will be the *Heaviside function*, defined as

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0. \end{cases}$$

Note that

$$\mathscr{L}\{u(t)\} = \frac{1}{s},$$

the same as for $\mathscr{L}\{1\}$, because we are only interested in functions on $[0, \infty)$. This yields that for any function $f(t)$ defined on $[0, \infty)$ it is true that $f(t) = u(t)f(t)$ if $t \geq 0$, and hence

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{u(t)f(t)\}.$$

Additionally to the Heaviside function, we consider shifted Heaviside function $u(t-a)$ for a non-negative $a \geq 0$ (see Fig. 2).

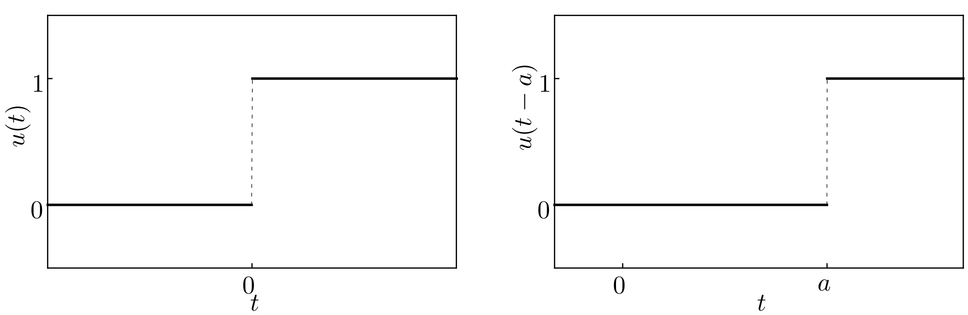

Figure 2: Heaviside function $u(t)$ (left) and shifted Heaviside function $u(t-a)$ (right).

It is a simple exercise to check, using the definition, that

$$\mathscr{L}\{u(t-a)\} = e^{-as}.$$

Now I can state the final property of the Laplace transform that we will use (there are more actually):

6° *Time shifting.* Let $\mathscr{L}\{f(t)\} = F(s)$, then

$$\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}F(s).$$

To prove it, consider

$$\mathscr{L}\{u(t-a)f(t-a)\} = \int_0^\infty u(t-a)f(t-a)e^{-st}\,\mathrm{d}t = \int_a^\infty f(t-a)e^{-st}\,\mathrm{d}t,$$

because $u(t-a) = 0$ when $t < a$. Now make the substitution $\xi = t - a$, then $\mathrm{d}t = \mathrm{d}\xi$ and if $t = a$ then $\xi = 0$. We obtain

$$\int_0^\infty f(\xi)e^{-s(\xi+a)}\,\mathrm{d}\xi = e^{-sa}\,\mathscr{L}\{f\}.$$

Let us find the Laplace transform of the function in Example 2.

Example 4 (Example 2 is continued)**.** Note that using the shifted Heaviside function we can construct for any $a < b$ the function

$$u(t-a) - u(t-b),$$

such that this function is equal to 1 when $t \in (a, b)$ and zero otherwise (think this out!) This means

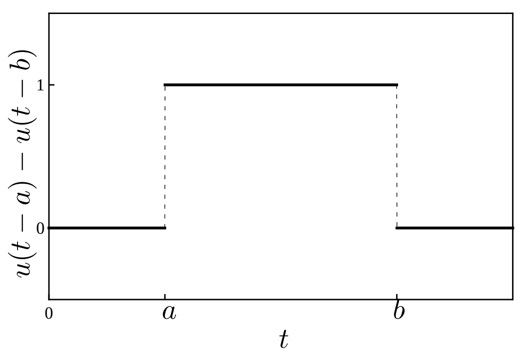

Figure 3: $u(t-a) - u(t-b)$ for $b > a$.

that for any $f(t)$ the function $f(t)\big(u(t-a) - u(t-b)\big)$ equals $f(t)$, when $t \in (a, b)$, and 0 otherwise. Now consider again

$$f(t) = \begin{cases} -1, & t \leq 4, \\ 1, & t > 4. \end{cases}$$

This, using the previous, can be represented as

$$f(t) = -1\big(u(t) - u(t-4)\big) + 1 \cdot u(t-4) = -u(t) + 2u(t-4).$$

Now, using Property 6°,

$$\mathscr{L}\{f\} = \mathscr{L}\{-u(t) + 2u(t-4)\} = -\frac{1}{s} + 2\frac{e^{-4s}}{s},$$

exactly as we already found in Example 2.

Example 5. Find the Laplace transform for

$$f(t)=\begin{cases}0, & t<1,\\ t^2, & 1<t<2,\\ 0, & t>2.\end{cases}$$

Again, using the properties of the Heaviside function, we can write

$$f(t)=t^2\big(u(t-1)-u(t-2)\big)=t^2u(t-1)-t^2u(t-2).$$

However, we cannot apply Property 6° directly to this expression since we need the expressions of the form $f(t-a)u(t-a)$. To deal with it, consider

$$t^2=(t-1+1)^2=(t-1)^2+2(t-1)+1.$$

Similarly,

$$t^2=(t-2+2)^2=(t-2)^2+4(t-2)+4.$$

Hence we have

$$f(t)=\big((t-1)^2+2(t-1)+1\big)u(t-1)-\big((t-2)^2+4(t-1)+4\big)u(t-2),$$

which implies that

$$\mathscr{L}\{f\}=\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)e^{-s}+\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right)e^{-2s}.$$

Now we are ready to start solving ODE with piecewise continuous right hand side.

Example 6. Solve

$$y''+2y'+y=f(t),\quad y(0)=1,\, y'(0)=0,$$

where

$$f(t)=\begin{cases}1, & t<2,\\ 0, & t>2.\end{cases}$$

We have

$$f(t)=u(t)-u(t-2).$$

Applying the Laplace transform to both sides, we have

$$(s^2+2s+1)Y-s-2=\frac{1}{s}(1-e^{-2s}),$$

or, after some rearrangement,

$$Y(s)=\frac{1}{s}-\frac{1}{s(s+1)^2}e^{-2s}.$$

Since, using the partial fractions,

$$\frac{1}{s(s+1)^2}=\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\,,$$

then we find

$$y(t) = \mathscr{L}^{-1}\{Y(s)\} = 1 - \left(1 - e^{-(t-2)} - e^{-(t-2)}(t-2)\right)u(t-2),$$

using Property 6°.

Note that we can rewrite our solution as

$$y(t) = \begin{cases} 1, & t < 2, \\ e^{2-t}(t-1), & t > 2. \end{cases}$$

The graph of the solution in given in Fig. 4.

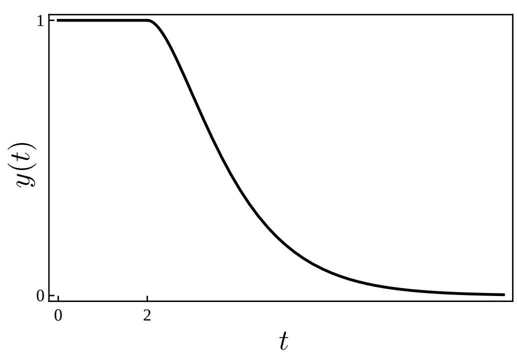

Figure 4: Solution $y(t)$ in Example 6.

Example 7. Solve

$$y'' + y' + y = f(t), \quad y(0) = 1,\, y'(0) = 0,$$

where $f(t)$ is shown in Fig. 5.

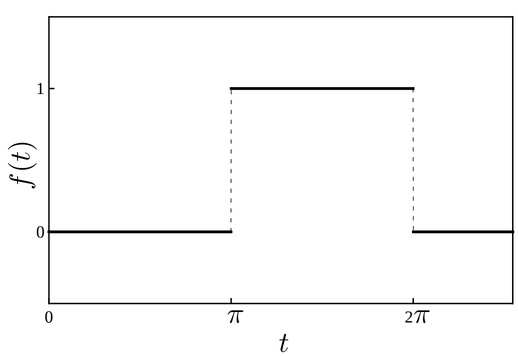

Figure 5: $f(t)$ in Example 7.

We find that

$$f(t) = u(t-\pi) - u(t-2\pi).$$

Applying the Laplace transform to the ODE, we find

$$s^2 Y - s + sY - 1 + Y = \frac{1}{s}(e^{-\pi s} - e^{-2\pi s}),$$

from where

$$Y(s) = \frac{s+1}{s^2+s+1} + \frac{1}{s(s^2+s+1)}(e^{-\pi s} - e^{-2\pi s}).$$

Note that using partial fraction decomposition, we have

$$\frac{1}{s(s^2+s+1)} = \frac{1}{s} - \frac{s+1}{s^2+s+1},$$

therefore, finally we have

$$Y(s) = \frac{1}{s}(e^{-\pi s} - e^{-2\pi s}) + \frac{s+1}{s^2+s+1}(1 - e^{-\pi s} + e^{-2\pi s}).$$

Now our task is to find the inverse Laplace transform for Y. We have $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = u(t)$, hence

$$\mathscr{L}^{-1}\left\{\frac{1}{s}(e^{-\pi s} - e^{-2\pi s})\right\} = u(t-\pi) - u(t-2\pi).$$

For the second term in $Y(s)$ consider first

$$\frac{s+1}{s^2+s+1} = \frac{s+1/2+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2} = \frac{s+1/2}{(s+1/2)^2+(\sqrt{3}/2)^2} + \frac{1}{\sqrt{3}}\frac{\sqrt{3}/2}{(s+1/2)^2+(\sqrt{3}/2)^2},$$

therefore

$$\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = e^{-t/2}\left(\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\right).$$

Finally, using Property 6°, we have

$$\begin{aligned} y(t) = \mathscr{L}^{-1}\{Y\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}(e^{-\pi s} - e^{-2\pi s}) + \frac{s+1}{s^2+s+1}(1 - e^{-\pi s} + e^{-2\pi s})\right\} = \\ &= u(t-\pi) - u(t-2\pi) + e^{-t/2}\left(\cos\frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2}\right)u(t) - \\ &- e^{-(t-\pi)/2}\left(\cos\frac{\sqrt{3}(t-\pi)}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}(t-\pi)}{2}\right)u(t-\pi) + \\ &+ e^{-(t-2\pi)/2}\left(\cos\frac{\sqrt{3}(t-2\pi)}{2} + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}(t-2\pi)}{2}\right)u(t-2\pi). \end{aligned}$$

The graph of the solution is shown in Fig. 6.

Example 8. Solve

$$y'' + y = f(t) = \begin{cases} \cos t, & t < \pi/2, \\ 0, & t > \pi/2, \end{cases} \quad y(0) = 3,\ y'(0) = -1.$$

We can rewrite the right hand side as

$$f(t) = \cos t\big(u(t) - u(t-\pi/2)\big).$$

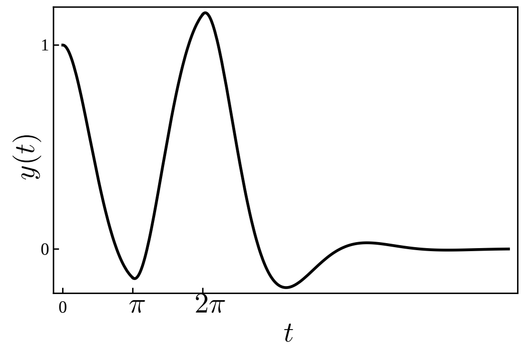

Figure 6: Solution $y(t)$ depending on time t in Example 7.

Note that to invoke Property 6° we need the function with the same argument as the argument of the Heaviside function.

$$f(t) = u(t)\cos t - \cos(t - \pi/2 + \pi/2)u(t - \pi/2) = u(t)\cos t + \sin(t - \pi/2)u(t - \pi/2),$$

where I used

$$\cos(a + b) = \cos a \cos b - \sin a \sin b.$$

Therefore, after applying the Laplace transform, I find

$$(s^2 + 1)Y(s) = 3s - 1 + \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1}e^{-\pi t/2},$$

or,

$$Y(s) = \frac{3s}{s^2 + 1} - \frac{1}{s^2 + 1} + \frac{s}{(s^2 + 1)^2} + \frac{1}{(s^2 + 1)^2}e^{-\pi t/2}.$$

To find the inverse Laplace transform, we will need

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2 + 1)^2}\right\}, \quad \mathscr{L}^{-1}\left\{\frac{1}{(s^2 + 1)^2}\right\}.$$

For this, using the property of the differentiation of the frequency, note that

$$\mathscr{L}\left\{t \sin t\right\} = \frac{2s}{(s^2 + 1)^2}, \quad \mathscr{L}\left\{t \cos t\right\} = \frac{s^2 - 1}{(s^2 + 1)}.$$

Hence

$$\mathscr{L}^{-1}\left\{\frac{s}{(s^2 + 1)^2}\right\} = \frac{1}{2}t \sin t.$$

For the second one

$$\frac{1}{(s^2 + 1)^2} = \frac{1}{2}\left(\frac{1}{s^2 + 1} - \frac{s^2 - 1}{(s^2 + 1)^2}\right),$$

hence

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2 + 1)^2}\right\} = \frac{1}{2}(\sin t - t \cos t).$$

Finally we find

$$y(t) = \mathscr{L}^{-1}\{Y\} = 3\cos t - \sin t + \frac{1}{2}t\sin t + \frac{1}{2}\big(\sin(t-\pi/2) - (t-\pi/2)\cos(t-\pi/2)\big)u(t-\pi/2).$$

By noting that $\sin(t-\pi/2) = -\cos t$ and $\cos(t-\pi/2) = \sin t$, we can simplify this expression as

$$y(t) = \begin{cases} 3\cos t - \sin t + \frac{1}{2}t\sin t, & t < 2, \\ \frac{5}{2}\cos t + \frac{\pi-4}{4}\cos t, & t > 2. \end{cases}$$

17.2 Delta-function

Consider the following problem:

$$y' + y = f(t), \quad y(0) = 0,$$

where f is defined as

$$f(t) = \begin{cases} \frac{I_0}{2\varepsilon}, & t \in (a-\varepsilon, a+\varepsilon), \\ 0, & \text{otherwise.} \end{cases}$$

In words, we have a mathematical model of some external action during the short time interval $(a-\varepsilon, a+\varepsilon)$ of the total intensity I_0 since the integral of this function is always I_0. I want to understand what happens with the solution if $\varepsilon \to 0$, i.e., if the this external action becomes instantaneous (think about a hit by a hammer) in the limit. Note that mathematically it is meaningless to pass to the limit $\varepsilon \to 0$ in f, since this limit, understood in the classical sense, does not exist. And yet physically we would like to understand what happens with our system if the external force acts at a certain time moment a.

The problem of course can be solved using the technique from the previous section. Taking the Laplace transform, I find that

$$Y(s) = \frac{I}{2\varepsilon}\frac{1}{s(s+1)}\left(e^{-(a-\varepsilon)s} - e^{-(a-\varepsilon)s}\right),$$

which, taking into account that $1/(s(s+1)) = 1/s - 1/(s+1)$, yields the solution

$$y(t) = \frac{I_0}{2\varepsilon}\big(u(t-(a-\varepsilon)) - u(t-(a+\varepsilon))\big) - \frac{I_0}{2\varepsilon}\big(u(t-(a-\varepsilon)e^{-t+(a-\varepsilon)} - u(t-(a+\varepsilon))e^{-t+(a+\varepsilon)})\big).$$

I can rewrite this solution as

$$y(t) = \begin{cases} 0, & t < a-\varepsilon, \\ \frac{I_0}{2\varepsilon}\big(1 - e^{-t+(a+\varepsilon)}\big), & a-\varepsilon < t < a+\varepsilon, \\ \frac{I_0}{2\varepsilon}\big(e^{-t+(a+\varepsilon)} - e^{-t+(a-\varepsilon)}\big), & t > a+\varepsilon. \end{cases}$$

Now passage to the limit $\varepsilon \to 0$ makes perfect sense. Since the middle expression above always goes from 0 to I_0 for the given t and since $(e^{\varepsilon} - e^{-\varepsilon})/(2\varepsilon) \to 1$ as $\varepsilon \to 0$, then I get in the limit

$$y(t) = \begin{cases} 0, & t < a, \\ I_0 e^{a-t}, & t > a, \end{cases}$$

or simply

$$y(t) = I_0 u(t-a)e^{a-t}.$$

Can the same solution be obtained in a different, less laborious, way? The most tedious part here, of course, is the passage to the limit. This can be done, however, almost automatically, if one realizes that for the family of functions

$$f_\varepsilon(t) = \begin{cases} \frac{1}{2\varepsilon}, & t \in (-\varepsilon, \varepsilon), \\ 0, & \text{otherwise}, \end{cases}$$

it is true that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f_\varepsilon(t) g(t)\, \mathrm{d}t = g(0)$$

for any continuous at 0 function g. I will leave it as an exercise to check the previous equality. Therefore, I can (and will) pass to the limit already at the time of applying the Laplace transform, and will not wait for the inversion. In other words, I have that

$$\lim_{\varepsilon \to 0} \mathscr{L}\left\{f_\varepsilon(t)\right\} = 1$$

since $e^0 = 1$.

In the following I also want to use a nicer notation for my "instantaneous action" and I will use the standard $\delta(t)$ to denote the action of the sequence $f_\varepsilon(t)$ and then passing to limit. Such an object is called *delta-function*, which I *defined* through the relation

$$\mathscr{L}\left\{\delta(t)\right\} = \int_0^\infty \delta(t) e^{-st}\, \mathrm{d}t = 1,$$

and hence by the shifting property

$$\mathscr{L}\left\{\delta(t-a)\right\} = e^{-as}$$

for the shifted delta function.

Again, using the introduced notation I have that

$$\int_{-\infty}^{\infty} \delta(t-x) g(t)\, \mathrm{d}t = g(x).$$

If someone is confused by all the manipulation that I performed, I would like to emphasize that all I am doing is "hiding under a rug" the passage to the limit that I explicitly did in my example.

Now, instead, I can do the following: Consider the IVP

$$y' + y = I_0 \delta(t-a), \quad y(0) = 0,$$

and take the Laplace transform:

$$Y(s) = I_0 \frac{e^{-as}}{s+1}.$$

Taking the inverse Laplace transform yields

$$y(t) = I_0 u(t-a) e^{a-t},$$

exactly as before, but algebraically much simpler since no explicit passage to the limit is required.