17 Laplace transform. Solving linear ODE with piecewise continuous right hand sides

In this lecture I will show how to apply the Laplace transform to the ODE \( Ly = f \) with piecewise continuous \( f \).

**Definition 1.** A function \( f(t) \) is piecewise continuous on the interval \( I = [a, b] \) if it is defined and continuous on this interval except, probably, a finite number of points, \( t_1, t_2, \ldots, t_k \), at each of which the left and right limits of this function exist (i.e., all the discontinuities of the first type, or “jump” discontinuities).

**Example 2.** Consider the function

\[
    f(t) = \begin{cases} 
        -1, & t \leq 4, \\
        1, & t > 4,
    \end{cases}
\]

the graph of which is given in the figure.

![Figure 1: Piecewise continuous function \( f(t) \) from Example 2](image)

You can see that there is one point of discontinuity, namely \( t = 4 \), at which \( f(t) \) has the discontinuity of the first type, or jump discontinuity. We have

\[
    \lim_{t \to 4^-} f(t) = -1, \quad \lim_{t \to 4^+} f(t) = 1,
\]

they both exist and are not equal to each other. The value of the jump is 2.

What is the Laplace transform of \( f(t) \)? We can find it using the definition:

\[
    \mathcal{L}\{f\} = \int_{0}^{\infty} f(t)e^{-st} \, dt
\]

\[
    = \int_{0}^{4} (-1)e^{-st} \, dt + \int_{4}^{\infty} 1 \cdot e^{-st} \, dt
\]

\[
    = \frac{1}{s}e^{-st}\bigg|_{0}^{4} - \frac{1}{s}e^{-st}\bigg|_{4}^{\infty}
\]

\[
    = \frac{1}{s}e^{-4s} - \frac{1}{s} + \frac{1}{s}e^{-4s}
\]

\[
    = \frac{1}{s}(2e^{-4s} - 1).
\]
**Example 3** (Heaviside function). The most important example for us will be the *Heaviside function*, defined as

\[ u(t) = \begin{cases} 
0, & t < 0, \\
1, & t > 0. 
\end{cases} \]

Note that

\[ \mathcal{L} \{u(t)\} = \frac{1}{s}, \]

the same as for \( \mathcal{L} \{1\} \), because we are only interested in functions on \([0, \infty)\). This yields that for any function \( f(t) \) defined on \([0, \infty)\) it is true that

\[ \mathcal{L} \{f(t)\} = \mathcal{L} \{u(t)f(t)\}. \]

Additionally to the Heaviside function, we consider shifted Heaviside function \( u(t - a) \) for a non-negative \( a \geq 0 \) (see the figure).

![Figure 2: Heaviside function \( u(t) \) (left) and shifted Heaviside function \( u(t - a) \) (right)](image)

It is a simple exercise to check, using the definition, that

\[ \mathcal{L} \{u(t - a)\} = e^{-as}. \]

Now I can state the final property of the Laplace transform that we will use (there are many more actually):

6° *Time shifting.* Let \( \mathcal{L} \{f(t)\} = F(s) \), then

\[ \mathcal{L} \{u(t - a)f(t - a)\} = e^{-as}F(s). \]

To prove it, consider

\[ \mathcal{L} \{u(t - a)f(t - a)\} = \int_0^\infty u(t - a)f(t - a)e^{-st} \, dt = \int_a^\infty f(t - a)e^{-st} \, dt, \]

because \( u(t - a) = 0 \) when \( t < a \). Now make the substitution \( \xi = t - a \), then \( dt = d\xi \) and if \( t = a \) then \( \xi = 0 \). We obtain

\[ \int_0^\infty f(\xi)e^{-s(\xi+a)} \, d\xi = e^{-sa} \mathcal{L} \{f\}. \]
Let us find the Laplace transform of the function in Example 2.

**Example 4** (Continue Example 2). Note that using the shifted Heaviside function we can construct for any \( a < b \) the function

\[
    u(t - a) - u(t - b),
\]

such that this function is equal to 1 when \( t \in (a, b) \) and zero otherwise (think this out!) This means

![Figure 3: u(t - a) - u(t - b) for b > a](image)

that for any \( f(t) \) the function \( f(t)(u(t - a) - u(t - b)) \) equals \( f(t) \), when \( t \in (a, b) \), and 0 otherwise.

Now consider again

\[
    f(t) = \begin{cases} 
    -1, & t \leq 4, \\
    1, & t > 4. 
    \end{cases}
\]

This, using the previous, can be represented as

\[
    f(t) = -1(u(t) - u(t - 4)) + 1 \cdot u(t - 4) = -u(t) + 2u(t - 4).
\]

Now, using Property 6\(^\circ\),

\[
    \mathcal{L} \{f\} = \mathcal{L} \{-u(t) + 2u(t - 4)\} = -\frac{1}{s} + 2\frac{e^{-4s}}{s},
\]

exactly as we already found in Example 2.

**Example 5.** Find the Laplace transform for

\[
    f(t) = \begin{cases} 
    0, & t < 1, \\
    t^2, & 1 < t < 2, \\
    0, & t > 2. 
    \end{cases}
\]

Again, using the properties of the Heaviside function, we can write

\[
    f(t) = t^2(u(t - 1) - u(t - 2)) = t^2u(t - 1) - t^2u(t - 2).
\]

However, we cannot apply Property 6\(^\circ\) directly to this expression since we need the expressions of the form \( f(t - a)u(t - a) \). To deal with it, consider

\[
    t^2 = (t - 1 + 1)^2 = (t - 1)^2 + 2(t - 1) + 1.
\]
Similarly,
\[ t^2 = (t - 2 + 2)^2 = (t - 2)^2 + 4(t - 2) + 4. \]

Hence we have
\[ f(t) = ((t - 1)^2 + 2(t - 1) + 1)u(t - 1) - ((t - 2)^2 + 4(t - 1) + 4)u(t - 2), \]
which implies that
\[ \mathcal{L} \{ f \} = \left( \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s} \right)e^{-s} + \left( \frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right)e^{-2s}. \]

Now we are ready to start solving ODE with piecewise continuos right hand side.

**Example 6.** Solve
\[ y'' + 2y' + y = f(t), \quad y(0) = 1, \; y'(0) = 0, \]
where
\[ f(t) = \begin{cases} 1, & t < 2, \\ 0, & t > 2. \end{cases} \]

We have
\[ f(t) = u(t) - u(t - 2). \]

Applying the Laplace transform to both sides, we have
\[ (s^2 + 2s + 1)Y - s - 2 = \frac{1}{s}(1 - e^{-2s}), \]
or, after some rearrangement,
\[ Y = \frac{1}{s} - \frac{1}{s(s + 1)}e^{-2s}. \]

Since, using the partial fractions,
\[ \frac{1}{s(s + 1)^2} = \frac{1}{s} - \frac{1}{s + 1} - \frac{1}{(s + 1)^2}, \]
then we find
\[ y(t) = \mathcal{L}^{-1} \{ Y \} = 1 - (1 - e^{-(t-2)} - e^{-(t-2)}(t - 2))u(t - 2), \]
using Property 6°.

Note that we can rewrite our solution as
\[ y(t) = \begin{cases} 1, & t < 2, \\ e^{2-t}(t - 1), & t > 2. \end{cases} \]

The graph of the solution is given in the figure below.

**Example 7.** Solve
\[ y'' + y' + y = f(t), \quad y(0) = 1, \; y'(0) = 0, \]
where \( f(t) \) as in the figure below.
We find that
\[ f(t) = u(t - \pi) - u(t - 2\pi). \]
Applying the Laplace transform to the ODE, we find
\[ s^2 Y - s + sY - 1 + Y = \frac{1}{s}(e^{-\pi s} - e^{-2\pi s}), \]
from where
\[ Y = \frac{s + 1}{s^2 + s + 1} + \frac{1}{s(s^2 + s + 1)}(e^{-\pi s} - e^{-2\pi s}). \]
Note that using partial fraction decomposition, we have
\[ \frac{1}{s(s^2 + s + 1)} = \frac{1}{s} - \frac{s + 1}{s^2 + s + 1}, \]
therefore, finally we have
\[ Y(s) = \frac{1}{s}(e^{-\pi s} - e^{-2\pi s}) + \frac{s + 1}{s^2 + s + 1}(1 - e^{-\pi s} + e^{-2\pi s}). \]
Now our task is to find the inverse Laplace transform for \( Y \). We have \( \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} = u(t) \), hence
\[ \mathcal{L}^{-1}\left\{ \frac{1}{s}(e^{-\pi s} - e^{-2\pi s}) \right\} = u(t - \pi) - u(t - 2\pi). \]
For the second term in \( Y(s) \) consider first

\[
\frac{s+1}{s^2+s+1} = \frac{s+1/2+1/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2} = \frac{s+1/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2} + \frac{1}{\sqrt{3}} \frac{\sqrt{3}/2}{(s+1/2)^2+\left(\sqrt{3}/2\right)^2},
\]

therefore

\[
\mathcal{L}^{-1}\left\{ \frac{s+1}{s^2+s+1} \right\} = e^{-t/2} \left( \cos \frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}t}{2} \right).
\]

Finally, using Property 6, we have

\[
y(t) = \mathcal{L}^{-1}\{Y\} = \mathcal{L}^{-1}\left\{ \frac{1}{s} (e^{-\pi s} - e^{-2\pi s}) + \frac{s+1}{s^2+s+1} (1 - e^{-\pi s} + e^{-2\pi s}) \right\}
\]

\[
= u(t-\pi) - u(t-2\pi) + e^{-t/2} \left( \cos \frac{\sqrt{3}t}{2} + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}t}{2} \right) u(t) -
\]

\[
- e^{-(t-\pi)/2} \left( \cos \frac{\sqrt{3}(t-\pi)}{2} + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}(t-\pi)}{2} \right) u(t-\pi) +
\]

\[
+ e^{-(t-2\pi)/2} \left( \cos \frac{\sqrt{3}(t-2\pi)}{2} + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}(t-2\pi)}{2} \right) u(t-2\pi).
\]

Below is the graph of the solution.

![Graph of the Solution](image)

Figure 6: Solution \( y(t) \) depending on time \( t \) in Example 7

**Example 8.** Solve

\[
y'' + y = f(t) = \begin{cases} 
\cos t, & t < \pi/2, \\
0, & t > \pi/2,
\end{cases} \quad y(0) = 3, \ y'(0) = -1.
\]

We can rewrite the right hand side as

\[
f(t) = \cos (u(t) - u(t - \pi/2)).
\]

Note that to invoke Property 6 we need the function with the same argument as the argument of the Heaviside function.

\[
f(t) = u(t) \cos t - \cos(t - \pi/2 + \pi/2)u(t - \pi/2) = u(t) \cos t + \sin(t - \pi/2)u(t - \pi/2),
\]
where I used 
\[ \cos(a + b) = \cos a \cos b - \sin a \sin b. \]

Therefore, after applying the Laplace transform, I will find
\[
(s^2 + 1)Y = 3s - 1 + \frac{s}{s^2 + 1} + \frac{1}{s^2 + 1}e^{-\pi t/2},
\]
or,
\[
Y(s) = \frac{3s}{s^2 + 1} - \frac{s}{(s^2 + 1)^2} + \frac{s}{(s^2 + 1)^2} + \frac{1}{(s^2 + 1)^2}e^{-\pi t/2}.
\]
To find the inverse Laplace transform, we will need
\[
\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 1)^2}\right\}, \quad \mathcal{L}^{-1}\left\{\frac{1}{(s^2 + 1)^2}\right\}.
\]
For this, using the property of the differentiation of the frequency, note that
\[
\mathcal{L}\{t \sin t\} = \frac{2s}{(s^2 + 1)^2}, \quad \mathcal{L}\{t \cos t\} = \frac{s^2 - 1}{(s^2 + 1)}.
\]
Hence
\[
\mathcal{L}^{-1}\left\{\frac{s}{(s^2 + 1)^2}\right\} = \frac{1}{2}t \sin t.
\]
For the second one
\[
\frac{1}{(s^2 + 1)^2} = \frac{1}{2}\left(\frac{1}{s^2 + 1} - \frac{s^2 - 1}{(s^2 + 1)^2}\right),
\]
hence
\[
\mathcal{L}^{-1}\left\{\frac{1}{(s^2 + 1)^2}\right\} = \frac{1}{2}(\sin t - t \cos t).
\]
Finally we find
\[
y(t) = \mathcal{L}^{-1}\{Y\} = 3 \cos t - \sin t + \frac{1}{2}t \sin t + \frac{1}{2}(\sin(t - \pi/2) - (t - \pi/2) \cos(t - \pi/2)) u(t - \pi/2).
\]
By noting that \(\sin(t - \pi/2) = -\cos t\) and \(\cos(t - \pi/2) = \sin t\), we can simplify this expression as
\[
y(t) = \begin{cases} 
3 \cos t - \sin t + \frac{1}{2} t \sin t, & t < 2, \\
\frac{5}{2} \cos t + \frac{\pi}{4} \cos t, & t > 2.
\end{cases}
\]

**Instead of conclusion**

I said that the examples in this lecture are the main reason we need the Laplace transform. This is indeed true, but only partially. There are two more cases when Laplace transform becomes indispensable theoretical and computational tool: First, when \(f(t)\) is an arbitrary periodic function different from simple sine and cosine functions. Second, when we consider solutions of the equations with impulsive right hand side (think of an instantaneous impact). Unfortunately, due to time limitation, we do not cover these exciting topics, and I refer you to the textbook.