21 Solving linear systems of ODE with constant coefficients. Part II. Complex eigenvalues

Consider

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y}, \quad \boldsymbol{A} = [a_{ij}]_{n\times n} \in \mathbf{M}_n(\mathbf{R}) \tag{1}$$

together with the initial condition

$$\boldsymbol{y}(t_0) = \boldsymbol{y}_0 \in \mathbf{R}^n. \tag{2}$$

From the previous lecture we know that the general solution to (1) has the form

$$\boldsymbol{y}(t) = C_1\boldsymbol{v}_1 e^{\lambda_1 t} + \ldots + C_n\boldsymbol{v}_n e^{\lambda_n t},$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are the corresponding eigenvectors, provided all the eigenvalues are *distinct*. However, even in this simple case we can have complex eigenvalues with complex eigenvectors. The goal here is to show that we still can choose a basis for the vector space of solutions such that all the vectors in it are real.

Proposition 1. *If $\boldsymbol{y}(t)$ is a solution to* (1) *then* $\operatorname{Re}\boldsymbol{y}(t)$ *and* $\operatorname{Im}\boldsymbol{y}(t)$ *are also solutions to* (1)*.*

Proof. Let $\boldsymbol{y}(t) = \boldsymbol{u}_1(t) + \mathrm{i}\boldsymbol{u}_2(t)$, where $\boldsymbol{u}_1(t) = \operatorname{Re}\boldsymbol{y}(t)$ and $\boldsymbol{u}_2(t) = \operatorname{Im}\boldsymbol{y}(t)$. Plug this into (1) and use the linearity to find

$$(\dot{\boldsymbol{u}}_1 - \boldsymbol{A}\boldsymbol{u}_1) + \mathrm{i}(\dot{\boldsymbol{u}}_2 - \boldsymbol{A}\boldsymbol{u}_2) = 0,$$

which implies that

$$\dot{\boldsymbol{u}}_1 = \boldsymbol{A}\boldsymbol{u}_1, \quad \dot{\boldsymbol{u}}_2 = \boldsymbol{A}\boldsymbol{u}_2.$$

■

Note that both $\boldsymbol{u}_1(t)$ and $\boldsymbol{u}_2(t)$ are real valued solutions, therefore, instead of one complex-valued solution $\boldsymbol{y}(t)$ we have two real valued solutions. To finalize the argument we need to show that the new basis, where two complex valued solutions corresponding to λ and $\overline{\lambda}$ are substituted with $\boldsymbol{u}_1(t)$ and $\boldsymbol{u}_2(t)$ is still a linearly independent set. To see this first note that if λ is a complex eigenvalue with eigenvector $\boldsymbol{v}$, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{\boldsymbol{v}}$. This follows from

$$\overline{\boldsymbol{A}\boldsymbol{v}} = \overline{\lambda\boldsymbol{v}} \implies \boldsymbol{A}\overline{\boldsymbol{v}} = \overline{\lambda}\overline{\boldsymbol{v}}.$$

Now we need to show that if $\{\boldsymbol{v}, \overline{\boldsymbol{v}}, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n\}$ is a linearly independent set, then $\{\operatorname{Re}\boldsymbol{v}, \operatorname{Im}\boldsymbol{v}, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_n\}$ is also a linearly independent set (left as an exercise).

Finally, let us see in details how our new real-valued solution looks like in coordinates. We have $\boldsymbol{y}(t) = \boldsymbol{v}e^{\lambda t}$ is a complex valued solution, here $\lambda = \alpha + \mathrm{i}\beta$, $\boldsymbol{v} = \boldsymbol{v}_1 + \mathrm{i}\boldsymbol{v}_2$. We find

$$\begin{aligned}
\boldsymbol{y}(t) &= (\boldsymbol{v}_1 + \mathrm{i}\boldsymbol{v}_2)e^{(\alpha+\beta\mathrm{i})t} \\
&= (\boldsymbol{v}_1 + \mathrm{i}\boldsymbol{v}_2)e^{\alpha t}(\cos\beta t + \mathrm{i}\sin\beta t) \\
&= e^{\alpha t}(\boldsymbol{v}_1\cos\beta t - \boldsymbol{v}_2\sin\beta t) + \mathrm{i}e^{\alpha t}(\boldsymbol{v}_1\sin\beta t + \boldsymbol{v}_2\cos\beta t).
\end{aligned}$$

Therefore, instead of two complex-valued solutions $\boldsymbol{v}e^{\lambda t}$ and $\overline{\boldsymbol{v}}e^{\overline{\lambda}t}$ we have two real-valued solutions

$$\begin{aligned}
\boldsymbol{u}_1(t) &= e^{\alpha t}(\boldsymbol{v}_1\cos\beta t - \boldsymbol{v}_2\sin\beta t), \\
\boldsymbol{u}_2(t) &= e^{\alpha t}(\boldsymbol{v}_1\sin\beta t + \boldsymbol{v}_2\cos\beta t).
\end{aligned}$$

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Example 2. Solve

$$\dot{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{y},$$

where

$$\boldsymbol{A} = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}.$$

First we find the characteristic polynomial

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \lambda^2 - 2\lambda + 5,$$

which means that we have two complex eigenvalues

$$\lambda_{1,2} = 1 \pm 2\mathrm{i}.$$

An eigenvector corresponding to λ_1 can be found as a solution to

$$\begin{bmatrix} 3 - 1 - 2\mathrm{i} & -2 \\ 4 & -1 - 1 - 2\mathrm{i} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or, in coordinates,

$$\begin{aligned} (1 - \mathrm{i})v_1 - v_2 &= 0, \\ 2v_1 - (1 + \mathrm{i})v_2 &= 0. \end{aligned}$$

This is not quite obvious from the first view that the two equations are equivalent, but they are (multiply the first by $1 + \mathrm{i}$), hence we have

$$v_2 = (1 - \mathrm{i})v_1,$$

if I take v_1 as a free variable. Therefore, any solution to our system is given by the vector $\bigl(v_1, (1-\mathrm{i})v_1\bigr)^\top$, and for my eigenvector I can choose, e.g., $v_1 = 1$, therefore,

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 - \mathrm{i} \end{bmatrix}.$$

I do not need to look for an eigenvector corresponding to λ_2, because, as it was shown above, $\boldsymbol{v}_2 = \overline{\boldsymbol{v}}_1$. By making these calculations I proved that my system has two linearly independent solutions

$$\boldsymbol{y}_1(t) = \begin{bmatrix} 1 \\ 1 - \mathrm{i} \end{bmatrix} e^{(1+2\mathrm{i})t}, \quad \boldsymbol{y}_2(t) = \begin{bmatrix} 1 \\ 1 + \mathrm{i} \end{bmatrix} e^{(1-2\mathrm{i})t},$$

and the general solution is

$$\boldsymbol{y}(t) = C_1 \boldsymbol{y}_1(t) + C_2 \boldsymbol{y}_2(t),$$

where C_1, C_2 are arbitrary constants. However, this solution is complex-valued. To find a real valued solution I will follow the algorithm from the first part of the lecture and take $\operatorname{Re} \boldsymbol{y}_1(t)$ and $\operatorname{Im} \boldsymbol{y}_2(t)$

as new two linearly independent real-valued solutions.

$$
\begin{aligned}
\boldsymbol{y}_1(t) &= \begin{bmatrix} 1 \\ 1-\mathrm{i} \end{bmatrix} e^{(1+2\mathrm{i})t} = \begin{bmatrix} 1 \\ 1-\mathrm{i} \end{bmatrix} e^{t}(\cos 2t + \mathrm{i}\sin 2t) \\
&= e^{t} \begin{bmatrix} \cos 2t + \mathrm{i}\sin 2t \\ \cos 2t + \sin 2t + \mathrm{i}(-\cos 2t + \sin 2t) \end{bmatrix} \\
&= \underbrace{e^{t} \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix}}_{\boldsymbol{u}_1(t)} + \mathrm{i}\, \underbrace{e^{t} \begin{bmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{bmatrix}}_{\boldsymbol{u}_2(t)} .
\end{aligned}
$$

Please do not include i into $\boldsymbol{u}_2(t)$!

Therefore, the general solution to the problem has the form

$$
\boldsymbol{y}(t) = C_1 e^{t} \begin{bmatrix} \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + C_2 e^{t} \begin{bmatrix} \sin 2t \\ -\cos 2t + \sin 2t \end{bmatrix},
$$

or

$$
\begin{aligned}
y_1(t) &= e^{t}(C_1 \cos 2t + C_2 \sin 2t), \\
y_2(t) &= e^{t}\big((C_1 - C_2)\cos 2t + (C_1 + C_2)\sin 2t\big),
\end{aligned}
$$

in coordinates.

Example 3. Solve the IVP

$$
\dot{\boldsymbol{y}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
$$

First we find the characteristic polynomial

$$
P_3(\lambda) = (1-\lambda)(\lambda^2 - 2\lambda + 2),
$$

which means that we have

$$
\lambda_1 = 1, \quad \lambda_{2,3} = 1 \pm \mathrm{i}.
$$

We can choose eigenvector $\boldsymbol{v}_1 = (1, 0, 0)^\top$ corresponding to λ_1. Similarly, we find that eigenvector $\boldsymbol{v}_2 = (0, \mathrm{i}, 1)^\top$ can be taken for λ_2. Since we are looking for a real-valued solution, we do not care about λ_3. We find that

$$
\boldsymbol{y}_2(t) = \begin{bmatrix} 0 \\ \mathrm{i} \\ 1 \end{bmatrix} e^{(1+2\mathrm{i})t} = e^{t} \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} + \mathrm{i}e^{t} \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix},
$$

therefore, the general solution to our problem can be written as

$$
\boldsymbol{y}(t) = C_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 0 \\ -\sin t \\ \cos t \end{bmatrix} e^{t} + C_3 \begin{bmatrix} 0 \\ \cos t \\ \sin t \end{bmatrix} e^{t} = e^{t} \begin{bmatrix} C_1 \\ -C_2 \sin t + C_3 \cos t \\ C_2 \cos t + C_3 \sin t \end{bmatrix}.
$$

Now we use the initial conditions and find that $C_1 = C_2 = C_3 = 1$. Thence, the final answer is

$$
\boldsymbol{y}(t) = e^{t} \begin{bmatrix} 1 \\ \cos t - \sin t \\ \cos t + \sin t \end{bmatrix}.
$$