23 Phase plane analysis for linear systems

23.1 Basic introduction

In this lecture we are going to talk about systems of two linear ODE of the first order in the form

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y, \\ \dot{y} &= a_{21}x + a_{22}y. \end{aligned} \tag{1}$$

Here I chose the notations $\big(x(t), y(t)\big)^\top$ for the unknown functions, and the independent variable is, as before, denoted by t, meaning "time." Note that *planar* system (1) is a particular case of the general nonlinear system

$$\begin{aligned} \dot{x} &= f(x, y), \\ \dot{y} &= g(x, y), \end{aligned} \tag{2}$$

where $f, g \in C^1(U; \mathbf{R})$, $U \subseteq \mathbf{R}^2$, are given functions of two variables. The common point between (1) and (2) is that their right-hand sides do not depend explicitly on t. They, of course, depend on t through the variables $x(t)$ and $y(t)$, but t itself is absent. Such systems are called *autonomous* (cf. autonomous first order ordinary differential equations).

Assume that system (2) (or (1)) has a solution $x = x(t; x_0), y = y(t; y_0)$, where $(x_0, y_0)^\top$ are the initial conditions. We can consider this solution as a parametrically defined curve: for each time moment t we have two numbers $(x, y) \in \mathbf{R}^2$, which can be represented as a point on the plane xy. If we change t, the point position will change, but since $x(t; x_0)$ and $y(t; x_0)$ are differentiable, then the change will be small, and we actually obtain a smooth curve. Moreover, by increasing or decreasing t we move on xy plane along this curve. Such curve with the direction of time increase on it is called an *orbit*, or a *trajectory* of system (2) (or system (1)). Our task here is to analyze the structure of orbits of system (1) on the plane xy, which is called the *phase plane*. Since some of the properties of the orbits of (1) hold in the general case (2), I will start with the more general system.

- If $(\hat{x}, \hat{y})$ are such that $f(\hat{x}, \hat{y}) = 0$ and $g(\hat{x}, \hat{y}) = 0$, then $x = \hat{x}$, $y = \hat{y}$ is a solution to (2), and the corresponding orbit is simply a point on the phase plane with coordinates $(\hat{x}, \hat{y})$. For the linear system (1) point $(\hat{x}, \hat{y})$ has to be a solution to

  $$\begin{aligned} 0 &= a_{11}x + a_{12}y, \\ 0 &= a_{22}x + a_{22}y, \end{aligned}$$

  i.e., a solution to a homogeneous system of two linear algebraic equations with the matrix

  $$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

  This system always has solution $(\hat{x}, \hat{y}) = (0, 0)$. This solution is unique if and only if $\det \boldsymbol{A} \neq 0$. If $\det \boldsymbol{A} = 0$ then we have infinitely many solutions. In order not to complicate the following discussion I will assume that $\det \boldsymbol{A} \neq 0$.

  Such points $(\hat{x}, \hat{y})$ are called *equilibrium points*, or *rest points*, or *stationary points*, or *critical points* of system (2). Hence the assumption for (1) that $\det \boldsymbol{A} \neq 0$ is equivalent to saying that system (1) has only one equilibrium at the origin.

- If $x = x(t)$, $y = y(t)$ is a solution to (2), then $\tilde{x} = x(t + c)$, $\tilde{y} = y(t + c)$ is a also a solution to (2) for any constant c.

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Proof. Note that if $(x(t), y(t))$ is a solution, then for the first equation in (2) it means that

$$\frac{dx}{dt}(t) = f\big(x(t), y(t)\big),$$

and since this is true for any t, it is true for $t + c$:

$$\frac{dx}{dt}(t + c) = f\big(x(t + c), y(t + c)\big),$$

which, due to the chain rule, can be rewritten as

$$\frac{dx}{d(t + c)}(t + c) = f\big(x(t + c), y(t + c)\big),$$

or, using new variable $\tau = t + c$,

$$\frac{dx}{d\tau}(\tau) = f\big(x(\tau), y(\tau)\big).$$

But since $x(\tau) = \tilde{x}(t)$, $y(\tau) = \tilde{y}(t)$, this exactly means that $\tilde{x}(t) = x(t + c), \tilde{y}(t) = y(t + c)$ is a solution to (2). ■

This simple and very important fact means that if $(x(t), y(t))$ is the solution to (2) with the initial condition $x(t_0) = x_0, y(t_0) = y_0$, then $(x(t + t_0), y(t + t_0))$ is the solution to system (2) with the initial condition $x(0) = x_0, y(0) = y_0$. From the geometric point of view it means that we can use different parametrizations to define the same curve on the phase plane.

For the linear system we can prove this fact explicitly. Recall that any solution to (1) is given by $e^{\boldsymbol{A}t}\boldsymbol{v}$ for some vector $\boldsymbol{v} \in \mathbf{R}^2$. Now consider $e^{\boldsymbol{A}(t+c)}\boldsymbol{v} = e^{\boldsymbol{A}t}\boldsymbol{u}$, where $\boldsymbol{u} = e^{\boldsymbol{A}c}\boldsymbol{v}$, which is clearly a solution to the linear system.

- *Orbits do not intersect.* Suppose contrary: there are two orbits $\big(x(t), y(t)\big)$ and $\big(\tilde{x}(t), \tilde{y}(t)\big)$ that pass through the same point (x_0, y_0) for different time moments t_1 and t_2: i.e.,

$$(x_0, y_0) = \big(x(t_1), y(t_1)\big) = \big(\tilde{x}(t_2), \tilde{y}(t_2)\big).$$

Since $\big(\tilde{x}(t), \tilde{y}(t)\big)$ is a solution, then, according to the previous property, $\big(\tilde{x}(t + (t_2 - t_1)), \tilde{y}(t + (t_2 - t_1))\big)$ is also a solution, which corresponds to the same orbit, but with a different time parametrization. On the other hand, the value of this solution at the point t_1 coincides with the value of $\big(x(t), y(t)\big)$ at the same point, which according to the uniqueness and existence theorem means that $\big(\tilde{x}(t+(t_2-t_1)), \tilde{y}(t+(t_2-t_1))\big)$ and $\big(x(t), y(t)\big)$ coincide, which yields that the existence of a common point for two orbits implies that these orbits coincide, hence no intersections.

This property, as well as the previous one, is not true for non-autonomous systems.

We obtained that the phase plane consists of orbits, which cannot intersect. It is impossible to depict all the orbits, but it is usually enough to draw only a few to get a general idea of the behavior of the solutions of system (2). In particular, it is always advisable first to plot equilibria. Several key orbits on the phase plane representing the general picture are called the *phase portrait*. It is usually quite difficult to draw the phase portrait of the general nonlinear system (2). For system (1), especially assuming that $\det \boldsymbol{A} \neq 0$, this problem can be solved completely, as I will show next.

23.2 Phase portraits of linear system (1)

There are only a few types of the phase portraits possible for system (1). Let me start with a very simple one:

$$\dot{x} = \lambda_1 x,$$
$$\dot{y} = \lambda_2 y.$$

This means that the matrix of the system has the diagonal form

$$\boldsymbol{A} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

i.e., it has real eigenvalues λ_1, λ_2 with the eigenvectors $(1,0)^\top$ and $(0,1)^\top$ respectively. The equations are decoupled and the general solution to this system is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda_1 t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\lambda_2 t}.$$

Note that this is a fancy way to write that

$$x(t) = C_1 e^{\lambda_1 t}, \quad y(t) = C_2 e^{\lambda_2 t}.$$

Now to figure out the behavior of the orbits (i.e., their qualitative form and the directions on them), we can argue as follows: Consider initially only special orbits corresponding to $C_1 = 0, \pm 1$ and $C_2 = 0, \pm 1$. First of all there is always the equilibrium $(0,0)$, which means that the corresponding orbit is a point at the origin. Next, take $C_1 = +1$ and $C_2 = 0$, which implies that $x(t) = e^{\lambda_1 t}$ and $y(t) = 0$. To get further information we need to specify the signs of λ_1 and λ_2.

Case $\lambda_1 > 0 > \lambda_2$. I repeat that I consider the case $C_1 = 1$ and $C_2 = 0$. This corresponds to the line, whose direction is given by the first eigenvector $\boldsymbol{v}_1 = (1,0)^\top$. If $t \to \infty$ then $x(t) \to \infty$ for $\lambda_1 > 0$, hence the orbit constitutes the half line $(x > 0, y = 0)$ with the direction from the origin to infinity (since also $x(t) \to 0$ if $t \to -\infty$). Similarly, taking $C_1 = -1, C_2 = 0$ we will find that $x(t) \to -\infty$ if $t \to \infty$ and $x(t) \to 0$ if $t \to -\infty$ on the half line $(x < 0, y = 0)$. Hence we fully described the orbit structure on the line corresponding to the direction of $\boldsymbol{v}_1$ (see the figure below): There are three orbits there, two half-lines separated by the equilibrium at the origin, and on both half lines the direction is *from* the origin (the corresponding eigenvalue is positive). Now take $C_1 = 0, C_2 = \pm 1$. In this case we find ourselves on the direction corresponding to $\boldsymbol{v}_2 = (0,1)^\top$, i.e., on y-axis. We again have three orbits there, but the direction is reversed, we approach the origin along these orbits because λ_2 is negative (see the figure below).

What about the case $C_1 \neq 0$ and $C_2 \neq 0$? For both $t \to \infty$ and $t \to -\infty$ one of the coordinate will approach infinity. Moreover, our intuition tells us that close orbits should behave similarly, therefore we do not have much choice as to obtain the orbit structure shown in the figure above. An equilibrium point for which we have two real eigenvalues, one is negative and one is positive, is called *saddle*.

I actually was quite vague about why the orbits not on the axes have this particular shape. Here is a proof. We have, again, that

$$x(t) = C_1 e^{\lambda_1 t}, \quad y(t) = C_2 e^{\lambda_2 t},$$

or

$$\frac{x}{C_1} = e^{\lambda_1 t}, \quad \frac{y}{C_2} = e^{\lambda_2 t}.$$

Raise the first equality to the power λ_2 and the second equality to the power λ_1. We find, by eliminating t, that

$$\frac{x^{\lambda_2}}{C_1^{\lambda_2}} = \frac{y^{\lambda_1}}{C_2^{\lambda_1}} \implies y = A x^{\frac{\lambda_2}{\lambda_1}},$$

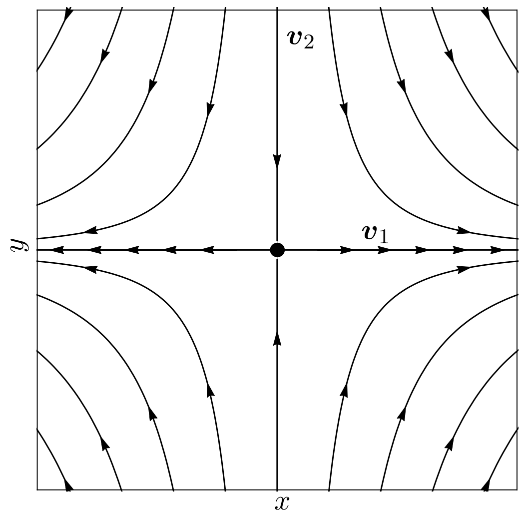

Figure 1: Saddle. The case $\lambda_1 > 0 > \lambda_2$. The first eigenvector corresponds to x-axis, and the second one corresponds to y-axis

where A is a new constant depending on C_1 and C_2. Since the eigenvalues have opposite signs, we find that orbits corresponds to "hyperbolas"

$$y = Ax^{-\gamma}, \quad \gamma > 0,$$

which we can see in the figure. By eliminating t we lost the information on the direction along the orbits. But since we already know the directions along the axes, we can restore it for our curves by continuity.

Q: Can you plot a phase portrait of the system with the diagonal matrix $\boldsymbol{A} = \operatorname{diag}(\lambda_1, \lambda_2)$ such that $\lambda_1 < 0 < \lambda_2$? What will be the difference with respect to the figure above?

Case $\lambda_2 > \lambda_1 > 0$. Formally, we have exactly the same general solution

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{\lambda_1 t} + C_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\lambda_2 t},$$

but note that opposite to the previous case here for any choice of C_1 and C_2 $(x, y) \to (0, 0)$ if $t \to -\infty$, hence geometrically all the orbits represent curves coming out from the origin and approaching infinity as $t \to \infty$. This is true in particular for the *characteristic directions* corresponding to the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (see the figure below, left panel).

The subtle question is however how exactly orbits approach the origin for $t \to -\infty$. For this we recall that the equation for the curves on the plane is given by

$$y = Ax^{\frac{\lambda_2}{\lambda_1}},$$

and since we assumed that $\lambda_2 > \lambda_1$ then we have

$$y = Ax^{\gamma}, \quad \gamma > 1,$$

which corresponds to "parabolas," i.e., to the curves that are tangent to the x-axis at the origin. Another way to see why x-axis is more important in this case is to note that when $t \to -\infty$, then $e^{\lambda_2 t}$ is much much smaller than $e^{\lambda_1 t}$, hence it is the first eigenvector that plays the most important role. For $t \to \infty$ the situation is opposite, since $e^{\lambda_2 t}$ is much much bigger than $e^{\lambda_1 t}$ and the second eigenvector shows the slope of the orbits.

An equilibrium point for which we have two real eigenvalues of the same sign is called *node*.

To obtain more intuition about how exactly the orbits approach the origin consider

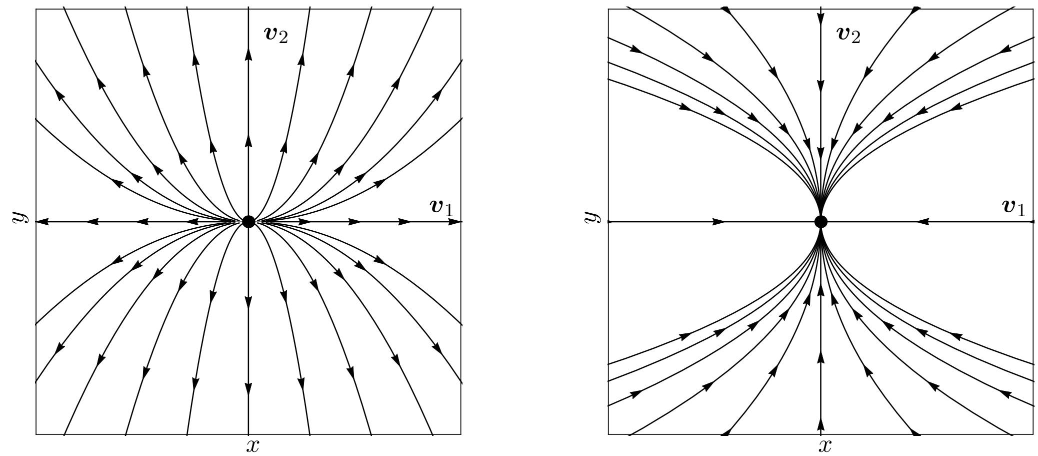

Figure 2: Left: Node. The case $\lambda_2 > \lambda_1 > 0$. The first eigenvector corresponds to x-axis, and the second one corresponds to y-axis. Right: Node. The case $\lambda_1 < \lambda_2 < 0$. The first eigenvector corresponds to x-axis, and the second one corresponds to y-axis

Case $\lambda_1 < \lambda_2 < 0$. Here the general solution shows that the direction on any orbits is from $\pm\infty$ to the origin as t goes from $-\infty$ to ∞. Now however when $t \to \infty$ $e^{\lambda_2 t}$ is much much bigger than $e^{\lambda_1 t}$ and hence the orbits behave as the second eigenvector $\boldsymbol{v}_2$, whereas for $t \to -\infty$ the first eigenvector becomes dominant, and therefore the orbits far from the origin are parallel to the first eigenvector (see the right panel of the figure above). The same conclusion can be seen from the equation for the phase curves

$$y = Ax^{\gamma}, \quad 0 < \gamma < 1.$$

It is a good exercise to consider two remaining cases $\lambda_2 < \lambda_1 < 0$ and $\lambda_1 > \lambda_2 > 0$.

Case $\lambda_1 = \lambda_2 < 0$. To make our discussion full, consider also the case of equal negative eigenvalues for the diagonal matrix $\boldsymbol{A} = \operatorname{diag}(\lambda_1, \lambda_2)$. Since the eigenvalues are negative, the direction on the orbits is *to* the origin (see the left panel in the figure below). *Q:* Do you know what to change on the figure to present a phase portrait for $\lambda_1 = \lambda_2 > 0$?

Up till now we discussed only diagonal matrix $\boldsymbol{A} = \operatorname{diag}(\lambda_1, \lambda_2)$, where λ_1 and λ_2 are real numbers. It turns out that not much changes if we consider a general matrix $\boldsymbol{A}$ with two real eigenvalues. Consider two examples.

Example 1. Consider the system

$$\begin{aligned} \dot{x} &= x + 3y, \\ \dot{y} &= x - y, \end{aligned}$$

which means that we have matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

with the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -2$ (cf. our very first example of the saddle point) and the corresponding eigenvectors $\boldsymbol{v}_1 = (3, 1)^\top$ and $\boldsymbol{v}_2 = (-1, 1)^\top$. Hence the general solution to our problem is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = C_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}.$$

The analysis of the general solution shows that on the characteristic direction corresponding to $\boldsymbol{v}_1$ the orbits point *from* the origin (because $\lambda_1 = 2 > 0$) and on the line with the direction $\boldsymbol{v}_2$ the orbits point to the origin

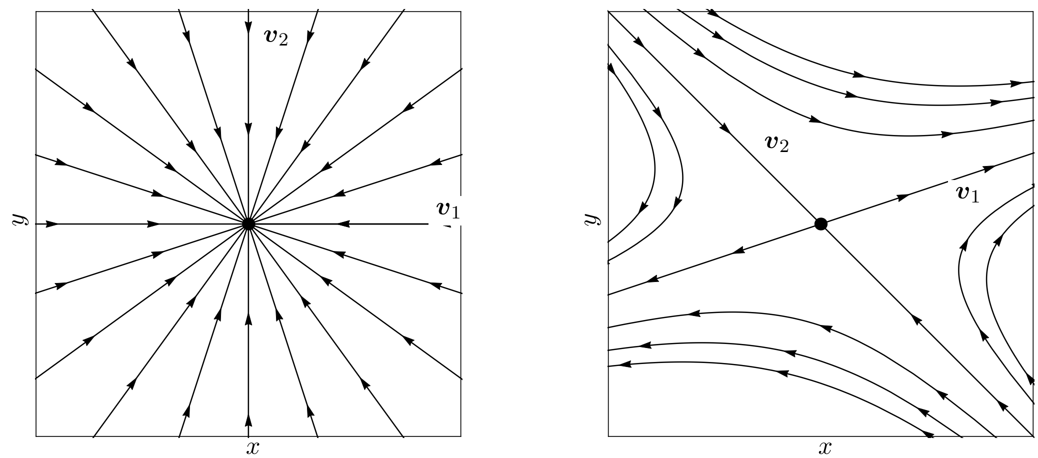

Figure 3: Left: Node. The case $\lambda_1 = \lambda_2 < 0$. The first eigenvector corresponds to x-axis, and the second one corresponds to y-axis. Right: Saddle. The case $\lambda_1 > 0 > \lambda_2$. Here the eigenvector directions do not coincides with the axes

(because $\lambda_2 = -2 < 0$). The rest of the orbits do not approach the origin either for $t \to \infty$ or for $t \to -\infty$, and can be plotted by continuity (see the right panel in the figure above).

So what is exactly different from the first example in this lecture, when we considered the diagonal matrix? Not much actually. You can see that the resulting picture in this example can be obtained from the figure of the first considered case by some stretching and rotation (without any tear!). It is true in general: If the eigenvalues of a general matrix $\boldsymbol{A}_2$ are real and have opposite sign, then the origin is a saddle. To plot it you first need to plot two lines with the directions corresponding to the eigenvectors, put arrows to the origin on the line which corresponds to the negative eigenvalue, and put the arrows from the origin on the line corresponding to the positive eigenvalue. The rest of the orbits are plotted by continuity and remembering that the orbits cannot intersect.

Example 2. Consider

$$\begin{aligned} \dot{x} &= -y, \\ \dot{y} &= 8x - 6y, \end{aligned}$$

hence our eigenvalues are $-2, -4$ with the eigenvectors $(1, 2)^\top, (1, 4)^\top$. Because both eigenvalues are negative, we know that all the orbits approach the origin when $t \to \infty$. Again, the only subtle thing here is to decide along which direction the orbits tend to the origin. Since we have that $\lambda_2 < \lambda_1 < 0$ then for $t \to \infty$ λ_1 is more important, hence the orbits will be parallel to $\boldsymbol{v}_1$ when the orbit is close to the origin. For $t \to -\infty$, far from the origin, λ_2 becomes dominant, and therefore the orbits will be parallel to $\boldsymbol{v}_2$ (see the figure, left panel).

Therefore, for any matrix with two real eigenvalues λ_1 and λ_2 with two distinct eigenvectors, we have the equilibrium point at the origin, which is called node. This point *attracts* orbits (in the sense that the direction on the orbits points to the origin) if eigenvalues are negative and *repels* them if eigenvalues are positive. To determine the direction along which the orbits approach the origin, you need to look for the eigenvector that corresponds to the eigenvalue that is closer to zero. Contrary, to see the behavior of the orbits far from the origin, we need to look for the direction of the eigenvector corresponding to the eigenvalue that is further from zero.

To conclude the discussion of the case when the matrix has real eigenvalues, recall that it is possible to have equal eigenvalues with only one eigenvector. Consider such a case with

$$\boldsymbol{A} = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

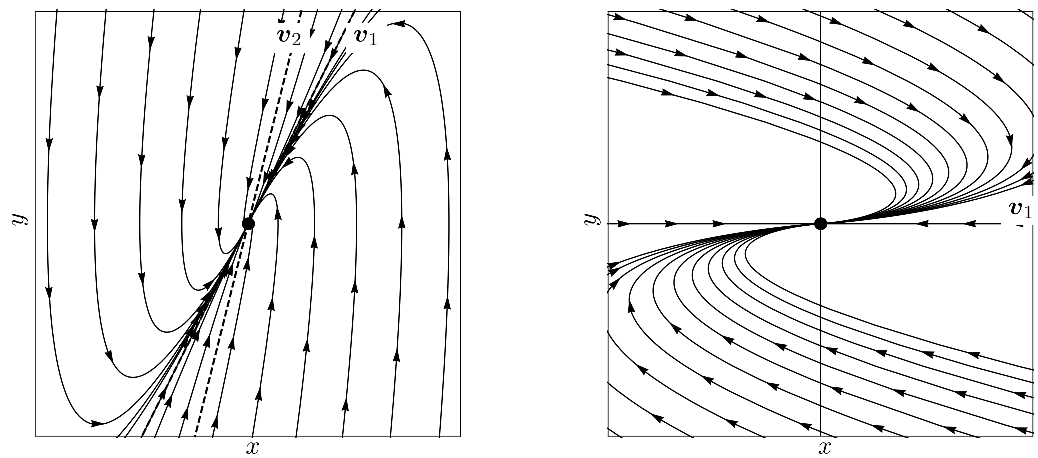

Figure 4: Left: Node. The case $\lambda_2 < \lambda_1 < 0$. Here the eigenvector directions do not coincides with the axes. Right: The case of equal eigenvalues with only one eigenvector

which has eigenvalue λ multiplicity 2 with the eigenvector $\boldsymbol{v} = (1, 0)^\top$. In this case we have only one characteristic direction, and the orbits approach the origin along it. Moreover, the direction on the orbits from the origin if $\lambda > 0$ and to the origin is $\lambda < 0$ (see figure, right panel for an example).

Now we switch to the case when the eigenvalues are not real.

Consider the system

$$\begin{aligned}\dot{x} &= ax - by,\\ \dot{y} &= bx + ay,\end{aligned}$$

for some constants a and b. Hence we have the matrix

$$\boldsymbol{A} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

We can analyze this case exactly as before, by finding the eigenvalues and eigenvectors (for example, the eigenvalues are $\lambda_{1,2} = a \pm \mathrm{i}b$). However, there is a simple way to figure out the behavior of the orbits. Multiply the second equation by i and add them, after some simplifications we will find that for $z = x + \mathrm{i}y$ it is true that:

$$\dot{z} = \lambda z, \quad \lambda = a + \mathrm{i}b.$$

Now I use the polar form of a complex number $z = \rho e^{\mathrm{i}\theta}$, from which (fill in the details)

$$\dot{\rho} = a\rho, \quad \dot{\theta} = b,$$

from which we find

$$\rho(t) = C_1 e^{at}, \quad \theta(t) = bt + C_2,$$

for the polar coordinates ρ and θ. To see how exactly the orbits look like, assume that $a > 0$ and $b > 0$. Therefore, we find that $\rho \to \infty$ as $t \to \infty$ and $\rho \to 0$ as $t \to -\infty$. For b positive it means that the polar angle changes in the positive direction, which is counterclockwise. Considering superposition of these two movements, we find that all the orbits (safe for the equilibrium point at the origin) are spiral, which the direction on them *from* the origin. An example for the same matrix with $a < 0$ and $b > 0$ is given on the right panel in the same figure. An equilibrium with such structure of orbits is called *focus* or *spiral*.

The general case is when the matrix has two complex conjugate eigenvalues $\lambda_{1,2} = a \pm b$. By the sign of a we know the direction on the orbits: If $a > 0$ then the direction is from the origin, and if $a < 0$ then the direction

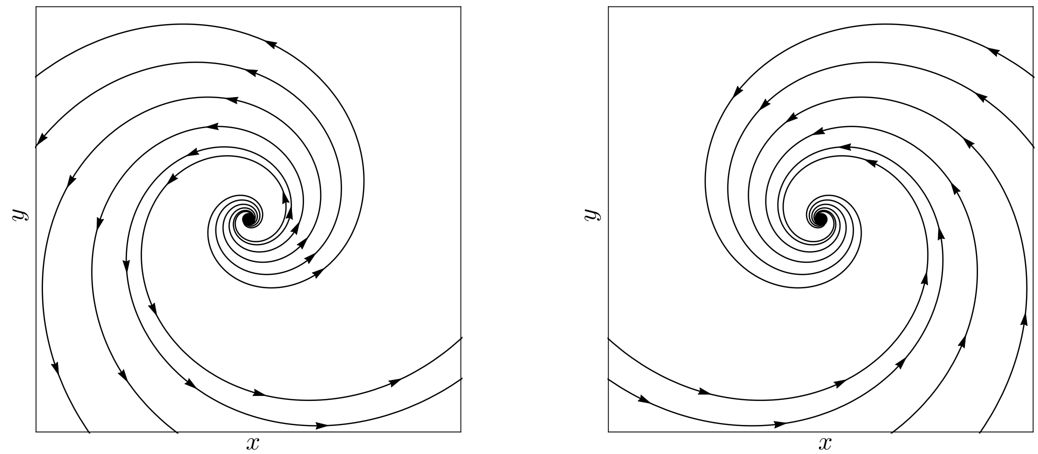

Figure 5: Left: Focus. $a > 0, b > 0$. Right: Focus. $a < 0, b > 0$

towards the origin. However, the subtle thing here is to determine whether the rotation occurs clockwise or counterclockwise. To make sure that you find the correct direction, it is useful to take any point $(x, y) \neq (0, 0)$ and find the direction at this point (this direction is given be the vector $(a_{11}x + a_{12}y, a_{21}x + a_{22}y)$). If you know whether the origin attracts or repels the orbits and one point with the precise direction, it becomes clear what is the whole picture on the phase plane.

As an example, and to make the discussion complete, consider the linear system with the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix},$$

with the eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\sqrt{3}$. Since $a = 0$ we find that the orbits neither approach nor leave the origin. Actually, it can be shown that all the orbits in this case are ellipses. To infer the direction of rotation, pick a point, e.g., $(x, y) = (1, 0)$. At this point we find the vector $(0, 1)^\top$, which points in the counterclockwise direction, hence the orbits look like ellipses with the counterclockwise directions on them (see the figure). An equilibrium with such phase portrait is called *center*.

23.3 Summary

In the previous section we found that it is possible to have the phase portrait around the origin that belongs to one of the following types:

- saddle (two real eigenvalues of opposite sign);
- node (two real eigenvalues of the same sign);
- degenerate node (one eigenvalue of multiplicity two with only one eigenvector);
- focus (two complex conjugate eigenvalues, moreover, $\operatorname{Re} \lambda \neq 0$);
- center (two imaginary eigenvalues, $\operatorname{Re} \lambda = 0$).

However, the analysis was mostly based on three types of matrices:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

To state why it is enough to consider only these three matrices, I will need the following

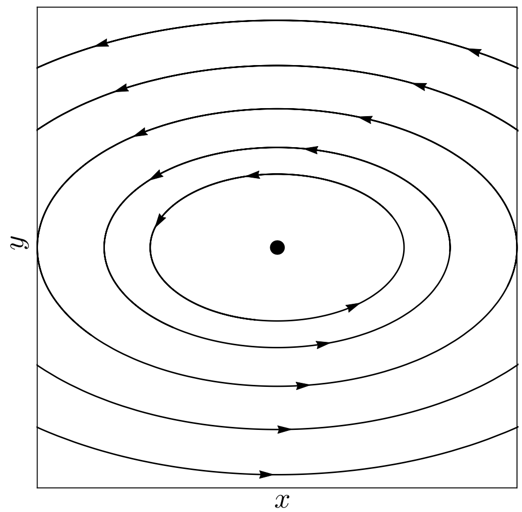

Figure 6: Center, the eigenvalues are purely imaginary

Definition 3. *Matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbf{M}_n(\mathbf{R})$ are called similar is there exists an invertible matrix $\boldsymbol{S}$, such that*

$$\boldsymbol{A} = \boldsymbol{S}^{-1}\boldsymbol{B}\boldsymbol{S}.$$

Similar matrices share a lot of properties. For example, they have the same eigenvalues (can you prove it?). The main fact is the next theorem, which I state without proof.

Theorem 4. *Any 2×2 matrix is similar to one of the three matrices above.*

As an important corollary we obtain that for any matrix $\boldsymbol{A} \in \mathbf{M}_2(\mathbf{R})$ such that $\det \boldsymbol{A} \neq 0$, the only possible phase portraits are given in the previous section.

23.4 Stability of the origin

Having at our disposable all the possible phase portraits of linear planar systems of ODE makes it very intuitively clear what it means to have the origin stable.

Definition 5. *The origin of the linear system*

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y, \\ \dot{y} &= a_{22}x + a_{22}y, \end{aligned}$$

with $\det \boldsymbol{A} \neq 0$ *is called*

- *Lyapunov stable, if any orbit, starting close enough to the origin, stays close to the origin for all positive t;*
- *asymptotically stable, if any orbit, starting close enough to the origin, is 1) Lyapunov stable, and 2) tends to the origin as $t \to \infty$;*
- *unstable, if there exits an orbit starting close enough to the origin that leaves a small neighborhood of the origin for some positive t.*

Using this definition we find that

- saddles are always unstable, since it is possible to find orbits close to the origin that eventually leave any neighborhood of the origin;
- nodes can be either asymptotically stable (this requires that both eigenvalues are negative) or unstable (both eigenvalues are positive);
- foci can be either asymptotically stable (if $\operatorname{Re}\lambda_{1,2} < 0$) or unstable ($\operatorname{Re}\lambda_{1,2} > 0$);
- center is Lyapunov stable, but not asymptotically stable, since the orbits do not approach the origin.

Putting everything together, we obtain a very important fact that says that

Theorem 6. *The origin of the linear planar system with the matrix $\boldsymbol{A}$ such that $\det \boldsymbol{A} \neq 0$, is stable if for all the eigenvalues $\operatorname{Re}\lambda \leq 0$ and unstable otherwise. Moreover, it is asymptotically stable if $\operatorname{Re}\lambda_{1,2} < 0$.*

Since there is only one equilibrium in the linear system, it is often said that the system is stable or asymptotically stable, meaning that the origin is stable or asymptotically stable.

It is convenient to summarize all the information about the linear planar systems using two parameters: trace and determinant of matrix $\boldsymbol{A}$. For an arbitrary matrix $\boldsymbol{A}$ the characteristic polynomial has the form

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = \lambda^2 - \operatorname{tr}\boldsymbol{A}\,\lambda + \det \boldsymbol{A},$$

where I used the notation $\operatorname{tr}\boldsymbol{A} := a_{11} + a_{22}$ for the *trace* of matrix $\boldsymbol{A}$, which is given by the sum of the elements on the main diagonal. Therefore, the eigenvalues can be found as

$$\lambda_{1,2} = \frac{\operatorname{tr}\boldsymbol{A} \pm \sqrt{(\operatorname{tr}\boldsymbol{A})^2 - 4\det\boldsymbol{A}}}{2}.$$

For example, if $\operatorname{tr}\boldsymbol{A} > 0$, $\det\boldsymbol{A} > 0$, and $\Delta = (\operatorname{tr}\boldsymbol{A})^2 - 4\det\boldsymbol{A} > 0$ then we obtain that there are two real eigenvalues of the same sign, which means that in this parameter region we have unstable node.

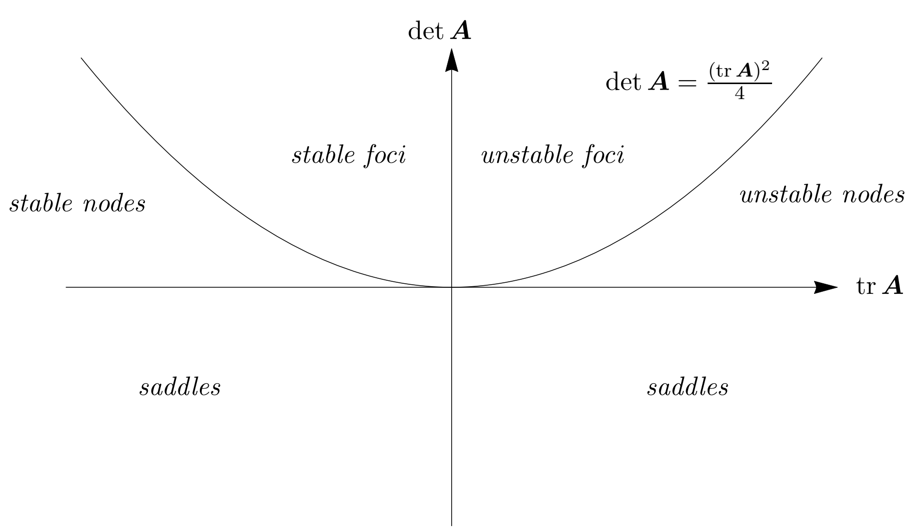

Figure 7: Parametric portrait of linear planar systems. There are six domains here, with the boundaries $\det\boldsymbol{A} = 0$, $\operatorname{tr}\boldsymbol{A} = 0$ and $\det A = (\operatorname{tr}\boldsymbol{A})^2/4$. Note that on the line $\operatorname{tr} A = 0$ when $\det\boldsymbol{A} > 0$ we have centers, and if $\det\boldsymbol{A} < 0$ we still have saddles. On the line $\det\boldsymbol{A} = (\operatorname{tr}\boldsymbol{A})^2/4$ we have two equal eigenvalues, and hence degenerate nodes

Note that the theorem above now can be restated as follows: The origin of the planar system with matrix for which $\det\boldsymbol{A} \neq 0$ is asymptotically stable if and only if $\operatorname{tr}\boldsymbol{A} < 0$ and $\det\boldsymbol{A} > 0$. No need to compute any eigenvalues!

23.5 Examples

Here we'll consider a few more detailed examples, where all the bits of the theory developed above will be used.

23.5.1 A mass on a spring revisited

We start with the system that we already analyzed in detail. Now we will look at it from a slightly different angle.

Recall that if we have a mass $m > 0$ on the spring with spring constant k and the damping coefficient is $c > 0$, then the position $x(t)$ of the center of mass satisfies the differential equation (we assume that no external force is applied to the mass)

$$m\ddot{x} + c\dot{x} + kx = 0.$$

Introducing the new variable $\dot{x} = v$ the second order equation can be rewritten as a system of two first order equations

$$\begin{aligned}\dot{x} &= v,\\ \dot{v} &= -\frac{c}{m}v - \frac{k}{m}x,\end{aligned}$$

with the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}.$$

The eigenvalues are the roots of characteristic equation

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0.$$

Assume first that $c = 0$, i.e., there is no damping in the system. Then the eigenvalues are

$$\lambda_{1,2} = \pm \mathrm{i}\sqrt{\frac{k}{m}} = \pm \mathrm{i}\omega_0,$$

and the origin in this case is a center. The only thing that needs to be checked is the direction of the phase orbits. Here if we take vector $\boldsymbol{e}_1 = (0, 1)$ then $\boldsymbol{A}\boldsymbol{e}_2 = \boldsymbol{e}_1 = (1, 0)$, that is, the motion is clockwise in the plane (x, v) (see top left phase portrait in Fig. 8). Clearly the origin is Lyapunov stable but not asymptotically stable in this case. This is the general situation: usually the Lyapunov stable equilibria correspond to the situations when either friction or damping are absent.

In reality we always have some kind of damping in our mechanical systems. Assume that not $0 < c$ is small. Note that in this situation $\operatorname{tr}\boldsymbol{A} < 0$ and $\det\boldsymbol{A} > 0$ for any parameter values, as expected. The eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2}\left(-c \pm \sqrt{c^2 - 4km}\right).$$

First, if $c^2 - 4km < 0$ (which is true for small c), then we have two complex conjugate eigenvalues with negative real parts, hence the origin is stable focus (the direction of motion also can be determined by taking a specific point and finding the direction at this point, or by continuity from the case $c = 0$), which correspond to the oscillations which eventually subside and the mass reaches the equilibrium (see Fig. 8, top right.)

If we keep increasing c then at some point $c^2 = 4km$ and we have one real negative eigenvalue $\lambda = -c(2m)^{-1}$ of multiplicity two. The corresponding eigenvector can be chosen as (check it)

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ -\frac{c}{2m} \end{bmatrix},$$

which also described the direction along which the orbits approach the origin (critically damped motion, Fig. 8, bottom left panel).

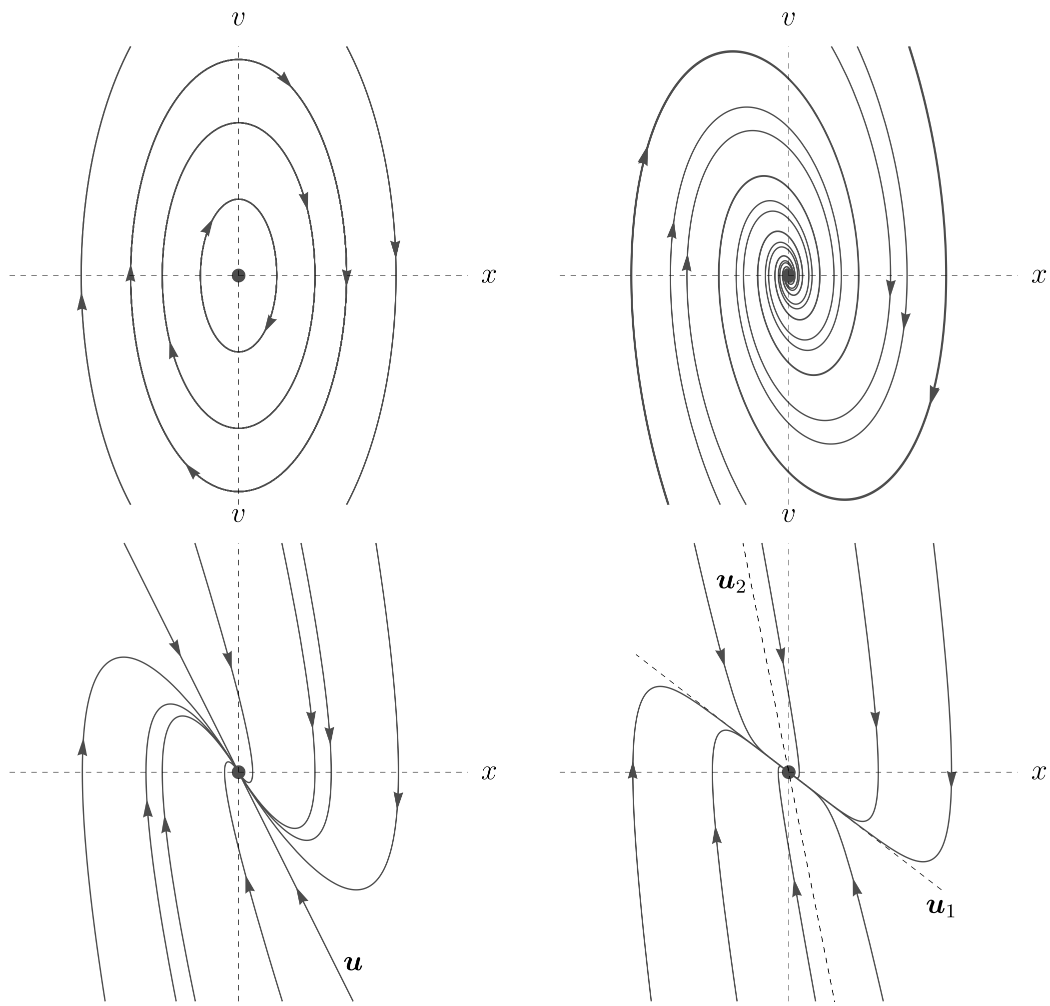

Figure 8: Various phase portraits of the mechanical system "mass on a spring." Top left: harmonic oscillations, no friction; top right: underdamped motion; bottom left: critically damped motion; bottom right: overdamped motion.

If we keep increasing c, we get two real negative eigenvalues with eigenvectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, which corresponds to overdamped motion (Fig. 8, bottom right panel). It is a good exercise to allow the parameters c and k to be negative and see what other kinds of phase portraits this system can produce.

23.5.2 Lanchester's law of combat

Consider a battle between two armies. Let $R(t)$ and $B(t)$ be the number of soldiers of these armies at time t respectively (so we have a "red" army and a "blue" army). Assume also that relative efficiencies of the armies (whatever it means) are given by constants b for blue and r for red. It is reasonable to assume that the rate of change (number of casualties per time unit) of each army if proportional to the product of the weapon efficiency and the army population. That is, the battle can be described by the system of ODE:

$$\dot{R} = -bB,$$
$$\dot{R} = -rR,$$

which is exactly in the form that we studied in this part of the course. The matrix of the system is given by

$$\boldsymbol{A} = \begin{bmatrix} 0 & -b \\ -r & 0 \end{bmatrix},$$

and hence the eigenvalues are $\lambda_{1,2} = \pm\sqrt{rb}$, which corresponds to the saddle at the origin. Let us try to be more specific about the structure of phase orbits.

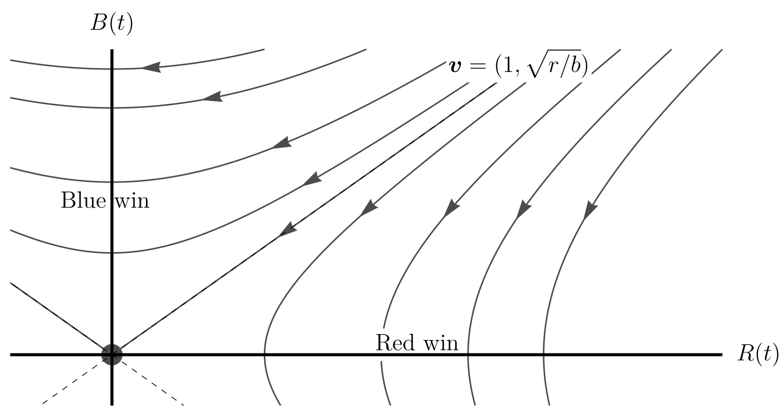

Figure 9: The phase portrait of the system $\dot{R} = -bB, \dot{B} = -rB$. The direction along the eigenvector $\boldsymbol{v}$ corresponds to the stalemate situation: both armies fight to the last soldier. This direction separates the set of initial conditions for which "blue" win from the set of initial conditions for which "red" win.

The interpretation of the model implies that we are only interested in the quadrant when $R(t), B(t) \geq 0$. Therefore we will consider only the eigenvalue $\lambda_1 = -\sqrt{rb}$, which corresponds to the saddle direction, along which the orbit approaches the origin. The corresponding eigenvector can be taken as $\boldsymbol{v} = (1, \sqrt{r/b})$ (see Fig. 9). Note that if the initial conditions are chosen such that we start exactly on the line with the direction $\boldsymbol{v}$, we call this situation "stalemate" since both armies loose all the soldiers. The same line divides the set of all the initial conditions into the set for which B win and R win. Specifically, if $B_0 > \sqrt{\frac{r}{b}}R_0$ then "blue" win, in the opposite case "red" win (see the figure).

An interesting property of this (very oversimplified) model is that at the stalemate one has

$$bB_0^2 = rR_0^2,$$

which is often called *Lanchester's square law of combat*, and which implies a quite nontrivial conclusion: To defeat the opponent that is twice as numerous, the other side must be at least four time more efficient.