5 Exact equations

In Lecture 3, where I discussed the geometric interpretation of the first order ODE, an attentive reader should have noted that the relation between the integral curves as the curves tangent to the given direction field and the graphs of the solutions to the equation $y' = f(x, y)$ is not as simple as it was stated. For instance, assuming that the solution has to be the function $y = y(x)$ we exclude all the directions parallel to the y-axis, even more, we also exclude all the integral curves that cross the lines parallel to the y-axis at more than one point. To overcome this difficulty, we can use the approach discussed in the previous lecture, when we switched the role of the dependent and independent variables, and instead of $y' = f(x, y)$ consider $x' = f_1(x, y)$, where $f_1(x, y) = 1/f(x, y)$, wherever $f(x, y)$ is undefined, but now the situation when $f(x, y) = 0$ becomes undefined. Therefore, it would be beneficial to have a more general form to write ODE.

The most general form of the first order ODE, for which the relation between the integral curves and the graphs of the solutions is most direct, is as follows:

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0, \tag{1}$$

where $M(x, y)$ and $N(x, y)$ are given functions. In this form the role of x and y is symmetric, and the direction field is defined everywhere, where $M(x, y)$ and $N(x, y)$ are defined and at least one of them is different from zero. Note that to go from (1) to the usual form $y' = f(x, y)$ one needs to set $f(x, y) = -\frac{M(x,y)}{N(x,y)}$.

Definition 1. *Equation* (1) *is called exact in the domain $D \subseteq \mathbf{R}^2$ if there exists a function $F(x, y) \in C^{(1)}(D; \mathbf{R})$ such that*

$$\mathrm{d}F(x, y) = M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = \frac{\partial F}{\partial x}\,\mathrm{d}x + \frac{\partial F}{\partial y}\,\mathrm{d}y, \quad (x, y) \in D \subseteq \mathbf{R}^2.$$

In other words, the expression $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y$ is a *total differential* if (1) is exact. Function F is often called a *potential*.

If an ODE is exact and we know the function $F(x, y)$ then its solution is immediate and given by

$$F(x, y) = C.$$

If, additionally to (1), we are also given the initial condition

$$y(x_0) = y_0, \tag{2}$$

then the solution to the IVP (1)–(2) is

$$F(x, y) = F(x_0, y_0).$$

When it is possible to express y as a function of x, it should be done, however, as even the simplest examples show, it is usually not possible to express one variable as a function of another, in this case the solution is kept as an implicit function.

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024

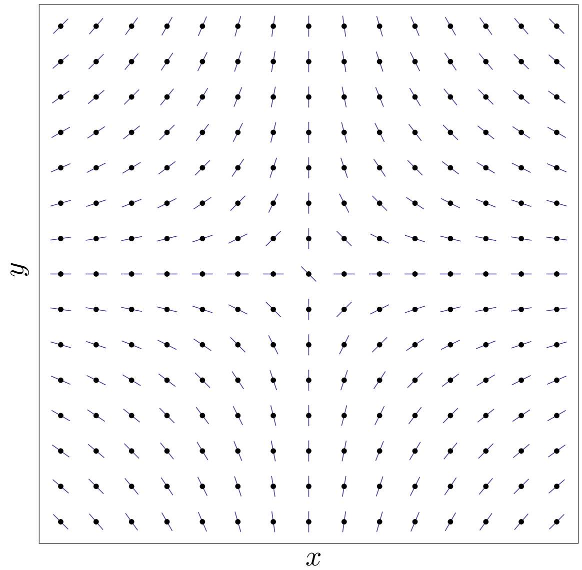

Figure 1: The direction field of the equation $y\,\mathrm{d}x + x\,\mathrm{d}y = 0$.

Example 2. Consider the equation

$$y\,\mathrm{d}x + x\,\mathrm{d}y = 0.$$

Since we must have that $\frac{\partial F}{\partial x} = y$, $\frac{\partial F}{\partial y} = x$, we can guess that in this case $F(x, y) = xy$. Hence the general solution to this equation is given by $xy = C$, which are the families of hyperbolas. The direction field is shown in the figure above. Note that the only point at which the direction field is not determined is the origin $(0, 0)$.

There are two main questions, which have to be answered, if a student encounters an ODE in the form (1). First, is this equation exact? Second, if the answer to the first question is "yes," how to find $F(x, y)$?

To answer question 1, you need to recall from your Calculus course that

Theorem 3. *The expression of the form* $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y$ *with* $M, N \in C^{(1)}(D; \mathbf{R})$ *is a total differential in an open simply connected set* D *if and only if*

$$\frac{\partial M}{\partial y}(x, y) \equiv \frac{\partial N}{\partial x}(x, y), \quad (x, y) \in D.$$

The set $D \subseteq \mathbf{R}^2$ is *open* if any point of the set is interior, and *simply connected* if it does not have holes. Once we know that our equation is exact, we can use this fact to figure out what $F(x, y)$ is. It is easier to show the procedure by an example.

Example 4. Solve the equation

$$(3x^2 + 6xy^2)\,\mathrm{d}x + (6x^2y + 4y^3)\,\mathrm{d}y = 0.$$

Here we have

$$M(x, y) = 3x^2 + 6xy^2, \quad N(x, y) = 6x^2y + 4y^3.$$

The theorem above works since

$$\frac{\partial M}{\partial y}(x,y) = 12xy \equiv 12xy = \frac{\partial N}{\partial x}(x,y)$$

for any $D \subseteq \mathbf{R}^2$. Hence our equation is exact. This fact implies that there exists function $F(x,y)$ such that

$$\frac{\partial F}{\partial x}(x,y) = 3x^2 + 6xy^2, \quad \frac{\partial F}{\partial y}(x,y) = 6x^2y + 4y^3.$$

Let us use the first of these equalities and integrate with respect to x:

$$\int \frac{\partial F}{\partial x}(x,y)\,\mathrm{d}x = \int (3x^2 + 6xy^2)\,\mathrm{d}x = x^3 + 3x^2y^2 + C(y).$$

Note that since I integrated with respect to x, the arbitrary constant in general depends on y. Now we take the derivative with respect to y from the obtained expression:

$$\frac{\partial}{\partial y}\big(x^3 + 3x^2y^2 + C(y)\big) = 6x^2y + C'(y),$$

and this has to be equal to $N(x,y)$:

$$6x^2y + C'(y) = 6x^2y + 4y^3 \implies C'(y) = 4y^3 \implies C(y) = y^4 + C_1.$$

Hence the final answer for our problem is (switching back to the usual notation C for the arbitrary constant)

$$x^3 + 3x^2y^2 + y^4 = C,$$

and the potential function (which is determined up to an additive constant) is

$$F(x,y) = x^3 + 3x^2y^2 + y^4.$$

Here we can express neither y nor x in the general solution. This example shows that sometimes it is easier to solve the problem than figure out the behavior of the integral curves.

Here are a few more exact equations with the corresponding solutions:

1. $(e^x \sin y + e^{-y})\,\mathrm{d}x - (xe^{-y} - e^x \cos y)\,\mathrm{d}y = 0$ with the general solution

$$e^x \sin y + e^{-y}x = C.$$

2. $\sin y\,\mathrm{d}x + x\cos y\,\mathrm{d}y = 0$ with the general solution

$$x \sin y = C.$$

3. $(3x^2 - 2x - y)\,\mathrm{d}x + (2y - x + 3y^2)\,\mathrm{d}y = 0$ with the general solution

$$x^3 - x^2 - xy + y^2 + y^3 = C.$$

You should practice your skills in solving these equations by the methods described above.

5.1 *Integrating factor

Sometimes we are given an ODE in the form (1) but

$$\frac{\partial M}{\partial y}(x,y)\neq\frac{\partial N}{\partial x}(x,y).$$

In this case the equation is not exact. However, we can always try to find (or guess) a function $\mu(x,y)$, which is called *an integrating factor*, such that after multiplication of the equation by this function we end up with an exact equation

$$\mu(x,y)M(x,y)\,\mathrm{d}x+\mu(x,y)N(x,y)\,\mathrm{d}y=0.$$

For the last equation to be exact we need to assume that

$$\frac{\partial}{\partial y}(\mu M)\equiv\frac{\partial}{\partial x}(\mu N).$$

Here I suppress the dependence on x, y to simplify the expressions. I will also denote the derivatives as $\mu'_x:=\frac{\partial \mu}{\partial x}$ and so on. Rearranging the last equality and evaluating the derivatives yield

$$\mu'_yM+M'_y\mu=\mu'_xN+N'_x\mu\implies\mu(M'_y-N'_x)+M\mu'_y-N\mu'_x=0.$$

In the last equation the unknown function is $\mu=\mu(x,y)$, which depends on two independent variables. Hence this is a differential equation, albeit not ordinary. This is an example of a *partial differential equation*, which, generally speaking, is a more complex object than the original ODE. So, did we learn something by these manipulations? Actually, yes, if we make some additional assumptions.

First, assume that an integrating factor is a function of x only. In this case the last equation can be rewritten as

$$\frac{\mu'_x}{\mu}=\frac{M'_y-N'_x}{N}\,.$$

For μ to be a function of x only, the right hand side has to depend only on x. Therefore, we obtain that if the expression

$$\frac{M'_y-N'_x}{N}$$

is a function of x only, then there exists an integrating factor $\mu=\mu(x)$, which can be found as a solution to the ordinary differential equation

$$\frac{\mu'_x}{\mu}=\frac{M'_y-N'_x}{N}\,.$$

In a similar vein (you should fill in the missing details), if the expression

$$\frac{N'_x-M'_y}{M}$$

is a function of only y, then there exists an integrating factor $\mu=\mu(y)$, which can be found as a solution to the ODE

$$\frac{\mu'_y}{\mu}=\frac{N'_x-M'_y}{M}\,.$$

Example 5. Consider the equation

$$x^2y\,\mathrm{d}x + x^3\,\mathrm{d}y = 0,$$

which, since $\frac{\partial}{\partial y}(x^2y) \neq \frac{\partial}{\partial x}(x^3)$, is not exact. Consider the expression

$$\frac{M'_y - N'_x}{N} = \frac{x^2 - 3x^2}{x^3} = -\frac{2}{x}\,,$$

which depends only on x. Hence we can find $\mu = \mu(x)$ as a solution to

$$\frac{1}{\mu}\mu' = -\frac{2}{x} \implies \mu(x) = \frac{1}{x^2}\,.$$

Note that because I need only one integrating factor, I do not include the arbitrary constant in my calculations. After multiplication of the original equation by $1/x^2$, we have

$$y\,\mathrm{d}x + x\,\mathrm{d}y = 0,$$

which we already solved at the beginning of the lecture:

$$xy = C.$$

Example 6. Consider the linear equation

$$y' + p(x)y = q(x).$$

We can rewrite it as

$$(p(x)y - q(x))\,\mathrm{d}x + \,\mathrm{d}y = 0,$$

with $M(x,y) = p(x)y - q(x)$ and $N(x,y) = 1$. This equation is not exact, however, the expression

$$\frac{M'_y - N'_x}{N} = p(x)$$

depends only on x. Therefore there is an integrating factor $\mu = \mu(x)$, which is a solution to

$$\frac{\mu'}{\mu} = p(x) \implies \mu(x) = \exp\int p(x)\,\mathrm{d}x,$$

exactly how we obtained earlier using completely different reasoning!

Example 7. Solve

$$2xy\log y\,\mathrm{d}x + \left(x^2 + y^2\sqrt{y^2+1}\right)\mathrm{d}y = 0.$$

This equation is not exact (check!). However, using our notations, we find that

$$\frac{N'_x - M'_y}{M} = -\frac{1}{y}\,,$$

which means that there exists an integrating factor depending on the variable y only:

$$\mu' = -\frac{\mu}{y} \implies \mu(y) = \frac{1}{y}\,.$$

Therefore, we have

$$2x \log y \, \mathrm{d}x + \frac{x^2}{y} \, \mathrm{d}y + y\sqrt{y^2+1} \, \mathrm{d}y = 0.$$

Now note that the first two terms, as well as the third one, can be readily simplified as

$$\mathrm{d}\left(x^2 \log y\right) + \frac{1}{3} \mathrm{d}\left(y^2+1\right)^{3/2} = 0,$$

therefore, the answer is

$$x^2 \log y + \frac{1}{3}\left(y^2+1\right)^{3/2} = C.$$

5.2 *For a mathematically inclined student: Orthogonal trajectories

I already discussed the fact that in general a first order ODE defines geometrically a one-parameter family of curves. For example, the ODE $x \, \mathrm{d}x + y \, \mathrm{d}y = 0$ defines the family of circles with the center at the origin $x^2 + y^2 = C$. The converse is also true: for a given one-parameter family of curves on the plane it is possible to find an ODE whose general solution coincides with this family.

Example 8. Find the ODE of the family of hyperbolas

$$\frac{x^2}{C^2} - y^2 = 1.$$

Here C is our parameter, for different C we have different hyperbolas (sketch them). Differentiate this expression with respect to x and obtain

$$\frac{2x}{C^2} - 2yy' = 0,$$

or, after multiplying by x,

$$\frac{x^2}{C^2} = xyy'.$$

Now eliminate the parameter using the original equation for the family of curves and finally obtain the sought ODE

$$1 - y^2 = xyy'.$$

In general, if one is given a one-parameter family of curves on the plane $\Phi(x, y, C) = 0$, then, after differentiating with respect to x, we find the system

$$\begin{aligned} \Phi(x, y, C) &= 0, \\ \frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y} y' &= 0, \end{aligned}$$

from which we can eliminate the parameter and find the ODE of the family of curves.

Now consider the family of curves given by $\Phi(x, y, C) = 0$ and another curve that forms a constant angle α with the curves of the family $\Phi(x, y, C) = 0$ at each point. Such curve is called an *isogonal trajectory*. In particular, if $\alpha = \pi/2$, then such curve is called an *orthogonal trajectory*.

Consider the problem of finding the family of orthogonal trajectories for a given family of curves $\Phi(x,y,C)=0$. Since the ODE for the family $\Phi(x,y,C)=0$ can be found by the methods described above as $y'=f(x,y)$, the ODE for the orthogonal family is given by

$$y' = -\frac{1}{f(x,y)}\,,$$

due to the fact that if two curves are orthogonal at some point x, then $y_1'(x)y_2'(x)=-1$ (*Q:* can you see why it is true?).

Example 9. Find the orthogonal trajectories to the family of circles

$$x^2+y^2=2Cx.$$

Note that the equation can be rewritten as

$$(x-C)^2+y^2=C^2,$$

which means that geometrically these are the circles with the centers on the x-axis and touching the y-axis. Differentiating with respect to x we find

$$2x+2yy'=2C.$$

Eliminating the parameter yields

$$x^2-y^2=2xyy' \implies y'=-\frac{x^2-y^2}{2xy}\,,$$

which means that the ODE for the orthogonal trajectories takes the form

$$y'=\frac{2xy}{x^2-y^2}\,.$$

Rewrite this equation as

$$-2xy\,\mathrm{d}x+(x^2-y^2)\,\mathrm{d}y=0,$$

which is not exact (check it). However, as can be proved, the function $\mu(y)=y^{-2}$ is an integrating factor here, and the equation

$$-\frac{2x}{y}\,\mathrm{d}x+\frac{x^2-y^2}{y^2}\,\mathrm{d}y=0$$

is exact. Its solution is given by

$$x^2+y^2=2Cy,$$

or

$$x^2+(y-C)^2=C^2,$$

which gives the family of circles with the centers on the y-axis touching the x-axis. In the figure you can see the original family (in red) and the orthogonal family (in blue):

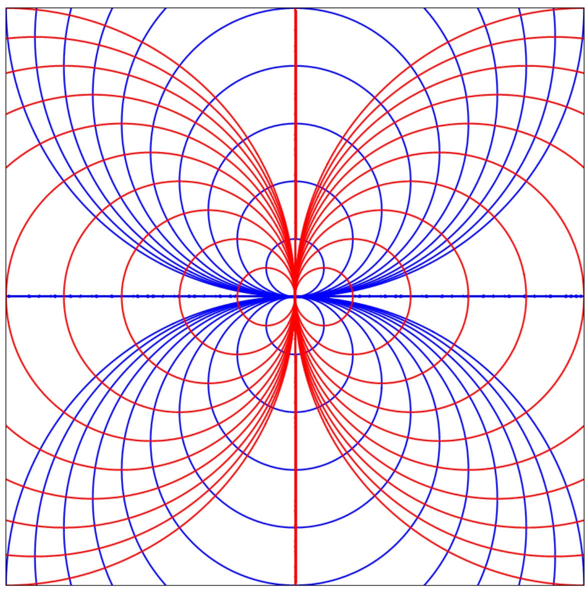

Figure 2: The orthogonal trajectories (in blue) to the original family of curves (in red).