6 Substitutions I

Consider again the first order ODE in the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0. \tag{1}$$

We learnt that we can solve (1) if it is either separable, linear, or exact. In some cases an integrating factor can be found. Let us reiterate the definitions of these three main classes of the first order ODE.

- Equation (1) is separable if and only if both $M(x,y)$ and $N(x,y)$ can be represented as products of functions depending on x only and functions depending on y only. Formally, (1) is separable $\Leftrightarrow$

  $$M(x,y) = M_1(x)M_2(y), \quad N(x,y) = N_1(x)N_2(y).$$

  To solve it we need to evaluate two integrals (and not to forget about special solutions which appear because we might divide by zero).

- Equation (1) is linear if it is possible to rearrange the terms in it such that we end up with either

  $$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x),$$

  or

  $$\frac{\mathrm{d}x}{\mathrm{d}y} + p(y)x = q(y).$$

  In the former case the equation is linear with respect to $y(x)$ and in the latter case it is linear with respect to $x(y)$. After this we can proceed either with the method of integrating factor or with the variation of the constant method.

- Equation (1) is exact if there exists a function $F(x,y)$ such that

  $$\mathrm{d}F(x,y) = \frac{\partial F}{\partial x}(x,y)\,\mathrm{d}x + \frac{\partial F}{\partial y}(x,y)\,\mathrm{d}y = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y.$$

  Recall that the necessary and sufficient condition for (1) to be exact is

  $$\frac{\partial M}{\partial y}(x,y) \equiv \frac{\partial N}{\partial x}(x,y),$$

  and general solution, provided we found $F(x,y)$, is given by

  $$F(x,y) = C,$$

  where C is an arbitrary constant.

Question: What should we do if (1) does not fall in any of the categories discusses above? Of course, for any particular case we can try to look for an integrating factor, which always exists (locally). However, in reality, this can be very tricky of even impossible to come up with an analytical expression for the integrating factor. Luckily, there is quite a universal method, which often leads to considerable

MATH266: Intro to ODE by Artem Novozhilov, e-mail: artem.novozhilov@ndsu.edu. Spring 2024

simplifications or even to full solutions. This methods consists in finding a correct substitution, after which our equation becomes either separable, or linear, or exact. Think about evaluating integrals in your Calculus course: There was the table of the basic integrals, and there were different substitutions that bring the integrals into the table form. A very similar situation is true for ODE.

In this lecture we consider two classes of ODE, which, after a substitution, can be analytically solved.

6.1 Bernoulli's equation

Definition 1. *First order ODE in the form*

$$y' + p(x)y = q(x)y^n, \quad n \in \mathbf{R}$$

is called Bernoulli's equation if $n \neq 0, 1$.

The equation is named after Jacob Bernoulli (1654–1705, Swiss mathematician), who is a brother of Johann Bernoulli (1667–1748) and relative of many other Bernoullis, who made different contributions to mathematics and related fields. Note the condition $n \neq 0, 1$, since in these cases the equation becomes linear, and we know how to solve it.

To solve Bernoulli's equation we rewrite it in the form

$$\frac{y'}{y^n} + p(x)\frac{1}{y^{n-1}} = q(x),$$

marking that $y = 0$ is a solution to our problem. Now the substitution is

$$z(x) = \frac{1}{y^{n-1}(x)} = y^{1-n}(x).$$

To make a correct substitution, we need to figure out how the derivative of $y(x)$ is related to the derivative of $z(x)$. One has

$$z'(x) = \frac{\mathrm{d}z}{\mathrm{d}x}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{y^{n-1}(x)} = (1-n)\frac{y'(x)}{y^n(x)} \implies \frac{y'(x)}{y^n(x)} = \frac{z'(x)}{1-n}.$$

Hence we have

$$\frac{z'}{1-n} + p(x)z = q(x),$$

or

$$z' + (1-n)p(x)z = q(x)(1-n),$$

which is a linear ODE, and which we know how to solve. One thing not to forget is to return to the original variable $y(x)$ at the very end.

Example 2. Solve the IVP

$$x^2y' + 2xy = \sqrt{y}, \quad y(1) = 1.$$

Here $n = \frac{1}{2}$. We have (note that I never remember the correct substitution by heart, contrary, I just divide the equation by y^n and after this figure out what to substitute)

$$\frac{y'}{y^{1/2}} + \frac{2}{x}y^{1/2} = \frac{1}{x^2}\,.$$

The substitution is $z = y^{1/2} \implies z' = \frac{1}{2}y^{-1/2}y' \implies \frac{y'}{y^{1/2}} = 2z'$. Hence,

$$2z' + \frac{2}{x}z = \frac{1}{x^2} \implies z' + \frac{1}{x}z = \frac{1}{2x^2}\,.$$

This is a linear equation. Let's solve it using, for a change, the variation of the constant method.

$$z' + \frac{1}{x}z = 0 \implies z = \frac{C}{x}.$$

Assume that

$$z(x) = \frac{C(x)}{x}\,,$$

and plug it into the original equation

$$\frac{C'(x)}{x} - \frac{C(x)}{x^2} + \frac{C(x)}{x^2} = \frac{1}{2x^2} \implies C(x) = \ln\sqrt{|x|} + C.$$

This means that

$$z(x) = \frac{\ln\sqrt{|x|}}{x} + \frac{C}{x}\,.$$

Now we are returning to the original variable $y(x)$:

$$y = z^2 \implies y(x) = \left(\frac{\ln\sqrt{|x|}}{x} + \frac{C}{x}\right)^2.$$

Hence the general solution to the problem is

$$y(x) = \left(\frac{\ln\sqrt{|x|}}{x} + \frac{C}{x}\right)^2 \quad \text{or} \quad y = 0.$$

To satisfy the initial conditions, we must have $C = 1$ (technically, from the last line we have two choices for $C = \pm 1$, however, if you recall that $z = \sqrt{y} \geq 0$, then only the choice $C = 1$ is correct). Therefore, the solution to the IVP is

$$y(x) = \left(\frac{\ln\sqrt{|x|}}{x} + \frac{1}{x}\right)^2.$$

6.2 Homogeneous equations

The second class of the first order ODE that can be solved after a correct substitution and which we consider un this lecture is the *homogeneous* equations.

Definition 3. *A first order ODE of the form*

$$y' = f\left(\frac{y}{x}\right)$$

is called homogeneous.

Hence the equation homogeneous, if the right hand side can be represented as a function of one variable only, which is the ratio of y/x.

To solve the homogeneous ODE make the substitution $z(x) = \frac{y(x)}{x}$, or $y(x) = z(x)x$. Remember that here x and y are not symmetric variables, since by the problem statement we assume that $y = y(x)$ is the dependent variable. From the substitution we find, using the product rule for the derivative, that

$$y'(x) = z'(x)x + z(x).$$

Putting this into the equation,

$$z'x + z = f(z),$$

or

$$z' = \frac{f(z) - z}{x},$$

which is a separable equation, and which we know how to solve. Not to loose any solutions, you should recall that every time we separate the variables, we also check whether we divide by a zero or not. Here we should look for zeros of $f(z) - z = 0$. Assume that $\hat{z}$ solves $f(z) = z$, therefore $z = \hat{z}$ is a solution to the separable equation, or, returning to the original variables, we obtain the solution $y = \hat{z}x$, which is a line through the origin. These solutions usually play a special rôle and divide families of similar integral curves.

Example 4. Solve

$$y' = \frac{x+y}{x-y}.$$

The right hand side of this equation can be rearranged as

$$y' = \frac{1 + y/x}{1 - y/x} = f(y/x),$$

hence the equation is homogeneous. Make the substitution $y/x = z$, $y' = z'x + z$:

$$z'x + z = \frac{1+z}{1-z} \implies z'x = \frac{1+z}{1-z} - z = \frac{1+z^2}{1-z}.$$

This is a separable equations

$$\int \frac{1-z}{1+z^2}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x \implies \arctan z - \frac{1}{2}\ln(1+z^2) = \ln|x| + C,$$

or, for the original variable y:

$$\arctan\frac{y}{x} = \frac{1}{2}\ln(x^2+y^2) + C.$$

The last is true since

$$\frac{1}{2}\ln\left(1 + \frac{y^2}{x^2}\right) + \ln|x| = \frac{1}{2}\ln(x^2+y^2).$$

Can you figure out how the integral curves look like? This is another example when an analytical solution does not add much to our knowledge on the behavior of the solution. Note that in this example we do not have special solutions of the form $y = \hat{z}x$.

Actually, the form of the integral curves in this case can be found by switching to the polar coordinates
$$x = \rho\cos\theta, \quad y = \rho\sin\theta.$$
Convince yourself that in the new coordinates we will find that
$$\rho = Ce^{\theta},$$
which is simply a spiral!

An important lesson from the last example is that it is not always obvious that the equation we are dealing with is homogeneous. Here is a test how to figure it out automatically. I start with an auxiliary definition: A function $f(x, y)$ is called *homogeneous of degree* m if
$$f(tx, ty) = t^m f(x, y).$$
For example $x + y$ is homogeneous of degree 1, since $tx + ty = t(x + y)$, and $x^3y - y^4$ is homogeneous of degree 4 (check). Now I can state

Proposition 5. *The first order ODE*
$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$$
is homogeneous (in the sense of Definition 3) if and only if the functions $M(x, y)$ and $N(x, y)$ are homogeneous (in the sense of the most recent definition) functions of the same degree.

I leave it for the reader to work out a proof of this statement. We can reformulate this condition for the ODE in the form
$$y' = f(x, y).$$
The ODE $y' = f(x, y)$ is homogeneous if and only if $f(x, y)$ is a homogeneous function of degree zero. You should deduce this fact from the proposition above.

Example 6. Is this equation
$$(y^4 - 2x^3y)\,\mathrm{d}x + (x^4 - 2xy^3)\,\mathrm{d}y = 0$$
homogenous?

Since both $M(x, y)$ and $N(x, y)$ are homogeneous of degree 4, we have that our equation is homogeneous and can be solved with the method described above.

To solve this equation, rewrite it as
$$y' = \frac{2x^3y - y^4}{x^4 - 2xy^3} = \frac{2\frac{y}{x} - \left(\frac{y}{x}\right)^4}{1 - 2\left(\frac{y}{x}\right)^3}.$$
Now
$$y = xz \implies y' = xz' + z,$$
and hence we have
$$xz' = \frac{2z - z^4}{1 - 2z^3} - z = \frac{z(1 + z^3)}{1 - 2z^3}.$$

This is a separable equation, which can be integrated as

$$\int \frac{1-2z^3}{z(1+z^3)}\,\mathrm{d}z = \log|x| + C.$$

At this point it is convenient to note that

$$\frac{1-2z^3}{z(1+z^3)} = \frac{1+z^3-3z^3}{z(1+z^3)} = \frac{1}{z} - \frac{3z^2}{1+z^3}\,.$$

Therefore,

$$\log|z| - \log|1+z^3| = \log|x| + C \implies \frac{z}{1+z^3} = Cx,$$

or, finally returning to the original variables,

$$\frac{xy}{x^3+y^3} = C.$$

This formula gives the general solution to the original equation. However, by inspecting the equation, we can note that both $y = 0$ and $x = 0$ are also solutions. Moreover, while solving the corresponding separable equation, we divided by $z(1+z^3)$ which has the real roots $\hat{z} = 0$ (already accounted for in $y = 0$) and $\hat{z}_1 = -1$. The second root implies that there is solution to the original homogeneous equation of the form $y = \hat{z}_2 x$ or $y = -x$. In the figure you can see the integral curves of the equation; the integral curves from the general solution are shown in green, and the blue ones indicate the special solutions that separates other families of integral curves.

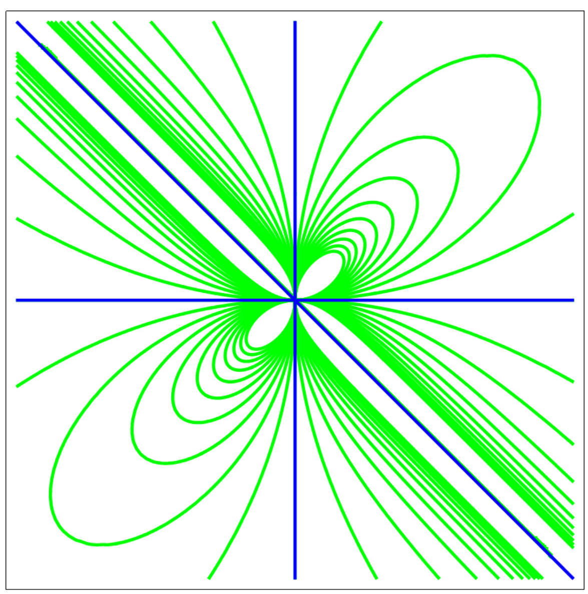

Figure 1: Integral curves of the equation $(y^4 - 2x^3y)\,\mathrm{d}x + (x^4 - 2xy^3)\,\mathrm{d}y = 0$.