3 Echelon form of a matrix

Recall from the first section that the elementary row operations are (i) switch the order of equations, (ii) multiply an equation by a non-zero constant, (iii) multiply an equation by a constant and add it to another equation. I will call them usually by these numbers (type 1, type 2, and type 3). Using these three operations I can always put any matrix into the so-called *row echelon form* and into the *reduced row echelon form*.

One can look at the elementary row operations as actually multiplication of a matrix by the so-called *elementary matrices* from the left. In particular, consider three *elementary matrices*:

Type (i) matrix is

$$\begin{bmatrix} 1 & & & & \\ & 0 & & 1 & \\ & & 1 & & \\ & 1 & & 0 & \\ & & & & 1 \end{bmatrix}.$$

Here I start with the identity matrix, the i-th and j-th diagonal entries are replaced by zeros, and at the i,j-th and j,i-th entries 1's are added.

Type (ii) matrix is

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \alpha & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \quad \alpha \neq 0.$$

Here in the identity matrix the i-th diagonal entry is replaced with constant or scalar $\alpha \neq 0$.

Type (iii) matrix is

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & \alpha & & 1 & \\ & & & & 1 \end{bmatrix}.$$

Here I start with the identity matrix and put α at the i,j-th position, $i \neq j$, $i > j$.

I claim that multiplication by these elementary matrices from the left amounts exactly to three elementary row operations:

- Multiplication by an elementary matrix of type (i) switches the i-th and j-th rows of the matrix;
- Multiplication by an elementary matrix of type (ii) multiplies the i-th row of the matrix by constant α;
- Multiplication by an elementary matrix of type (iii) takes the j-th row of a matrix, multiplies it by α, and adds the result to the i-th row.

One of course should check these rules (e.g., by looking how these matrices work on column vectors).

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Exercise 1. Check how the elementary matrices act on an arbitrary column vector.

Exercise 2. What if instead of an elementary matrix of type 3 I will take a very similar matrix, but with $i < j$? Will anything change?

Lemma 3.1. *Elementary matrices are invertible and their inverses are also elementary matrices.*

Proof. We can always "undo" the row operations by switching equations i and j again, multiplying by $1/\alpha$, and by taking the j-th equation, multiplying by $-\alpha$, and adding to the i-th row, and therefore we have an explicit form for the inverse matrices. ■

Remark 3.2. The proof I gave above is quite vague, but so are many other proofs. Importantly, it contains the main ideas, which the student should be able to extend to a full careful proof if needed. As an example, let me prove that Type 1 elementary matrix is invertible in a somewhat more rigorous manner.

Let me denote this matrix $\boldsymbol{E}$. Since multiplication of type 1 matrix corresponds to switching rows i and j of the matrix, then, $\boldsymbol{EI}$ means that I switch the i-th and j-th rows of $\boldsymbol{I}$. Let me consider $\boldsymbol{EEI} = \boldsymbol{I}$ because I switch rows i and j twice and hence do not change anything. Therefore, $\boldsymbol{E}$ is the inverse to itself, and hence any elementary matrix of type 1 is invertible.

Exercise 3. Expand the proof of the lemma above for the elementary matrices of type 2 and 3; i.e., by writing explicitly the inverse matrix for these types.

Now it should be clear that when I talk about elementary row operations, what I am actually doing mathematically is multiplying the original matrix $\boldsymbol{M}$ from the left by a sequence of elementary matrices:

$$\boldsymbol{M}' = \boldsymbol{E}_k \dots \boldsymbol{E}_2 \boldsymbol{E}_1 \boldsymbol{M},$$

where each $\boldsymbol{E}_i$ is an elementary matrix of one of three discussed types.

This process of simplifying the original matrix is called *the row reduction* and it consists of three sub-steps:

1. Find the leftmost non-zero column of the matrix;

2. Make sure, by switching the order of equations, that the upper entry is non-zero. This entry will be called *pivot*;

3. Get rid of all the nonzero entries below the pivot by using the third type elementary operations.

I apply this main step first to the original matrix, then I leave the first row alone and apply the same step to all the rows except the first one, after this to the rows rows one and two and so on. And since the number of rows is finite, I will always end up with a matrix in a *row echelon form.*

Example 3.3. Consider, for example, the following system

$$\begin{aligned} x_1 + x_2 + 2x_3 + x_4 &= 5, \\ x_1 + x_2 + 2x_3 + 6x_4 &= 10, \\ x_1 + 2x_2 + 5x_3 + 2x_4 &= 7. \end{aligned}$$

I start by writing the augmented matrix and them proceed with the row reduction

$$\left[\begin{array}{cccc|c}1&1&2&1&5\\1&1&2&6&10\\1&2&5&2&7\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&2&1&5\\0&0&0&5&5\\0&1&3&1&2\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&2&1&5\\0&1&3&1&2\\0&0&0&5&5\end{array}\right].$$

We have three pivots, and the solution can be read from this matrix. I notice that $x_4 = 1$, and x_1 and x_2 can be expressed through x_3, which is a *free variable* in my case. After some simplification I get

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}3+t\\1-3t\\t\\1\end{bmatrix}=\begin{bmatrix}3\\1\\0\\1\end{bmatrix}+t\begin{bmatrix}1\\-3\\1\\0\end{bmatrix},\quad t\in\mathbf{R},$$

where the notation $t \in \mathbf{R}$ means that t is an arbitrary real number.

To determine exactly when my matrix in the row echelon form, I notice that two conditions must be satisfied:

1. All zero rows are below nonzero rows;
2. For any nonzero row its pivot is strictly to the right of the pivot from the previous row.

Instead of making the back substitution as above, I could have proceeded with further simplifications of my matrix, to put it into the *reduced row echelon form.* A matrix is in *reduced row echelon form* if it is in the row echelon form and, additionally,

1. All the pivots are equal to 1.
2. All the entries above and below pivots are zero.

Clearly I can always achieve this by the second and third type elementary row operations.

Continuing my example, I get

$$\left[\begin{array}{cccc|c}1&0&-1&0&3\\0&1&3&1&2\\0&0&0&1&1\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&-1&0&3\\0&1&3&0&1\\0&0&0&1&1\end{array}\right].$$

Here you can see that the only column that has no pivot is column 3, and hence I have only one *free* variable, x_3.

I proved in the first section that elementary row operations do not produce new solutions. The proof there was somewhat lame, mainly due to the lack of convenient definitions and notations. Let me prove this fact again, using the introduced notations for the elementary matrices.

Proposition 3.4. *Consider two linear systems* $\boldsymbol{Ax}=\boldsymbol{b}$ *and* $\boldsymbol{A}'\boldsymbol{x}=\boldsymbol{b}'$, *such that the second one is obtained from the first by elementary row operations. Then they have the same solutions.*

Proof. Let me denote the augmented matrices as $\boldsymbol{M}=[\boldsymbol{A}|\boldsymbol{b}]$ and $\boldsymbol{M}'=[\boldsymbol{A}'|\boldsymbol{b}']$ of the first and second systems respectively. I know that

$$\boldsymbol{M}'=\boldsymbol{E}_k\ldots\boldsymbol{E}_1\boldsymbol{M}=\boldsymbol{PM}.$$

Matrix $\boldsymbol{P}$ is invertible as a product of invertible matrices, with the inverse $\boldsymbol{P}^{-1}$. Now, if $\hat{\boldsymbol{x}}$ solves the first system, i.e., $\boldsymbol{A}\hat{\boldsymbol{x}} = \boldsymbol{b}$, then it also solves the second one, since it is given by $\boldsymbol{PA}\hat{\boldsymbol{x}} = \boldsymbol{Pb}$. In the opposite direction, if $\tilde{\boldsymbol{x}}$ solves the second system then it also solves the first one, since it is obtained as $\boldsymbol{P}^{-1}\boldsymbol{A}'\tilde{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{b}'$. ■

To conclude, if one needs to solve a system of linear equations, then first an augmented matrix must be written, this matrix must be put into the reduced row echelon form, the columns without pivots correspond to the free variables that must be moved to the right hand sides. All other variables are expressed uniquely through the free variables. After assigning arbitrary values to the free variables we obtain all possible solutions to the original system. In short, the augmented matrix of the system of linear equations in the reduced row echelon form tells me everything about possible solutions.

Proposition 3.5. *Let $\boldsymbol{M}' = [\boldsymbol{A}'|\boldsymbol{b}']$ be an augmented matrix in the reduced row echelon form. Then the system $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$ has a solution if and only if there are no pivots in the last column of $\boldsymbol{M}'$. If the system has a solution (it is consistent), then this solution is unique if there are no free variables; and there are infinitely many solutions if free variables are present.*

Proof. I will prove only the first claim leaving the rest to the reader. Recall that if you are required to prove something with the wording "if and only if" this means that you need to consider two cases or two directions. Assume first that we have a pivot in the last column. This gives me an equation, e.g., $0x_1 + \ldots + 0x_n = 1$, which cannot hold for any $\boldsymbol{x} = [x_1 \ \ldots \ x_n]$, hence no solution. If there is no pivot in the last column then I can always take some arbitrary values for my free variables, and all other variables are determined uniquely, and hence I have a solution. ■

Exercise 4. Prove the remaining statements in the proposition above.

A little more terminology. A system of linear algebraic equations $\boldsymbol{Ax} = \boldsymbol{b}$ is called *nonhomogeneous* if $\boldsymbol{b} \neq \boldsymbol{0}$ and *homogeneous* if $\boldsymbol{b} = \boldsymbol{0}$. Note that a homogeneous system always has a (trivial) solution $\boldsymbol{x} = \boldsymbol{0}$.

Proposition 3.6. *Every system $\boldsymbol{Ax} = \boldsymbol{0}$ with m equations and n unknowns has a non trivial solution if $m < n$.*

Proof. Since $m < n$ the number of pivots is less than the number of variables n, and hence I have some free variables, which can be assigned arbitrary values. ■

Lemma 3.7. *A square matrix $\boldsymbol{M}$ in the reduced row echelon form is either the identity matrix $\boldsymbol{I}$ or has a bottom row of zeros.*

Proof. Since I have n columns and n rows then I have at most n pivots. If there are n of them then the reduced row echelon form is $\boldsymbol{I}$, if there are fewer then n pivots then some rows are zero, including the bottom row. ■

Theorem 3.8. *The following conditions are equivalent for a square matrix $\boldsymbol{A}$*

(*A*) *The reduced row echelon form for $\boldsymbol{A}$ is the identity matrix*;

(*B*) *$\boldsymbol{A}$ is a product of elementary matrices*;

(*C*) *$\boldsymbol{A}$ is invertible.*

Proof. $(A) \implies (B)$. I assume that there are elementary matrices such that

$$\boldsymbol{E}_k \dots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I}.$$

Now I can multiply both sides by $\boldsymbol{E}_1^{-1} \dots \boldsymbol{E}_k^{-1}$ to get

$$\boldsymbol{A} = \boldsymbol{E}_1^{-1} \dots \boldsymbol{E}_k^{-1}.$$

$(B) \implies (C)$ This is true since the product of invertible matrices is invertible.

$(C) \implies (A)$ First I note that an invertible matrix cannot have a whole row of zeros. If $\boldsymbol{A}$ is invertible, so is its reduced row echelon form $\boldsymbol{A}'$, which also cannot have a row of all zeros, and hence, by Lemma 3.7, is the identity matrix. ■

Theorem 3.8, among other things, provides an algorithm to compute the inverse matrix. I have that for an invertible matrix I must have

$$\boldsymbol{E}_k \dots \boldsymbol{E}_1 \boldsymbol{A} = \boldsymbol{I},$$

or

$$\boldsymbol{E}_k \dots \boldsymbol{E}_1 \boldsymbol{I} = \boldsymbol{A}^{-1}.$$

Hence to compute the inverse matrix one must apply the elementary row operations, reducing the original matrix to the identity matrix, to $\boldsymbol{I}$. As shown above, the same sequence of operations, which applied to the identity matrix $\boldsymbol{I}$, yields $\boldsymbol{A}^{-1}$.

Example 3.9. To find the inverse matrix for

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$$

I write first

$$\left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right]$$

and perform the elementary operations

$$\left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & -4 & -2 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 5 & 1 & 0 \\ 0 & 1 & 1/2 & -1/4 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & -3/2 & 5/4 \\ 0 & 1 & 1/2 & -1/4 \end{array}\right].$$

Exercise 5. Check that the found matrix is indeed the inverse.

Finally I can state and prove the main theorem about the system of linear equations with a square matrix.

Theorem 3.10. *The following are equivalent for a square matrix* $\boldsymbol{A}$*:*

- (A) $\boldsymbol{A}$ *is invertible*;
- (B) *The system* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ *has a unique solution for any* $\boldsymbol{b}$;
- (C) *The system of homogeneous equations* $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ *has only the trivial solution* $\boldsymbol{x} = \boldsymbol{0}$*.*

Proof. $(A) \implies (B)$. If $\boldsymbol{A}$ is invertible then its reduced row echelon form $\boldsymbol{A}'$ is the identity matrix, and hence the equivalent system $\boldsymbol{A}'\boldsymbol{x} = \boldsymbol{b}'$ has the unique solution $\boldsymbol{b}'$.

It should be clear that $(B) \implies (C)$.

To show that $(C) \implies (A)$ I will consider logically equivalent not (A) implies not (C). Much more on logic will be discussed later.

If $\boldsymbol{A}$ is not invertible then its row reduced echelon form will have a row of zeros and hence there are fewer than n pivots, and hence there are some free variables and therefore the homogeneous system has a non-trivial solution. ■

Remark 3.11. Note that I also proved that if a homogeneous system has only the trivial solution then the corresponding non-homogenous system has a unique solution for any $\boldsymbol{b}$.

Finally, let me mention that all the same elementary operations can be performed on columns. The easy way to see it is to introduce the operation of taking the *transpose* of a matrix.

Definition 3.12. *Let $\boldsymbol{A}$ be an $m \times n$ matrix. Its transpose, denoted $\boldsymbol{A}^\top$, is defined as*

$$[a_{ij}^\top] = [a_{ji}],$$

and hence an $n \times m$ matrix.

The role of rows and columns is interchanged by the transpose operation on matrices, as the following lemma shows.

Lemma 3.13. *Let $\boldsymbol{A}, \boldsymbol{B}$ be two matrices such that the product $\boldsymbol{AB}$ is defined. Then*

$$(\boldsymbol{AB})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top.$$

Exercise 6. Prove this lemma.

Exercise 7. Prove that

$$\begin{aligned}
(\boldsymbol{A}^\top)^\top &= \boldsymbol{A}, \\
(\alpha \boldsymbol{A})^\top &= \alpha \boldsymbol{A}^\top, \\
(\boldsymbol{A} + \boldsymbol{B})^\top &= \boldsymbol{A}^\top + \boldsymbol{B}^\top,
\end{aligned}$$

and finally if $\boldsymbol{A}$ is invertible, so is $\boldsymbol{A}^\top$ and $(\boldsymbol{A}^\top)^{-1} = (\boldsymbol{A}^{-1})^\top$.

Now we can deduce the facts about *right multiplication* from the corresponding facts about *left multiplication*. In particular, the right multiplications by the elementary matrices corresponds to three elementary column operations.

Exercise 8. Think the last statement out carefully.

For future reference I finish this section with two more results, which I will use (directly or indirectly) later. The proof of the first result is assigned as a homework exercise.

First, for an arbitrary matrix $\boldsymbol{A}$ (note, I do not assume that $\boldsymbol{A}$ is square) I define *the left inverse* $\boldsymbol{L}$ as a matrix for which $\boldsymbol{LA} = \boldsymbol{I}$, and the *right inverse* $\boldsymbol{R}$ as a matrix for which $\boldsymbol{AR} = \boldsymbol{I}$.

Theorem 3.14. *Let* $\boldsymbol{A}$ *be a square matrix that has either a left inverse or a right inverse, a matrix* $\boldsymbol{B}$ *such that either* $\boldsymbol{BA} = \boldsymbol{I}$ *or* $\boldsymbol{AB} = \boldsymbol{I}$*. Then* $\boldsymbol{A}$ *is invertible and* $\boldsymbol{B}$ *its inverse.*

And the second result I mentioned is (intuitively obvious)

Theorem 3.15. *The row reduced echelon form of matrix* $\boldsymbol{A}$ *is unique.*