12 Liouville’s theorem. Fundamental theorem of algebra

One of the immediate consequences of Cauchy’s integral formula is Liouville’s theorem, which states that an entire (that is, holomorphic in the whole complex plane \( \mathbb{C} \)) function cannot be bounded if it is not constant. This profound result leads to arguably the most natural proof of \emph{Fundamental theorem of algebra}. Here are the details.

12.1 Liouville’s theorem

\textbf{Theorem 12.1.} Let \( f \) be entire and bounded. Then \( f \) is constant.

\textit{Proof.} Take two arbitrary points \( a, b \in \mathbb{C} \) and let \( \gamma_R \) be the circle \( \partial B(0, R) \), where \( R \) is chosen so big that \( |z - a| \geq R/2 \) and \( |z - b| \geq R/2 \) for all points \( z \in \gamma_R \). For both points Cauchy’s integral formula holds:

\[ f(a) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z-a} \, dz, \quad f(b) = \frac{1}{2\pi i} \int_{\gamma_R} \frac{f(z)}{z-b} \, dz. \]

Therefore,

\[ |f(a) - f(b)| = \left| \frac{a-b}{2\pi} \int_{\gamma_R} \frac{f(z)}{(z-a)(z-b)} \, dz \right|. \]

By the assumption \(|f(z)| \leq M\) for some constant \( M \) and hence on \( \gamma_R \)

\[ \left| \frac{f(z)}{(z-a)(z-b)} \right| \leq \frac{4M}{R^2}. \]

This implies, by the ML-inequality,

\[ |f(a) - f(b)| \leq \frac{4M |a-b|}{R^2} \text{length } \gamma_R = |a-b| \frac{4M}{R} \to 0, \quad R \to \infty. \]

Therefore \( f(a) = f(b) \), and since \( a, b \) were arbitrary, \( f \) must be constant. \hfill \Box

12.2 Fundamental theorem of algebra

Recall that a polynomial \( p: \mathbb{C} \to \mathbb{C} \) is a function of the form

\[ p(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0, \]

where \( c_0, \ldots, c_n \) are some given complex constant. I assume that \( c_n \neq 0 \). If \( n \geq 1 \) then the polynomial is non-constant. Any polynomial is an entire function, and therefore continuous. Complex number \( \hat{z} \) is called a root of polynomial \( p \) if \( p(\hat{z}) = 0 \). Now I am ready to state and, more importantly, prove the following theorem.

\textbf{Theorem 12.2.} Every non-constant polynomial \( p \) has at least one root \( \hat{z} \in \mathbb{C} \). 

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Proof. I will prove this theorem by contradiction assuming that there is a nonconstant polynomial \( p \) with no root. First I note that for any non-constant polynomial \( p(z) \to \infty \) as \( z \to \infty \) (recall that complex plane has only one infinity). Indeed,

\[
p(z) = z^n \left( c_n + \frac{c_{n-1}}{z} + \ldots + \frac{c_0}{z^n} \right),
\]

and since \( z^n \to \infty \) as \( z \to \infty \) I obtain the required conclusion. Now take sufficiently large ball \( B(0, R) \). By the proven \( |p(z)| > M \) outside this ball for some constant \( M \), and I can always choose \( R \) big enough to guarantee that

\[
\frac{1}{|p(z)|} < 1,
\]

for all \( z \notin B(0, R) \). Since I assumed that \( p \) has no roots, function

\[
z \mapsto \frac{1}{p(z)}
\]

is entire, and in particular it is holomorphic inside \( B(0, R) \), and hence continuous on \( \overline{B}(0, R) \). Continuous functions on compact sets attain their maximum and minimum values and in particular bounded, hence

\[
\frac{1}{|p(z)|} \leq B,
\]

for some constant \( B \) and \( z \in \overline{B}(0, R) \). This implies that the entire function \( 1/p \) is bounded in \( \mathbb{C} \) and hence, by Liouville’s theorem, constant, which contradicts the assumption that \( p \) is non-constant.

Remark 12.3. In a more algebraic fashion the previous theorem sometimes is stated as “The field of complex numbers \( \mathbb{C} \) is algebraically closed.”

Here is an important consequence of this theorem, which sometimes also called the fundamental theorem of algebra.

Theorem 12.4. Any complex polynomial \( p: \mathbb{C} \to \mathbb{C} \) can be uniquely factored (up to the permutation of the factors) as

\[
p(z) = c_n(z - \hat{z}_1)^{\alpha_1}(z - \hat{z}_2)^{\alpha_2}\ldots(z - \hat{z}_k)^{\alpha_k},
\]

where \( \hat{z}_1, \ldots, \hat{z}_k \) are the roots of \( p \), and \( \alpha_1, \ldots, \alpha_k \) are the corresponding multiplicities, that satisfy \( \alpha_1 + \ldots + \alpha_k = n \).

In other words, every complex polynomial of degree \( n \) has exactly \( n \) complex roots counting each root according to its multiplicity.

Proof. Let \( p \) be a polynomial of degree \( n \). If \( n = 0 \) we are done. If \( n \geq 1 \) by the fundamental theorem of algebra there must be \( \hat{z} \in \mathbb{C} \) such that

\[
p(\hat{z}) = 0.
\]

I claim that this is equivalent to the fact that polynomial \( p \) can be written as the product of \( (z - \hat{z}) \) and another polynomial \( q \) of degree \( n - 1 \):

\[
p(z) = (z - \hat{z})q(z).
\]
Indeed, one direction is obvious (let \( \hat{z} \) be such that \( p(\hat{z}) = 0 \) then \( (z - \hat{z})q(z) = 0 \) as well). In the other direction, I will show even more. Specifically, no matter what the number \( \hat{z} \in \mathbb{C} \) is (non necessarily a root), there exist complex constant \( a_1, \ldots, a_n \) such that polynomial can be written as

\[
p(z) = a_0 + a_1(z - \hat{z}) + \ldots + a_n(z - \hat{z})^n.
\]

The proof is direct, by construction. Take \( w = z - \hat{z}, \) therefore \( z = w + \hat{z}, \) and \( p(z) = p(w + \hat{z}) = \tilde{p}(w) = c_0 + \ldots + c_n(w + \hat{z})^n. \) Raise all the terms to the corresponding power, simplify, and end up with \( \tilde{p}(w) = a_0 + a_1w + \ldots + a_nw^n; \) hence \( p(z) = \tilde{p}(z - \hat{z}) \) has the required form. Note that \( c_n = a_n \) and \( a_0 = p(\hat{z}). \)

Now I can rearrange my polynomial as follows

\[
p(z) = a_0 + (z - \hat{z})(a_1 + a_2(z - \hat{z}) + \ldots + a_n(z - \hat{z})^{n-1}) = a_0 + (z - \hat{z})q(z),
\]

where \( q \) has degree \( n - 1. \) From the last expression I obtain the required conclusion that if \( p(z) = (z - \hat{z})q(z) \) then \( \hat{z} \) must be a root (since \( a_0 = 0 = p(\hat{z}). \))

We are basically done, since, due to fact that \( q \) is a complex polynomial of degree \( n - 1, \) it is either constant \( (n = 1) \) or, by fundamental theorem of algebra, has a complex root \( \hat{w}, \) therefore,

\[
q(z) = (z - \hat{w})h(z),
\]

where \( h \) has degree \( n - 2, \) and so on.

I will leave to prove the uniqueness of such factorization to the reader.

\[\square\]

**Remark 12.5.** In complex analysis very often the term “root” is replaced with the term “zero.” And in general zeros of functions and their corresponding multiplicities are studied. In the following I will stick to this terminology.

**Example 12.6.** Consider

\[
p(z) = z^4 + (2 + i)z^3 - (2 - 5i)z^2 - (4 + 5i)z + (3 - i),
\]

which is a polynomial of degree 4. Therefore there must be 4 (counting multiplicities) complex roots. Indeed, one can check that \( i \) is a root of multiplicity 2 and \( 1, -3 + i \) are the roots multiplicity 1, that is,

\[
p(z) = (z + i)^2(z - 1)(z + 3 - i).
\]