13 Lyapunov functions

13.1 Definition and main theorem

Up till now, for a general system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in \mathbf{R}^2 \tag{1}$$

we have two methods to get insight about the structure of the phase portrait. First, we can study stability of an equilibrium using linearization of (1) around this equilibrium. Second, we can draw the nullclines to infer some global behavior. There methods, as should be clear from some of the examples in class and homework, have a lot of limitations. Nullclines, for instance, often do not lead to precise conclusions; linearization can be used only if the equilibrium is hyperbolic, that is, if all the eigenvalues of the Jacobi matrix evaluated at the studied equilibrium have no zero real parts. Moreover, linearization is an essentially *local* method and does not say anything about, e.g., the *basin of attraction* of an asymptotically stable equilibrium.

In this section I introduce yet another powerful device to study autonomous systems of ODE — the so-called *Lyapunov functions.* Let me first introduce a positive definite function. Function $V\colon U \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$ is said to be *positive definite* in U if (*i*) $V(0) = 0$ and (*ii*) $V(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in U$ such that $\boldsymbol{x} \neq 0$. Similarly I can define a *negative definite* function.

Example 1. Clearly,

$$V(x, y) = x^2 + y^2$$

is positive definite on all $\mathbf{R}^2$.

Function

$$V(x, y) = x^2 + y^2 - y^3$$

is positive definite only in a small strip along the x-axis.

Function

$$V(x, y) = x + y^2$$

is no positive definite in any open U containing the origin.

Example 2. A flexible family of positive definite functions is given by

$$V(x, y) = ax^2 + 2bxy + cy^2,$$

where the parameters must satisfy the conditions $a > 0$ and $ac - b^2 > 0$ (prove it).

To get a general idea what is the geometry of a positive definite function, note that positive definite V has a strict isolated minimum at $\boldsymbol{x} = 0$. Therefore, intuitively I can conclude that the level sets of my function, i.e., the sets of the form

$$\{\boldsymbol{x} \in \mathbf{R}^2 \colon V(x) = k\}$$

are closed curves at least for some small k (see Fig. 1).

Now geometrically it should be clear that if the vector field of system (1) points inside the level sets of a positive definite function at each point than there is no way the orbits can leave a small

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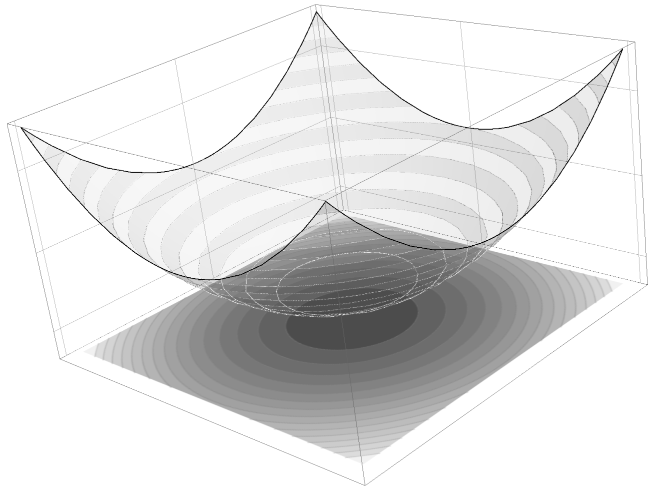

Figure 1: Graph of a positive definite function V and its level sets.

neighborhood of the origin. Algebraically, I can express this condition as the derivative of V along the vector field defined by (1). Namely,

$$\begin{aligned}\dot V(x(t),y(t)) &= \frac{\partial V}{\partial x}(x(t),y(t))\dot x(t) + \frac{\partial V}{\partial y}(x(t),y(t))\dot y(t)\\ &= \frac{\partial V}{\partial x}(x(t),y(t))f(x(t),y(t)) + \frac{\partial V}{\partial y}(x(t),y(t))g(x(t),y(t))\\ &= \nabla V(x,y)\cdot \boldsymbol{f}(x,y),\end{aligned}$$

where I used the notations for the gradient $\nabla V = (\partial_x V, \partial_y V)$ and the usual dot product $\boldsymbol{x}\cdot\boldsymbol{y}$ of two vectors $\boldsymbol{x},\boldsymbol{y}\in\mathbf{R}^2$.

Recall that

$$\boldsymbol{x}\cdot\boldsymbol{y} = x_1y_1 + x_2y_2 = |\boldsymbol{x}||\boldsymbol{y}|\cos\theta,$$

and since the vector gradient points in the same direction as the normal to the level set, then it is (again, intuitively, there are a lot of subtle details here) expected if $\nabla V\cdot\boldsymbol{f}\leq 0$ then we can say samething about the stability of our equilibrium point. Indeed,

Theorem 3. *Let $\hat{\boldsymbol{x}} = 0$ be an equilibrium of* (1) *and V be a positive definite function on some neighborhood of* 0*. Then*

1. *If $\dot V\leq 0$ for all $\boldsymbol{x}\in U$, $\boldsymbol{x}\neq 0$ then $\hat{\boldsymbol{x}}$ is Lyapunov stable;*
2. *If $\dot V < 0$ for all $\boldsymbol{x}\in U$, $\boldsymbol{x}\neq 0$ then $\hat{\boldsymbol{x}}$ is asymptotically stable;*
3. *If $\dot V > 0$ for all $\boldsymbol{x}\in U$, $\boldsymbol{x}\neq 0$ then $\hat{\boldsymbol{x}}$ is unstable.*

Proof. I will give a proof of the first claim leaving the rest as (similar) exercises.

Pick an $\epsilon > 0$ such that the ball $B_\epsilon\colon |\boldsymbol{x}|\leq\epsilon\subseteq U$. Let S_ϵ be the boundary of B_ϵ. Since V is continuous there must be a point on S_ϵ where V attains its minimum, denote this minimum as $\beta = \min_{\boldsymbol{x}\in S_\epsilon} V(\boldsymbol{x})$, and note that $\beta > 0$. Consider yet another ball $B_\delta\colon |\boldsymbol{x}|\leq\delta\subseteq U$. Since $V(0)=0$

and V continuous I can always choose $\delta > 0$ such that $V(\boldsymbol{x}) < \beta$ for all $\boldsymbol{x} \in B_\delta$. My goal is to show that if $\boldsymbol{x}_0 \in B_\delta$ then $\boldsymbol{x}(t, \boldsymbol{x}_0) \in B_\epsilon$ for all $t > 0$. Since $\dot{V}(\boldsymbol{x}) \leq 0$ then $V(\boldsymbol{x}_0) < \beta$ implies that $V(\boldsymbol{x}(t, \boldsymbol{x}_0)) < \beta$ for all $t > 0$ because non-positive derivative implies that V does not increase along the orbit with the initial condition $\boldsymbol{x}_0$. This, in its turn, implies that $\boldsymbol{x}(t, \boldsymbol{x}_0)$ cannot cross S_ϵ since $V(\boldsymbol{x}) \geq \beta$ for all $\boldsymbol{x} \in S_\epsilon$. ■

Remark 4. To distinguish between the first and second cases, the Lyapunov function that satisfies (ii) is usually called *strict Lyapunov function*.

Remark 5. Note that the third condition of the theorem above is quite strong, since for the instability it is enough just one orbit to leave a neighborhood of the origin, whereas I ask that *all* the orbits will leave this neighborhood. There exists a nice development along these lines, which allows to prove the instability for more general cases. Please google *Chetaev's theorem* for more information.

Lyapunov function actually allows to gain some information about the global behavior of orbits. First, let me define *the basin of attraction* of an asymptotically stable equilibrium at the set of all initial conditions leading to the long term behavior that approach this equilibrium. More technically, the basin of attraction $B(\hat{\boldsymbol{x}})$ of the equilibrium $\boldsymbol{x}$ is

$$B(\hat{\boldsymbol{x}}) = \{\boldsymbol{x}_0 \in \mathbf{R}^2 \colon \boldsymbol{x}(t, \boldsymbol{x}_0) \to \hat{\boldsymbol{x}} \text{ as } t \to \infty\}.$$

Now the idea is to look at the sets of the form $U_\alpha = \{\boldsymbol{x} \colon V(\boldsymbol{x}) \leq \alpha\}$. Again, geometrically intuitively correct (but requires proof!), is to expect that if V is strict in U and $U_\alpha \subseteq U$ for some α then $U_\alpha \subseteq B(\hat{\boldsymbol{x}})$. Even more impressive, in some cases I can prove that $\hat{\boldsymbol{x}}$ is *globally asymptotically stable*, which means that no matter what my initial conditions are, the solution eventually will approach $\hat{\boldsymbol{x}}$. Specifically, if I assume that V is a strict Lyapunov function for $\mathbf{R}^2$, and $V(\boldsymbol{x}) \to \infty$ for $|\boldsymbol{x}| \to \infty$ then $\hat{\boldsymbol{x}}$ is globally asymptotically stable.

13.2 Examples

Example 6. I start with the simplest possible example

$$\begin{aligned}\dot{x} &= -x,\\ \dot{y} &= -y.\end{aligned}$$

Of course, I can simply solve this system, but let me look instead at

$$V(x, y) = \frac{x^2 + y^2}{2},$$

which is a positive definite function on all $\mathbf{R}^2$.

I calculate

$$\dot{V}(x, y) = -(x^2 + y^2),$$

which is negative for all $(x, y) \neq 0$. Therefore, according to my theorem, I can conclude that the equilibrium $(\hat{x}, \hat{y}) = (0, 0)$ is asymptotically stable. Moreover, since $\dot{V} < 0$ for all $\mathbf{R}^2$ and clearly $V(x, y) \to \infty$ if $\sqrt{x^2 + y^2} \to \infty$, I can say that my equilibrium is globally asymptotically stable.

Example 7. Consider the system

$$\dot{x} = -x + y^2,$$
$$\dot{y} = -2y + 3x^2.$$

This system has two equilibria, one at the origin, which is locally asymptotically stable (by linearization), and hence another one unstable (why?). Can we somehow estimate the basin of attraction of $(0, 0)$? Consider the Lyapunov function

$$V(x, y) = \frac{x^2}{2} + \frac{y^2}{4},$$

which is positive definite. Now calculate

$$\dot{V}(x, y) = -x^2 + xy^2 - y^2 + \frac{3}{2}yx^2 = -x^2(1 - \frac{3}{2}y) - y^2(1 - x),$$

which is negative for all (x, y) that satisfy $x < 1$ and $y < 2/3$. This clearly indicates, as we know, that the origin is asymptotically stable. To gain an idea of the basin of attraction, we must find the largest region around $(0, 0)$ where $V(x, y) \leq \alpha$ and still be negative definite. Since the level sets of V are the ellipses with the axes $\sqrt{2\alpha}$ and $2\sqrt{\alpha}$ hence we must have that $\sqrt{2\alpha} < 1$ and $2\sqrt{\alpha} < 2/3$, which implies that $\alpha < 1/9$, which means that the largest region that we can be sure lays inside the basin of attraction is the ellipse

$$\frac{x^2}{2} + \frac{y^2}{4} = \frac{1}{9}.$$

However, as numerical illustration shows, the actual basin of attraction is way bigger, but still smaller than the whole plane (see Fig. 2).

Example 8. Now let us consider again our familiar model of the pendulum

$$\ddot{x} + \sin x = 0,$$

which can be written as the system of two first order equations

$$\dot{x} = y,$$
$$\dot{y} = -\sin x.$$

We already know the phase portrait of this system (see Lecture 9), but here let me use the knew machinery of Lyapunov functions to establish that the origin is Lyapunov stable.

As a candidate of Lyapunov function let me take

$$V(x, y) = \frac{y^2}{2} + 1 - \cos x.$$

Note that in a small neighborhood of $(0, 0)$ my V is positive definite. Now

$$\dot{V}(x, y) = y \sin x + y(-\sin x) = 0,$$

and hence my V is an example of a Lyapunov function, but not strict Lyapunov function. Therefore, I can conclude, as I already know, that the origin is Lyapunov stable.

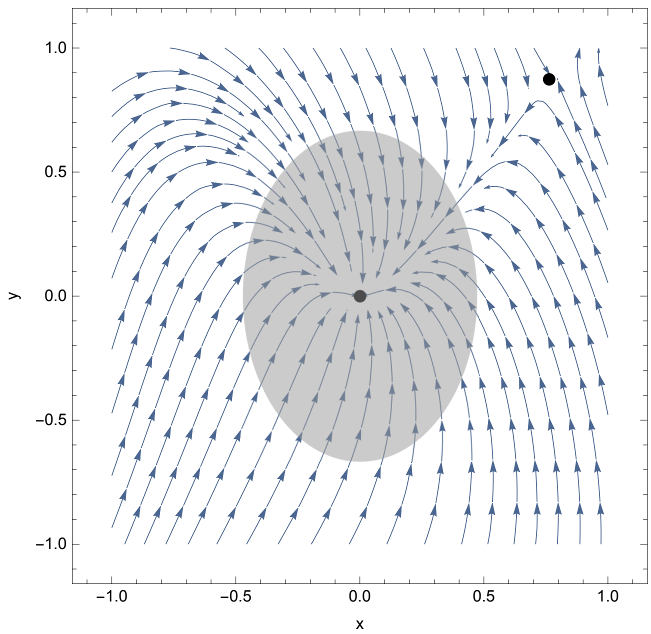

Figure 2: Phase portrait of the system from Example 2. The shaded area is the identified subset of the basin of attraction of the origin. The thick circles are two equilibria.

This example is quite special because $\dot{V} = 0$ for all (x, y). Such functions, for which $\dot{V} = 0$ with respect to the orbits of some system of differential equations, are called *first integrals*. Note that as a first integral I picked the total energy of the system. It can be proved that if V is a first integral then the orbits are laying in the level sets of V: Indeed, the gradient ∇V is directed along the normal to the level sets, whereas the dot product equals to zero means that two vectors are orthogonal, and hence the vector field is orthogonal to the normal, and therefore belongs to the level set.

Now I will include friction in my problem:

$$\ddot{x} + s\dot{x} + \sin x = 0, \quad s > 0.$$

I can rewrite my equation as the system

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -sy + \sin x, \end{aligned}$$

which has the same equilibria as the conservative pendulum equation. Here, however, I can use the standard linearization approach to conclude that the origin is locally asymptotically stable (do this).

Let me try the same Lyapunov function

$$V(x, y) = \frac{y^2}{2} + 1 - \cos x.$$

Now I calculate

$$\dot{V}(x, y) = -sy^2,$$

which is unfortunately not strict. Hence the only thing I can conclude by Lyapunov function approach is that the origin is Lyapunov stable. There is a very efficient way to deal with this case, please google *LaSalle invariance principle* if you are interested.

Example 9. Finally, we will be able to fill one of the gaps in our previous analysis. Recall that Lotka–Volterra predator-prey model takes the form

$$\dot{x} = x(1 - y),$$
$$\dot{y} = y(-\gamma + x),$$

if I do not include the intraspecific competition. We already know that there is an internal equilibrium, which is Lyapunov stable but not asymptotically stable (and of course the linearization cannot detect it).

Let me take the following function

$$H(x, y) = \hat{x} \log x - x + \hat{y} \log y - y,$$

where $(\hat{x}, \hat{y}) = (\gamma, 1)$ are the coordinates of the internal equilibrium.

This function is not positive definite, but if I use it to build

$$V(x, y) = H(\hat{x}, \hat{y}) - H(x, y),$$

then one can check that V is positive definite everywhere in the inside of $\mathbf{R}^2_+$ and $V(\hat{x}, \hat{y}) = 0$, so I can try it as a candidate for a Lyapunov function. Indeed,

$$\dot{V} = -\hat{x}(1-y)+x(1-y)-\hat{y}(-\gamma+x)+y(-\gamma+x) = (1-y)(x-\hat{x})+(x-\gamma)(y-\hat{y}) = (\hat{y}-y)(x-\hat{x})+(x-\hat{x})(y-\hat{y}) = 0,$$

where I used the fact $\hat{x} = \gamma, \hat{y} = 1$.

Hence my V is a Lyapunov function, and I can conclude that the internal equilibrium is Lyapunov stable. Moreover, H and V are both first integrals, and hence my orbits are laying on their level sets.

Now I add the intraspecific competition and must deal with

$$\dot{x} = x(1 - \alpha x - y),$$
$$\dot{y} = y(-\gamma + x - \beta y).$$

Recall that $\alpha\gamma < 1$ then there is the internal equilibrium $(\hat{x}, \hat{y})$, that solves the system

$$1 = \alpha\hat{x} + \hat{y},$$
$$\gamma = \hat{x} - \beta\hat{y}.$$

We found, by linearization, that this equilibrium is locally asymptotically stable, but the nullcline analysis did not allow us to conclude that all the orbits are attracted to this equilibrium.

Let me take again the same V, note that $\hat{x}, \hat{y}$ are different. Then,

$$\dot{V}(x, y) = (x - \hat{x})(1 - \alpha x - y) + (y - \hat{y})(-\gamma + x + \beta y) = -\alpha(x - \hat{x})^2 - \beta(x - \hat{y})^2,$$

which satisfies $\dot{V}(x, y) < 0$ for all $(x, y) \in \mathbf{R}^2_+$, $x \neq 0, y \neq 0$ and $x \neq \hat{x}, y \neq \hat{y}$. This implies that no possible closed orbit can exist because it would contradict the strictness of the Lyapunov function (can you supply more detailed arguments?) and moreover, since the level sets of V actually fill the whole interior of $\mathbf{R}^2_+$ hence the equilibrium $(\hat{x}, \hat{y})$ attracts all the orbits from the interior of $\mathbf{R}^2_+$.

13.3 Linearization theorem

I introduced the linearization technique by alluding to the Taylor series and dropping some terms. This is by no means a proof. Here, however, I am able to prove at least some part of the linearization theorem using a carefully chosen Lyapunov function.

First I will need some auxiliary linear algebra facts. Recall that matrix $\boldsymbol{B}$ is called *positive definite* if $\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x} = \sum_{i,j=1}^{2} b_{ij}x_i x_j > 0$ for all $\boldsymbol{x} \neq 0$. The expressions of the form $\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x}$ with positive definite $\boldsymbol{B}$ are natural candidates for Lyapunov functions.

Proposition 10. *Let $\boldsymbol{A}$ be a 2×2 real matrix with eigenvalues with negative real parts. Then there exists a positive definite matrix $\boldsymbol{B}$ such that $\boldsymbol{A}^\top\boldsymbol{B} + \boldsymbol{B}\boldsymbol{A} = -\boldsymbol{I}$.*

Proof. Recall that no matter what matrix $\boldsymbol{A}$ is, if it has eigenvalues with negative real parts, there exists an invertible linear transformation $\boldsymbol{P}$ such that $\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{J}$, where $\boldsymbol{J}$ in one of the three matrices

$$\boldsymbol{J}_1 = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}, \quad \boldsymbol{J}_2 = \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}, \quad \boldsymbol{J}_3 = \begin{bmatrix} -\alpha & \beta \\ -\beta & -\alpha \end{bmatrix},$$

where all the parameters are positive.

Without loss of generality I can assume that my $\boldsymbol{A}$ is in one of the three forms. Now take for cases 1 and 3

$$\boldsymbol{B}_1 = \begin{bmatrix} 1/(2\lambda_1) & 0 \\ 0 & -1/(2\lambda_2) \end{bmatrix}, \quad \boldsymbol{B}_3 = \begin{bmatrix} 1/(2\alpha) & 0 \\ 0 & 1/(2\alpha) \end{bmatrix},$$

and get the desired result. For matrix J_2 I first need to note that I can transform $\boldsymbol{A}$ into

$$\begin{bmatrix} -\lambda & \epsilon \\ 0 & -\lambda \end{bmatrix}$$

by the invertible transformation

$$\begin{bmatrix} 1 & 0 \\ 0 & \epsilon \end{bmatrix}.$$

Now, taking $\boldsymbol{D} = 1/(2\lambda)\boldsymbol{I}$, I note that

$$\begin{bmatrix} -\lambda & \epsilon \\ 0 & -\lambda \end{bmatrix}^\top \boldsymbol{D} + \boldsymbol{D}\begin{bmatrix} -\lambda & \epsilon \\ 0 & -\lambda \end{bmatrix} = \begin{bmatrix} -1 & \epsilon/(2\lambda) \\ \epsilon/(2\lambda) & -1 \end{bmatrix}.$$

Noticing that the result is similar to the identity matrix, I conclude that in this case I also can find such matrix $\boldsymbol{B}$ that satisfies the conditions. ∎

Now I am ready to prove the main result.

Theorem 11. *Assume that the matrix of Jacobi evaluated at the equilibrium $\hat{\boldsymbol{x}}$ of the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ has all the eigenvalues satisfy $\operatorname{Re}\lambda_{1,2} < 0$. Then this equilibrium is locally asymptotically stable.*

Proof. Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in \mathbf{R}^2$$

that has an equilibrium $\hat{\boldsymbol{x}} = 0$ (if the equilibrium is not zero we simply introduce new variable $\boldsymbol{y} + \boldsymbol{x}$ and for variable $\boldsymbol{y}$ we'll have zero equilibrium).

Using the Taylor's formula I can write that

$$\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\hat{\boldsymbol{x}}) + \boldsymbol{f}'(\hat{\boldsymbol{x}})(\boldsymbol{x} - \hat{\boldsymbol{x}}) + \boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}),$$

where $\boldsymbol{f}'(\hat{\boldsymbol{x}})$ is the jacobi matrix evaluated at $\hat{\boldsymbol{x}}$ and $\boldsymbol{g}(0) = 0, \boldsymbol{g}'(0) = 0$. So in general I can transform my system into

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{g}(\boldsymbol{x}), \quad \boldsymbol{A} = \boldsymbol{f}'(\hat{\boldsymbol{x}}).$$

Now let me take $V(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{x}$, where $\boldsymbol{B}$ is the symmetric positive definite matrix that satisfies $\boldsymbol{A}^\top \boldsymbol{B} + \boldsymbol{B}\boldsymbol{A} = -\boldsymbol{I}$, which always exists by the previous proposition. Calculating the derivative I have

$$\dot{V}(\boldsymbol{x}) = -\boldsymbol{x}^\top \boldsymbol{x} + 2\boldsymbol{x}^\top \boldsymbol{B}\boldsymbol{g}(\boldsymbol{x}).$$

From the conditions on $\boldsymbol{g}$, I have that there is $m > 0$ such that $|\boldsymbol{g}(\boldsymbol{x})| \leq m|\boldsymbol{x}|$ for all $|x| < \delta$. Let β be the largest eigenvalue of $\boldsymbol{B}$, then

$$\dot{V}(\boldsymbol{x}) \leq -(1 - 2\beta m)\boldsymbol{x}^\top \boldsymbol{x}$$

and therefore negative if $m < 1/(2\beta)$. This means that I constructed a strict Lyapunov function and therefore my equilibrium is asymptotically stable. ■