1 What are PDE?

1.1 Basic definitions and the general philosophy of the course

Since the main prerequisite for this course is a basic course of Ordinary Differential Equations (ODE), and everyone in class is accustomed with the idea to solve an equation where the unknown is some function, I will start directly with

Definition 1.1. *Partial Differential Equation (abbreviated in the following as PDE in both singular and plural usage) is an equation for an unknown function of two or more independent variables that involves partial derivatives.*

Since there is some vagueness in the given definition, I can give a mathematically more satisfactory definition as

Definition 1.2. *A PDE is an equation of the form*

$$F(x, y, \dots, u, u_x, u_y, \dots, u_{xx}, u_{xy}, u_{yy}, \dots) = 0$$

for the given function F and the unknown function u of several variables $x, y, \dots$.

In Definition 1.2 I used the notation

$$u_x = \frac{\partial u}{\partial x}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \dots$$

for the partial derivatives. Sometimes other notations are used, in particular

$$\partial_x u = \frac{\partial u}{\partial x}, \quad \partial_{xx} u = \frac{\partial^2 u}{\partial x^2}, \dots$$

but I will usually stick to the notation with subscripts.

Definition 1.3. *The order of PDE is the order of the highest derivative in it.*

Example 1.4. Here is an example of a second order PDE:

$$u_t = u_{xx} + u_{yy} + u,$$

where, as should be clear from the equation itself, the unknown function u is a function of three independent variables (t, x, y). In the following I will save variable t to denote almost exclusively time and x, y, z to denote the Cartesian coordinates.

It is nice to have a general and mathematically rigorous Definition 1.2, however, already at this point I would like to state in a slightly incorrect and provocative form that

There exists no general mathematical theory of partial differential equations.

Math 483/683: Partial Differential Equations by Artem Novozhilov
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Moreover, the historical trend for studying various problems involving PDE shows that particular specific examples of PDE, motivated by physical (geometrical, biological, etc) situations, are the driving force for the development of abstract mathematical theories, and this is how I would like to proceed in my course: From specific examples to necessary mathematical tools to the properties of the solutions. There are a lot of good reasons picking such *modus operandi*, but instead of giving my own arguments I will present two quotations.

The first one is from Preface to the first volume of *Methods of mathematical physics* by Courant and Hilbert:

> Since the seventeenth century, physical intuition has served as a vital source for mathematical problems and methods. Recent trends and fashions have, however, weakened the connection between mathematics and physics; mathematicians, turning away from the roots of mathematics in intuition, have concentrated on refinement and emphasized the postulational side of mathematics, and at times have overlooked the unity of their science with physics and other fields. In many cases, physicists have ceased to appreciate the attitudes of mathematicians. This rift is unquestionably a serious threat to science as a whole; the broad stream of scientific development may split into smaller and smaller rivulets and dry out. It seems therefore important to direct our efforts toward reuniting divergent trends by clarifying the common features and interconnections of many distinct and diverse scientific facts. Only thus can the student attain some mastery of the material and the basis be prepared for further organic development of research.

The second quotation is from Preface to *Lectures on Partial Differential Equations* by Vladimir Arnold:

> In the mid-twentieth century the theory of partial differential equations was considered the summit of mathematics, both because of the difficulty and significance of the problems it solved and because it came into existence later than most areas of mathematics.
>
> Nowadays many are inclined to look disparagingly at this remarkable area of mathematics as an old-fashioned art of juggling inequalities or as a testing ground for applications of functional analysis. Courses in this subject have even disappeared from the obligatory program of many universities [...] The cause of this degeneration of an important general mathematical theory into an endless stream of papers bearing titles like "On a property of a solution of a boundary-value problem for an equation" is most likely the attempt to create a unified, all-encompassing, superabstract "theory of everything."
>
> The principal source of partial differential equations is found in the continuous-medium models of mathematical and theoretical physics. Attempts to extend the remarkable achievements of mathematical physics to systems that match its models only formally lead to complicated theories that are difficult to visualize as a whole [...]
>
> At the same time, general physical principles and also general concepts such as energy, the variational principle, Huygens' principle, the Lagrangian, the Legendre transformation, the Hamiltonian, eigenvalues and eigenfunctions, wave-particle duality, dispersion relations, and fundamental solutions interact elegantly in numerous highly important problems of mathematical physics. The study of these problems motivated the development of large areas of mathematics such as the theory of Fourier series and integrals, functional analysis,

algebraic geometry, symplectic and contact topology, the theory of asymptotics of integrals, microlocal analysis, the index theory of (pseudo-)differential operators, and so forth. Familiarity with these fundamental mathematical ideas is, in my view, absolutely essential for every working mathematician. The exclusion of them from the university mathematical curriculum, which has occurred and continues to occur in many Western universities under the influence of the axiomaticist/scholastics (who know nothing about applications and have no desire to know anything except the "abstract nonsense" of the algebraists) seems to me to be an extremely dangerous consequence of Bourbakization of both mathematics and its teaching. The effort to destroy this unnecessary scholastic pseudoscience is a natural and proper reaction of society (including scientific society) to the irresponsible and self-destructive aggressiveness of the "superpure" mathematicians educated in the spirit of Hardy and Bourbaki.

Following the spirit of these two citations (one is from 1924 and another is from 2004) in these notes I will try to use the physical intuition and concentrate on the specific examples rather than on the general theory as much as possible.

Now let me solve a few simple PDE to get an idea what complications we can meet in the future.

Example 1.5. I assume that the function u of two independent variables (x,y) satisfies the PDE

$$u_x = 0.$$

How can I solve it? By a simple integration, of course:

$$u(x,y) = \int 0 \,\mathrm{d}x = f(y),$$

where f is an arbitrary function of variable y. Hence the first conclusion: while the *general solutions* to ODE usually depend on the arbitrary *constants* (the number of which usually coincides with the order of the equations), for PDE the general solution depends on the arbitrary *functions*. This fact alone should convince you in a bigger complexity of PDE.

Next important (and very non-obvious) moment here is whether in the previous example I can take *any* function f for my general solution. Jumping way ahead, I would like to state that "What does it mean to solve a PDE?" is a very difficult question. This difficulty notwithstanding, most of the time we will be contend to live with a much easier specific concept which is called the *classical solution*:

Definition 1.6. *The function $u\colon D \longrightarrow \mathbf{R}$ is called a classical solution to a k-th order PDE if it satisfies this equation at every point of its definition and belongs to the set $\mathcal{C}^{(k)}(D;\mathbf{R})$.*

Recall that the notation $\mathcal{C}^{(k)}(U;V)$ means the set of functions $u\colon U \longrightarrow V$ whose all k-th order derivatives are continuous (it is said that the function u is k times continuously differentiable). Therefore (returning to Example 1.5) my general solution $u(x,y) = f(y)$ will be a classical solution to $u_x = 0$ only if $f \in \mathcal{C}^{(1)}(\mathbf{R};\mathbf{R})$, which, for instance, implies that $u(x,y) = y$ is a classical solution and $u(x,y) = |y|$ is not because function $y \mapsto |y|$ is not differentiable at the point $y = 0$ and hence not in class $\mathcal{C}^{(1)}$.

Exercise 1. Can you find the general classical solution[1] to

$$u_{xy} = 0?$$

The same question about the differential equations

$$u_{xx} = 0, \quad u_{xx} + u = 0,$$

where I assume that function u is a function of two variables, say x and y.

Example 1.7. Function

$$u(t, x) = t + \frac{1}{2}x^2$$

is a classical solution to

$$u_t = u_{xx},$$

because $u \in \mathcal{C}^{(2)}(\mathbf{R}^2; \mathbf{R})$ and satisfies the equation (check it). *Q:* Can you come up with other classical solutions to this equation?[2]

The four basic equations we will be studying in this course are:

- *One-dimensional transport equation:*

  $$u_t + cu_x = 0.$$

  Here $c \in \mathbf{R}$ is a given constant.

- *Wave equation:*

  $$u_{tt} = \Delta u.$$

  Recall that Δ is called the *Laplace operator* and is given by (∇ is the *del* operator, in this particular case $\nabla = (\partial_x, \partial_y, \partial_z)$)

  $$\Delta u = \operatorname{div} \operatorname{grad} u = \nabla^2 u,$$

  in particular in the Cartesian coordinates it is written for $u \colon \mathbf{R}^3 \longrightarrow \mathbf{R}$ as

  $$\Delta u = u_{xx} + u_{yy} + u_{zz},$$

  it should be clear how to write the Laplace operator for the functions defined on the plane and on the line.

- *Heat or diffusion equation:*

  $$u_t = \Delta u.$$

- *Laplace equation:*

  $$\Delta u = 0.$$

The first one is a *linear* first order equation and the other three are *linear* second order equations. It is simply staggering how much modern mathematics was developed in the attempts to solve these equations (or their close relatives). We will see only a tiny part of this.

[1]Solutions to the exercises the student finds in these notes are given at the end of each section. I encourage you to try to solve these exercises first on your own.

[2]The letter Q throughout the notes stand for a simple question, which the student should try to answer before moving forward.

1.2 More examples

Here are more examples of PDE that appear in various applications:

- *Linear transport equation:*

$$u_t + \sum_{i=1}^{k} b_i u_{x_i} = 0, \quad b_i \in \mathbf{R}.$$

- *Helmholtz's equation:*

$$\Delta u = \lambda u, \quad \lambda \in \mathbf{R}.$$

- *Schrödinger's equation:*

$$\mathrm{i} u_t + \Delta u = 0.$$

  Here i is the imaginary unit, $\mathrm{i}^2 = -1$.

- *Telegraph equation:*

$$u_{tt} + 2du_t - u_{xx} = 0, \quad d > 0.$$

- *Beam equation:*

$$u_{tt} + u_{xxxx} = 0.$$

  All the examples above are *linear*. Here are some *nonlinear* examples:

- *Hopf's equation:*

$$u_t + uu_x = 0.$$

- *The eikonal equation* (from German word for *image*):

$$(u_x)^2 + (u_y)^2 = 1.$$

- *Hamilton–Jacobi equation:*

$$u_t + H(u_x, x) = 0,$$

  where H is a given nonlinear function, which is called the Hamiltonian.

- *Korteweg–de Vries (KdV) equation* (Q: Have you heard the word "soliton"?):

$$u_t + uu_x + u_{xxx} = 0.$$

- *Reaction–diffusion equation:*

$$u_t = f(u) + \Delta u,$$

  where f is a given nonlinear function.

  It is also often necessary and important to study *systems* of PDE:

- *Maxwell's equations of classical electrodynamics:*

$$\begin{aligned} \boldsymbol{E}_t &= \operatorname{curl} \boldsymbol{B}, \\ \boldsymbol{B}_t &= -\operatorname{curl} \boldsymbol{E}, \\ \operatorname{div} \boldsymbol{B} &= \operatorname{div} \boldsymbol{E} = 0. \end{aligned}$$

  Here $\boldsymbol{E} = (E_1, E_2, E_3)$, $\boldsymbol{B} = (B_1, B_2, B_3)$,

$$\operatorname{div} \boldsymbol{F} = \nabla \cdot \boldsymbol{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z},$$

  and

$$\operatorname{curl} \boldsymbol{F} = \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{vmatrix},$$

  for the vector field $\boldsymbol{F} = (F_1, F_2, F_3)$.

- *Navier–Stokes equations of hydrodynamics:*

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \Delta \boldsymbol{u} &= \nabla p, \\ \operatorname{div} \boldsymbol{u} &= 0, \end{aligned}$$

  where $\boldsymbol{u} = (u_1, u_2, u_3)$. Here there are four nonlinear equations for four unknown functions u_1, u_2, u_3, p. There are still a lot of open questions about this system, most famous of which is the existence of *global* solutions. This is one of the six unsolved "million" millennium problems by Clay Institute.

Exercise 2. Consider a convex closed curve in the plane with coordinates (x, y) (*convex* means that if you connect any pair of points on this curve with a straight line, the interval between these points will be *inside* the curve). Outside the region bounded by the curve consider function u whose value at each point is the distance from that point to the given curve. This function is smooth. Convince yourself that this function satisfies the eikonal equation. (I do not require a rigorous proof, a heuristic but plausible reasoning is sufficient.)

Exercise 3. Find all solutions $r \mapsto v(r)$ of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ that depend only on the radial coordinate $r = \sqrt{x^2 + y^2}$.

1.3 Test yourself

These notes will contain a number of sections titled "Test yourself." In these sections the students are asked to answer a number of very basic questions related to the studied material. If you have issues with answering any of these questions, it indicates that more work and time are required to master the material.

1.1. Give a definition of the gradient $\nabla u = \operatorname{grad} u$ of a function u. Give a definition of level sets of function u. What is the most important property of the gradient? (It may be useful to consider an example, say $u(x, y) = x^2 + y^2$, and see its level sets, gradient at a certain point, and how they are related).

1.2. The two most basic ODEs are

$$u' = u \quad \text{and} \quad u'' + u = 0.$$

Write down the general solutions to these equations.

1.3. Find the general solution of the Laplace equation for function u that depends only on one spatial variable x. (Note that in this case the Laplace equation is an ODE.)

1.4. Find all the solutions to the two dimensional Laplace equation $u_{xx} + u_{yy} = 0$ of the form

$$u(x, y) = Ax^2 + Bxy + Cy^2.$$

(You are asked to determine for which A, B, C this particular function solves the equation.)

Is this a classical solution?

1.5. Can you *define* what we call "linear" equation? (I will give a precise definition later in the course, here I am asking you to try to explain precisely why some equations are called linear and some are called nonlinear). Give an example of a linear PDE, give an example of a nonlinear PDE (without looking through these notes).

1.4 Solutions to the exercises

Exercise 1. Denoting $v(x, y) = u_x(x, y)$ I have that

$$v_y = 0,$$

which has the general solution $v(x, y) = f(x)$, for an arbitrary $\mathcal{C}^{(1)}$ function f. Now I have

$$u_x = f(x),$$

which integrates to $u(x, y) = F(x)+G(y)$, where $F' = f$ and G is an arbitrary $\mathcal{C}^{(2)}$ function of variable x. Finally,

$$u(x, y) = F(x) + G(y), \quad F, G \in \mathcal{C}^{(2)}(\mathbf{R}; \mathbf{R}).$$

It is very advisable for a student to solve at this point $u_{xx} = 0$ and $u_{xx} + u = 0$ for $u\colon \mathbf{R}^2 \longrightarrow \mathbf{R}$, I will not supply a solution here. ∎

Exercise 2. In this exercise I do not require a rigorous proof. Recall that ∇u at a given point points in the direction of the fastest increase of u and moreover the length (magnitude) of ∇u, which I denote as $\|\nabla u\|$, gives the rate of this increase (on the Euclidian plane $\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2}$, for $\boldsymbol{x} = (x_1, x_2) \in \mathbf{R}^2$). These facts are usually given justifications in Calc III. Now (a good idea is to make a sketch at this point) it should be clear geometrically that if $\boldsymbol{a}$ is a vector along the line connecting the point (x, y) with the closest point on the curve in the question, then $\nabla u(x, y)$ points in the direction opposite to $\boldsymbol{a}$ because of the convexity of the curve. Let $(\tilde{x}, \tilde{y})$ be a point on the same line further away from the curve, then the rate of increase of u at (x, y) is by definition

$$\frac{|u(x, y) - u(\tilde{x}, \tilde{y})|}{\|(x, y) - (\tilde{x}, \tilde{y})\|},$$

when $(\tilde{x}, \tilde{y}) \to (x, y)$. Since u is *the distance*, then the last ratio is of course simply 1. To finish this heuristic reasoning, the student should recognize that the eikonal equation is nothing else other than

$$\|\nabla u\| = 1$$

in my notation. ■

Exercise 3. It it possible to approach this exercise from different perspectives. For the first solution I will assume that the students have some experience working with the polar coordinates.

I start by introducing the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$

or

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}\,.$$

The goal is to rewrite the equation for u in terms of partial derivatives of v, where

$$u(x, y) = u(r\cos\theta, r\sin\theta) = v(r, \theta).$$

The fact that I only look for solutions that do not depend on θ simplifies the computations since $\partial_\theta v = 0$. For instance, by the multivariable chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos\theta,$$

because $\partial_\theta r = \cos\theta$. Next,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(v_r\cos\theta) = (v_r\cos\theta)_r r_x + (v_r\cos\theta)_\theta \theta_x = v_{rr}\cos^2\theta + \frac{1}{r}v_r\sin^2\theta.$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = v_{rr}\sin^2\theta + \frac{1}{r}v_r\cos^2\theta,$$

or finally

$$u_{xx} + u_{yy} = v_{rr} + \frac{1}{r}v_r = 0.$$

Now I note that, after multiplying by r, I can rewrite my equation for unknown v as

$$(rv_r)_r = 0,$$

hence

$$v_r(r) = \frac{A}{r}\,,$$

or, finally,

$$v(r) = A\log r + B,$$

where A, B are arbitrary constants. In words, all radially symmetric solutions to the two-dimensional Laplace equation are constants and, if I exclude the point $r = 0$, multiples of natural logarithm.

There is a more fundamental approach to this exercise that works in any number of dimensions. To practice the multivariate chain rule, I will apply this approach to the general Laplace equation $\Delta u = 0$ in arbitrary n dimensions.

I am looking for the solution to $\Delta u = 0$ in the form

$$u(x_1, \ldots, x_n) = v(r), \quad r = \sqrt{x_1^2 + \ldots + x_n^2}\,.$$

I have

$$\frac{\partial u}{\partial x_j} = \frac{\partial v}{\partial x_j} = v'(r)\frac{\partial r}{\partial x_j} = v'(r)\frac{x_j}{r}\,,$$

since

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{\sqrt{x_1^2 + \ldots + x_n^2}}\,.$$

Similarly,

$$\frac{\partial^2 u}{\partial x_j^2} = \frac{\partial}{\partial x_j}\left(v'(r)\frac{x_j}{r}\right) = \frac{\partial}{\partial x_j}\left(\frac{v'(r)}{r}\right)x_j + \frac{f'(r)}{r}\,.$$

Since

$$\frac{\partial}{\partial x_j}\left(\frac{v'(r)}{r}\right) = \frac{v''(r)r - v'(r)}{r^2}\frac{\partial r}{\partial x_j} = \frac{v''(r)r - v'(r)}{r^2}\frac{x_j}{r}\,,$$

whence

$$\frac{\partial^2 u}{\partial x_j^2} = \frac{v''(r)r - v'(r)}{r^3}x_j^2 + \frac{v'(r)}{r}\,.$$

Finally, recalling that $\Delta u = u_{x_1x_1} + \ldots + u_{x_nx_n}$, I get

$$\frac{v''(r)r - v'(r)}{r^3}r^2 + \frac{v'(r)}{r}n = v''(r) + \frac{n-1}{r}v'(r) = 0,$$

which is of course coincides with my first result for $n = 2$. I will leave it as a (recommended) exercise to find general solution to this differential equation for arbitrary n. ■