7 Solving the wave equation. Some extensions

7.1 Linear versus nonlinear equations

I would like to start this lecture with a discussion what is called linear in mathematics. Recall that I used the presentation

\[ \left( \frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) u = 0 \]

to write the wave equation in a form that hints to the possible ways to solve it. The expression inside the parenthesis is called a differential operator. If I denote this differential operator as \( L \) then I can use the notation

\[ Lu = 0 \]

to write my wave equation in a concise way.

In general, any differential equation, either ordinary of partial, can be written in the form \( Lu = f \), where \( L \) is some differential operator and \( f \) some function that does not depend on \( u \). For example, for the simple harmonic oscillator \( u'' + u = 0 \) I have that \( L = \frac{d^2}{dt^2} + 1 \). For the linear transport equation with variable speed \( u_t + c(x)u_x = 0 \), my operator is \( \partial_t + c(x)\partial_x \) and so on. (If I cannot separate my \( L \) from \( u \) completely, it hints that the equation is nonlinear, for instance, in Hopf’s equation \( u_t + uu_x = 0 \) the differential operator involves multiplication by \( u \) itself, and hence I cannot provide a stand alone expression for \( L \), in such cases I write \( L : f \mapsto f_t + ff_x \) indicating that my operator \( L \) maps any function \( f \) from its domain to the expression \( f_t + ff_x \).)

**Definition 7.1.** A (differential) operator \( L \) is called linear if for any \( u_1 \) and \( u_2 \) from its domain and any constants \( \alpha_1 \) and \( \alpha_2 \) I have

\[ L(\alpha_1u_1 + \alpha_2u_2) = \alpha_1Lu_1 + \alpha_2Lu_2. \]

For example, the wave operator \( L = \partial_t - c^2\partial_x \), the simple harmonic oscillator operator \( L = \partial_t + 1 \), and the operator from linear transport equation \( L = \partial_t + c(x)\partial_x \) are linear (check this!). However, the differential operator of the Hopf equation \( u_t + uu_x = 0 \) is not linear. Indeed, let me apply this operator to \( \alpha u \) for some constant \( \alpha \). The expression \( \alpha u \) will be mapped to \( \alpha u_t + \alpha^2 u u_x \), which is not equal to \( \alpha(u_t + uu_x) \) as necessary for the operator to be linear. Hence now I have a precise way to show that Hopf’s equation is nonlinear.

Now I can give a rigorous definition of a linear differential equation.

**Definition 7.2.** Let \( L \) be a differential operator. Then the differential equation

\[ Lu = f, \quad \text{(7.1)} \]

where \( f \) does not depend on \( u \), is called linear if \( L \) is a linear operator. It is called linear homogeneous if \( f = 0 \) and inhomogeneous (or nonhomogeneous) otherwise. If \( L \) is not linear then equation \((7.1)\) is called nonlinear.

The linearity of the equation is very important, since for the linear equations holds the so-called superposition principle, which is a consequence of the following simple and yet very important proposition.

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Math 483/683: Partial Differential Equations by Artem Novozhilov
e-mail: artem.novozhilov@ndsu.edu. Spring 2020
Proposition 7.3. Consider a linear inhomogeneous equation

\[ Lu = f, \]

and its homogeneous counterpart

\[ Lu = 0. \]

(i) If \( u_1, u_2 \) solve (7.3) then \( \alpha_1 u_1 + \alpha_2 u_2 \) also solves (7.3).

(ii) If \( u_1, u_2 \) solve (7.2) then \( u_1 - u_2 \) solves (7.3).

(iii) If \( u_1 \) solves \( Lu = f_1 \) and \( u_2 \) solves \( Lu = f_2 \) then \( \alpha_1 u_1 + \alpha_2 u_2 \) solves \( Lu = \alpha_1 f_1 + \alpha_2 f_2 \).

(iv) A general solution to (7.2) is given as a sum of general solution to (7.3) and a particular solution to (7.2).

Proof. Since most of the stated facts follow directly from the definition, I will only prove (iv).

It is important to understand what exactly is stated in the theorem above. It is actually an if and only if statement, and requires proving two (simple) directions. Elaborating, the statement says that (a) if \( u_h \) is any solution to (7.3) and \( u_p \) is a particular solution to (7.2), then their sum \( u_h + u_p \) will solve (7.2). This clearly follows from linearity. In other direction, (b) if \( u \) is any solution to (7.2) then it can be written as \( u_h + u_p \), where now \( u_h \) is some solution to (7.3) and \( u_p \) solves (7.2). This is also true since \( u - u_p \), due to (ii), must solve (7.3), and hence for some solution \( u_h = u - u_p \) or \( u = u_h + u_p \) as required.

7.2 Solving an inhomogeneous wave equation. Duhamel’s principle

Consider an inhomogeneous wave equation on the infinite string:

\[ u_{tt} = c^2 u_{xx} + F(t, x), \]
\[ u(0, x) = f(x), \]
\[ u_t(0, x) = g(x), \]

(7.4)

where \( F, f, g \) are given sufficiently smooth functions. First I will use the linearity of this equation to divide it into simpler problems. To wit, consider

\[ v_{tt} = c^2 v_{xx}, \]
\[ v(0, x) = f(x), \]
\[ v_t(0, x) = g(x), \]

(7.5)

and

\[ w_{tt} = c^2 w_{xx} + F(t, x), \]
\[ w(0, x) = 0, \]
\[ w_t(0, x) = 0. \]

(7.6)

That is, I divided my original problem into the initial value problem for the homogeneous wave equation and inhomogeneous problem with zero initial conditions.

Lemma 7.4. Let \( v \) solve (7.5) and \( w \) solve (7.6). Then \( u = v + w \) solves (7.4).
Proof. Exercise.

I know how to solve problem (7.5), for this I can simply use d’Alembert’s formula. Hence I only need to figure out how to solve inhomogeneous problem (7.6). For this I will use the so-called Duhamel’s principle, which generally works for linear differential equations. The idea is to reduce the inhomogeneous problem to a series of homogeneous ones with specific initial conditions and after it sum (integrate) everything together by using the linearity of the equation. To perform this program I should be able to solve, along with (7.5), the following problem

\[ v_{tt} = c^2 v_{xx}, \quad t > \tau, \]
\[ v(\tau, x) = f(x), \]
\[ v_t(\tau, x) = g(x), \]

which differs from what I studied before only by the initial time moment: the moment zero is replaced with moment \( \tau \).

Lemma 7.5. Problem (7.7) has the solution (note that I include the parameter \( \tau \) also in my formula and in the variables of function \( v \))

\[ v(t, x; \tau) = \frac{f(x - c(t - \tau)) + f(x + c(t - \tau))}{2} + \frac{1}{2c} \int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s) ds. \]

Proof. To prove the result one can use new variable \( \eta = t - \tau \), for which the initial condition is given at zero but the equation does not change, use d’Alembert’s formula and return to the original variables. The details are left as an (easy) exercise.

All the main auxiliary work is done and I am ready to prove

Lemma 7.6. Consider yet another initial value problem for the wave equation:

\[ r_{tt} = c^2 r_{xx}, \quad t > \tau \]
\[ r(\tau, x; \tau) = 0, \]
\[ r_t(\tau, x; \tau) = F(\tau, x), \]

and let \( r(t, x; \tau) \) solve this problem. Then

\[ w(t, x) = \int_0^t r(t, x; \tau) d\tau \]

solves problem (7.6).

Proof. To prove the lemma I will use Leibnitz integral rule, which says that

\[ \frac{d}{dx} \int_{a(x)}^{b(x)} g(s, x) ds = \int_{a(x)}^{b(x)} \frac{d}{dx} g(s, x) ds + g(b(x), x) b'(x) - g(a(x), x) a'(x). \]

I have

\[ w_t = r(t, x; t) + \int_0^t r_t(t, x; \tau) d\tau = \int_0^t r_t(t, x; \tau) d\tau. \]
Next
\[ w_{tt} = r_t(t, x; t) + \int_0^t r_{tt}(t, x; \tau) d\tau = F(t, x) + \int_0^t r_{tt}(t, x; \tau) d\tau. \]

I also have
\[ w_{xx} = \int_0^t r_{xx}(t, x; \tau) d\tau = \frac{1}{c^2} \int_0^t r_{tt}(t, x; \tau) d\tau. \]

Hence I get \( w_{tt} - c^2 w_{xx} = F(t, x) \) and \( w(0, x) = w_t(0, x) = 0 \), which concludes the proof.

Now I can put everything together. According to Lemma (7.5) solution to (7.8) is given by
\[ r(t, x; \tau) = \frac{1}{2c} \int_{x-c(t-\tau)}^{x+ct} F(\tau, \eta) d\eta. \]

Using d’Alembert’s formula, Lemma 7.4, and Lemma 7.6 I have that the solution to (7.4) is given by
\[ u(t, x) = \frac{f(x-ct) + f(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds + \frac{1}{2c} \int_0^t \int_{x-c(t-\tau)}^{x+ct} F(\tau, \eta) d\eta d\tau, \]
which is the sought solution to the inhomogeneous wave equation with given initial conditions.

**Remark 7.7.** The same solution obtained in the textbook by switching to the characteristic coordinates \( \xi = x - ct, \eta = x + ct \). I invite the students to read through this derivation.

**Remark 7.8.** I will use Duhamel’s principle again later in the course, so I summarize here the main idea of this principle. It proceeds in two steps:

1. Construct a family of solutions of the homogeneous initial value problem with variable initial moment \( \tau > 0 \) and the initial data \( F(\tau, x) \).

2. Integrate the above family with respect to the parameter \( \tau \).

**Remark 7.9.** The double integral in the final solution shows that the disturbance at the point \((t, x)\) is obtained through the disturbances from every point of the characteristic triangle (that is, of the triangle with vertex \((t, x)\) and two sides given by characteristics \(x - ct\) and \(x + ct\)). Hence if we have only initial displacement, the signal comes from just two points; if we have initial velocity nonzero that the signal comes from the interval (domain of dependence); if we have nonhomogeneous equation, the signal is summed (integrated) throughout the whole characteristic triangle.

**Exercise 1.** Apply the Duhamel principle to the first order linear ODE \( u' + p(t)u = q(t), u(0) = u_0 \).

**Solution.** This equation in an introductory ODE course is usually solved by either using variation of the constant method or integrating factor method. Let me show how the same result is obtained by using Duhamel’s method.

First, of course, I note that problem \( v' + p(t)v = 0, v(0) = u_0 \) has the solution \( v(t) = u_0 e^{-\int_0^t p(s) ds} \). Hence I am left with the problem \( w' + p(t)w = q(t), w(0) = 0 \). Instead of the last problem I consider a series of problems
\[ r' + p(t)r = 0, t > \tau, r(\tau; \tau) = q(\tau). \]
By direct integrating I have
\[ r(t; \tau) = q(\tau)e^{-\int_0^t p(s)ds}. \]
As before, I claim that \( w(t) = \int_0^t r(t; \tau)d\tau \) solves \( w' + p(t)w = q(t), \ w(0) = 0. \) This is checked directly. Hence I got the final answer
\[ u(t) = v(t) + w(t) = u_0e^{-\int_0^t p(s)ds} + \int_0^t q(\tau)e^{-\int_0^\tau p(s)ds}d\tau. \]

To check this answer let me use the variation of the constant method. First I solve the homogeneous equation \( u' + p(t)u = 0 \) getting
\[ u(t) = Ce^{-\int_0^t p(s)ds}. \]
Now I assume that \( C \) is actually a function of \( t \) and plug it back in the original equation to find
\[ C'(t)e^{-\int_0^t p(s)ds} = q(t) \implies C(t) = \int_0^t q(\tau)e^{\int_0^\tau p(s)ds}d\tau + A, \]
and hence (after using the initial condition)
\[ u(t) = e^{-\int_0^t p(s)ds} \left( u_0 + \int_0^t q(\tau)e^{\int_0^\tau p(s)ds}d\tau \right), \]
which, after some simplification, reduces to what I obtained using Duhamel’s method. □