7 Solving the wave equation. Some extensions

7.1 Linear versus nonlinear equations

I would like to start this lecture with a discussion what is called *linear* in mathematics. Recall that in the previous lecture I used the presentation

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = 0$$

to write the wave equation in a form that hints to the possible ways to solve it. The expression inside the parenthesis is called a *differential operator*. If I denote this differential operator as L then I can use the notation

$$Lu = 0$$

to write my wave equation in a concise way.

In general, any differential equation, either ordinary of partial, can be written in the form $Lu = f$, where L is some differential operator applied to function u and f is a given function that does not depend on u. For example, for the simple harmonic oscillator $u'' + u = 0$ I have that $L = \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 1$. For the linear transport equation with variable speed $u_t + c(x)u_x = 0$ my operator is $\partial_t + c(x)\partial_x$ and so on. (Jumping ahead, if I cannot separate my L from u completely, it hints that the equation is nonlinear, for instance, in Hopf's equation $u_t + uu_x = 0$ the differential operator involves multiplication by u itself, and hence I cannot provide a stand alone expression for L, in such cases I write $L\colon u \mapsto u_t + uu_x$ indicating that my operator L maps function u that must belong to the domain of L to the expression $u_t + uu_x$.)

Definition 7.1. *A (differential) operator L is called linear if for any u_1 and u_2 from its domain and any constants α_1 and α_2 I have*

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L u_1 + \alpha_2 L u_2.$$

For example, the wave operator $L = \partial_{tt} - c^2\partial_{xx}$, the simple harmonic oscillator operator $L = \partial_{tt} + 1$, and the operator from linear transport equation $L = \partial_t + c(x)\partial_x$ are linear (check this!). However, the differential operator of the Hopf equation $u_t + uu_x = 0$ is not linear. Indeed, let me apply this operator to αu for some constant α. The expression αu will be mapped to $\alpha u_t + \alpha^2 uu_x$, which is *not* equal to $\alpha(u_t + uu_x)$ as necessary for the operator to be linear. Hence now I have a precise way to show that Hopf's equation is nonlinear.

Now I can give a rigorous definition of a linear differential equation.

Definition 7.2. *Let L be a differential operator. Then the differential equation*

$$Lu = f, \tag{7.1}$$

where f does not depend on u, is called linear if L is a linear operator. It is called linear homogeneous if $f = 0$ and inhomogeneous (or nonhomogeneous) otherwise. If L is not linear then equation (7.1) *is called nonlinear.*

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The linearity of the equation is very important, since for the linear equations holds the so-called *superposition principle*, which is a consequence of the following simple and yet very important proposition.

Proposition 7.3. *Consider a linear inhomogeneous equation*

$$Lu = f, \tag{7.2}$$

and its homogeneous counterpart

$$Lu = 0. \tag{7.3}$$

(*i*) *If* u_1, u_2 *solve* (7.3) *then* $\alpha_1 u_1 + \alpha_2 u_2$ *also solves* (7.3).

(*ii*) *If* u_1, u_2 *solve* (7.2) *then* $u_1 - u_2$ *solves* (7.3).

(*iii*) *If* u_1 *solves* $Lu = f_1$ *and* u_2 *solves* $Lu = f_2$ *then* $\alpha_1 u_1 + \alpha_2 u_2$ *solves* $Lu = \alpha_1 f_1 + \alpha_2 f_2$.

(*iv*) *A general solution to* (7.2) *is given as a sum of general solution to* (7.3) *and a particular solution to* (7.2).

Proof. Since most of the stated facts follow directly from the definition, I will only prove (*iv*).

It is important to understand what exactly is stated in the theorem above. It is actually an if and only if statement, and requires proving two (simple) directions. Elaborating, the statement says that (*a*) if u_h is *any* solution to (7.3) and u_p is a (given) particular solution to (7.2), then their sum $u_h + u_p$ will solve (7.2). This clearly follows from linearity. In other direction, (*b*) if u is *any* solution to (7.2) then it can be written as $u_h + u_p$, where now u_h is picked from the set of all the solutions to (7.3) (and which is clearly *not* empty) and u_p solves (7.2) and, again, is given. This is also true since $u - u_p$, due to (*ii*), must solve (7.3), and hence for some solution $u_h = u - u_p$ or, equivalently, $u = u_h + u_p$ as required. ■

Exercise 1. Prove the remaining parts of Proposition 7.3.

7.2 Solving an inhomogeneous wave equation. Duhamel's principle

Consider an inhomogeneous wave equation on the infinite string:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} + F(t,x), \\ u(0,x) &= f(x), \\ u_t(0,x) &= g(x), \end{aligned} \tag{7.4}$$

where F, f, g are given sufficiently smooth functions. First I will use the linearity of this equation to divide it into simpler problems. To wit, consider

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \\ v(0,x) &= f(x), \\ v_t(0,x) &= g(x), \end{aligned} \tag{7.5}$$

and

$$\begin{aligned} w_{tt} &= c^2 w_{xx} + F(t,x), \\ w(0,x) &= 0, \\ w_t(0,x) &= 0. \end{aligned} \tag{7.6}$$

That is, I divided my original problem into the initial value problem for the homogeneous wave equation and inhomogeneous problem with zero initial conditions.

Lemma 7.4. *Let v solve* (7.5) *and w solve* (7.6). *Then $u = v + w$ solves* (7.4).

Proof. Exercise. ∎

I know how to solve problem (7.5), for this I can simply use d'Alembert's formula. Hence I only need to figure out how to solve inhomogeneous problem (7.6). For this I will use the so-called *Duhamel's principle*, which generally works for *linear* differential equations. The idea is to reduce the inhomogeneous problem to a series of homogeneous ones with specific initial conditions and after it sum (integrate) everything together by using the linearity of the equation. To perform this program I should be able to solve, along with (7.5), the following problem

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \quad t > \tau, \\ v(\tau, x) &= f(x), \\ v_t(\tau, x) &= g(x), \end{aligned} \tag{7.7}$$

which differs from what I studied before only by the initial time moment: the moment zero is replaced with moment τ.

Lemma 7.5. *Problem* (7.7) *has the solution (note that I include the parameter τ also in my formula and in the variables of function v)*

$$v(t, x; \tau) = \frac{f\big(x - c(t-\tau)\big) + f\big(x + c(t-\tau)\big)}{2} + \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} g(s)\mathrm{d}s.$$

Proof. To prove the result one can use new variable $\eta = t - \tau$, for which the initial condition is given at zero but the equation does not change, use d'Alembert's formula and return to the original variables. The details are left as an exercise. ∎

Exercise 2. Prove Lemmas 7.4 and 7.5.

All the main auxiliary work is done and I am ready to prove

Lemma 7.6. *Consider yet another initial value problem for a family of wave equations depending on the parameter τ:*

$$\begin{aligned} r_{tt} &= c^2 r_{xx}, \quad t > \tau \\ r(\tau, x; \tau) &= 0, \\ r_t(\tau, x; \tau) &= F(\tau, x), \end{aligned} \tag{7.8}$$

and let $r(t, x; \tau)$ be its solution for each fixed τ. Then

$$w(t, x) = \int_0^t r(t, x; \tau)\mathrm{d}\tau$$

solves problem (7.6).

Proof. To prove the lemma I will use the Leibnitz integral rule, which says that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} g(s,x)\mathrm{d}s\right) = \int_{a(x)}^{b(x)} \frac{\mathrm{d}g}{\mathrm{d}x}(s,x)\mathrm{d}s + g(b(x),x)b'(x) - g(a(x),x)a'(x).$$

I have

$$w_t = r(t,x;t) + \int_0^t r_t(t,x;\tau)\mathrm{d}\tau = \int_0^t r_t(t,x;\tau)\mathrm{d}\tau.$$

Next

$$w_{tt} = r_t(t,x;t) + \int_0^t r_{tt}(t,x;\tau)\mathrm{d}\tau = F(t,x) + \int_0^t r_{tt}(t,x;\tau)\mathrm{d}\tau.$$

I also have

$$w_{xx} = \int_0^t r_{xx}(t,x;\tau)\mathrm{d}\tau = \frac{1}{c^2}\int_0^t r_{tt}(t,x;\tau)\mathrm{d}\tau.$$

Hence I get $w_{tt} - c^2 w_{xx} = F(t,x)$ and $w(0,x) = w_t(0,x) = 0$, which concludes the proof. ■

Now I can put everything together. According to Lemma (7.5) solution to (7.8) is given by

$$r(t,x;\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\tau,\eta)\mathrm{d}\eta.$$

Using d'Alembert's formula, Lemma 7.4, and Lemma 7.6 I have that the solution to (7.4) is given by

$$u(t,x) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\mathrm{d}s + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(\tau,\eta)\mathrm{d}\eta\mathrm{d}\tau,$$

which is the sought solution to the inhomogeneous wave equation with given initial conditions.

Remark 7.7. The same solution can be obtained by switching to the characteristic coordinates $\xi = x - ct$, $\eta = x + ct$. I invite the reader to try this rout.

Remark 7.8. I will use Duhamel's principle again later in the course, so I summarize here the main idea of this principle. It proceeds in two steps:

Step 1. Construct a family of solutions of the homogeneous initial value problem with variable initial moment $\tau > 0$ and the initial data $F(\tau, x)$.

Step 2. Integrate the above family with respect to the parameter τ.

In simple words, Duhamel's principle can be stated (somewhat vaguely) as follows: "If one can solve an initial value problem for a homogeneous linear differential equation then an inhomogeneous linear differential equation can be solved as well."

Remark 7.9. The double integral in the final solution shows that the disturbance at the point (t,x) is obtained through the disturbances from every point of the characteristic triangle (that is, of the triangle with the vertex (t,x) and two sides given by the corresponding characteristics $x - ct$ and $x + ct$). Hence if I have a homogeneous equation with only the initial displacement nonzero, the signal comes from just two initial points; if I, again for the homogeneous equations, have the initial velocity nonzero then the signal comes from the interval (domain of dependence) of the initial values; if, finally, I have an inhomogeneous equation, the signal is summed (integrated) throughout the whole characteristic triangle with the vertices $(0,\xi),(0,\eta),(t,x)$ (and it could be helpful to go back to the previous section and glance at the last figure in it).

Exercise 3. Apply Duhamel's principle to the first order linear ODE $u'+p(t)u=q(t)$, $u(0)=u_0$.

7.3 Classification of PDE

There exist various classifications of PDE, here is our first one. I already used the terms linear, semilinear, quasilinear in different contexts, but, except for the first one, never defined them exactly. Now that I have an easy to test definition of linearity, I can provide precise definitions. I will do it by using the first order PDE, but the same classification holds for any order PDE if the definitions below applied to the terms with the highest derivatives. Again, for simplicity, I will use only two independent variables, but I hope that the generalization to any number of independent variables is straightforward.

So, let me start. The general form of the first order PDE is

$$F(x,y,u,u_x,u_y)=0.$$

A first order PDE is called *linear* if it has the form

$$a(x,y)u_x+b(x,y)u_y+f(x,y)u=f(x,y)$$

for some given functions a,b,c,f. An example is the linear transport equation

$$u_t+cu_x=0.$$

(The student should check that linearity defined in this way coincides with the definition of being linear given at the beginning of this section.)

A first order PDE is called *semilinear* if it is not linear and has the form

$$a(x,y)u_x+b(x,y)u_y=f(x,y,u),$$

for some given functions a,b,f. In other words (which should be easily transferred to higher order equations), a semilinear PDE is such that the coefficients at the highest derivatives are functions of independent variables only. An example is, for instance,

$$u_x+u_y=u(1-u).$$

Next, a first order PDE is called *quasilinear*, if it is not linear or semilinear, and has the form

$$a(x,y,u)u_x+b(x,y,u)u_y=f(x,y,u),$$

for some given functions a,b,f. Rephrasing, for the quasilinear equations the highest derivatives, taking separately from everything else, are included in a "linear way." An example is Hopf's equation

$$u_t+uu_x=0.$$

Finally, if a first order PDE is not linear, semilinear, or quasilinear, it is called *fully nonlinear*, and example is the eikonal equation

$$(u_x)^2+(u_y)^2=1.$$

Note that semilinear, quasilinear, and fully nonlinear equations are nonlinear.

Exercise 4. Give a similar classification of second order PDE.

7.4 Test yourself

7.1. Is the equation $u_t = \alpha u_{xx} + f(t,x)$ linear? Can you *prove* it?

7.2. If the equation $u_t = \alpha u_{xx} + u(1-u)$ linear? Can you *prove* it?

7.3. Solve the problem for $t > 0, x \in \mathbf{R}$

$$u_{tt} = c^2 u_{xx} + t \sin x, \quad u(0,x) = (1+x^2)^{-1}, \quad u_t(0,x) = 0.$$

7.5 Solutions to the exercises

Exercise 1. These are direct consequences of the definition of linearity. Assume, say, that u_1, u_2 solve $Lu = 0$, then $u = \alpha_1 u_1 + \alpha_2 u_2$ satisfies $Lu = L(\alpha_1 u_1 + \alpha_2 u_2) =$ (byleariry) $= \alpha_1 Lu_1 + \alpha_2 Lu_2 = 0$ (by the assumption), and (i) has been proved. To prove (ii) start again with the basic assumption that $Lu_1 = f$ and $Lu_2 = f$. Hence for $u = u_1 - u_2$ I have $Lu = L(u_1 - u_2) = Lu_1 - Lu_2 = f - f = 0$, as required, and a similar line of reasonings proves (iii). ■

Exercise 2. To prove Lemma 7.4 just plug $u = v + w$ into (7.4) and use (7.5) and (7.6) to reach the required conclusion.

Instead of proving Lemma 7.5 directly, let me consider instead (to type a little less) the initial value problem for ODE $u' = au$, $u(\tau) = u_0$. Of course I can solve it directly, but let me make a change of the independent variable $t = \tau + \eta$. By the change rule for $u(t) = u(\tau + \eta) = v(\eta) = v(t - \tau)$ I have $\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\eta}$ and the initial condition becomes $v(0) = u_0$ for v. Now, $v' = av, v(0) = u_0$ yields $v(\eta) = u_0 e^{a\eta}$, and hence $u(t) = u_0 e^{a(t-\tau)}$. Exactly the same argument and the use of d'Alembert's formula prove Lemma 7.5. ■

Exercise 3. This equation in an introductory ODE course is usually solved by either using *variation of the constant method* or *integrating factor* method. Let me show how the same result is obtained by using Duhamel's method.

First, of course, I note that problem $v' + p(t)v = 0, v(0) = u_0$ has the solution $v(t) = u_0 e^{-\int_{t_0}^t p(s)\mathrm{d}s}$. Hence I am left with the problem $w' + p(t)w = q(t)$, $w(0) = 0$. Instead of the last problem I consider a series of problems

$$r' + p(t)r = 0, t > \tau, r(\tau; \tau) = q(\tau).$$

By direct integrating I have

$$r(t; \tau) = q(\tau) e^{-\int_\tau^t p(s)\mathrm{d}s}.$$

As before, I claim that $w(t) = \int_0^t r(t; \tau)\mathrm{d}\tau$ solves $w' + p(t)w = q(t)$, $w(0) = 0$. This is checked directly. Hence I got the final answer

$$u(t) = v(t) + w(t) = u_0 e^{-\int_0^t p(s)\mathrm{d}s} + \int_0^t q(\tau) e^{-\int_\tau^t p(s)\mathrm{d}s}.$$

To check this answer let me use the variation of the constant method. First I solve the homogeneous equation $u' + p(t)u = 0$ getting

$$u(t) = Ce^{-\int_0^t p(s)\mathrm{d}s}.$$

Now I *assume* that C is actually a function of t and plug it back in the original equation to find

$$C'(t)e^{-\int_0^t p(s)\mathrm{d}s} = q(t) \implies C(t) = \int_0^t q(\tau)e^{\int_0^\tau p(s)\mathrm{d}s}\mathrm{d}\tau + A,$$

and hence (after using the initial condition)

$$u(t) = e^{-\int_0^t p(s)\mathrm{d}s}\left(u_0 + \int_0^t q(\tau)e^{\int_0^\tau p(s)\mathrm{d}s}\mathrm{d}\tau\right),$$

which, after some simplification, reduces to what I obtained using Duhamel's method. ■

Exercise 4. The general form of a second order PDE with two independent variables is

$$F(x, y, u, u_x, u_y, u_{xy}, u_{xx}, u_{yy}) = 0.$$

It is linear if it has the form

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = g(x, y).$$

It is semilinear if it has the form

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y, u, u_x, u_y).$$

It is quasilinear if it has the form

$$a(x, y, u, u_x, u_y)u_{xx} + b(x, y, u, u_x, u_y)u_{xy} + c(x, y, u, u_x, u_y)u_{yy} = f(x, y, u, u_x, u_y).$$

The rest of the equations are fully nonlinear. ■