10 Motivation for Fourier series

Donkeys and scholars in the middle!

French soldiers,
forming defensive squares in Egypt,
during Napoleon's campaign

The central topic of our course is the so-called Fourier method or method of separation of variables to solve linear PDE with constant coefficients. This method historically required a detailed analysis of the question when a given arbitrary function can be represented as a linear combination of trigonometric functions, and partial answers to this question eventually turned into a wide mathematical field that is called nowadays Fourier Analysis. To motivate the appearance of Fourier series, i.e., infinite linear combination of trigonometric functions, I will use the problem of deriving the temperature inside an insulated ring. I will look at this problem in the most general settings that involve the notion of *functions of complex variable*, probably unfamiliar to most of the students. Many of the steps I present here will be repeated again later in a somewhat more "real" examples.

10.1 Fourier series appear for the first time

Consider an insulated circular rod with some initial temperature distribution along it. I am interested in answering the question how the temperature changes with time at each point of the rod. Using the experience from the previous lecture I can write that my temperature at the point x at the time t, which as before I denote $u(t,x)$, must satisfy the heat equation

$$u_t = \alpha^2 u_{xx}, \quad t > 0, \quad -\pi < x \leq \pi, \tag{10.1}$$

note that I assumed that my spatial variable changes from $-\pi$ to π, using the geometry of my rod.

I also have the initial condition

$$u(0,x) = g(x), \quad -\pi < x \leq \pi. \tag{10.2}$$

Clearly, none of the considered in the previous lecture boundary conditions would work in this particular case. A little thought, however, shows that it is natural to set here *periodic boundary conditions*

$$\begin{aligned} u(t,-\pi) &= u(t,\pi), \quad t > 0, \\ u_x(t,-\pi) &= u_x(t,\pi), \quad t > 0, \end{aligned} \tag{10.3}$$

such that the profile of the temperature in my rod is continuously differentiable at every point.

To solve problem (10.1)–(10.3) I make, following the giants of the 18th century, an ingenious assumption that I can look for the solution in the form

$$u(t,x) = T(t)X(x),$$

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i.e., as the product of two functions, the first one depends only on t and the second one depends only on x (I will say more about this assumption later on in the course). Using this *ansatz* ("ansatz" in mathematics is an educated guess) yields

$$T'(t)X(x) = \alpha^2 T(t)X''(x),$$

where the prime denotes the derivatives with respect to the corresponding variables. Rearranging implies

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}\,,$$

i.e., the right hand side of this equality depends only on t, and the right hand side depends only on x. What is an immediate consequence of this fact? Both sides must be equal to the same constant! Indeed, fix some t and hence the left hand side is constant, which implies that the right hand side is constant *for any* x, now go in the opposite direction. Hence, I can write

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = \mu,$$

where μ is some in general *complex* constant, $\mu \in \mathbf{C}$. The consequence of the last two equalities is the following two differential equations

$$T' = \alpha^2 \mu T \tag{10.4}$$

and

$$X'' = \mu X. \tag{10.5}$$

Before proceeding further, let me check what my assumption on the structure of solutions of the heat equation implies for the initial and boundary conditions. The initial condition does not give me much insight at this point, but the boundary conditions now read

$$T(t)X(-\pi) = T(t)X(\pi), \quad T(t)X'(-\pi) = T(t)X'(\pi),$$

which implies, naturally assuming that $T(t) \neq 0$, that

$$X(-\pi) = X(\pi), \quad X'(-\pi) = X'(\pi). \tag{10.6}$$

Problem (10.5), (10.6) is an ODE boundary value problem, and my task is to determine for which complex constants μ this problem has a non-zero solution (zero solution exists for any μ but it is of no interest to me assuming that $g(x) \neq 0$), and how exactly this solution looks for the given μ.

I start with equation (10.5) and use a few facts from the theory of linear ODE. I do hope that the students can solve this problem for real values of μ.

Exercise 1. Solve problem (10.5), assuming that $\mu \in \mathbf{R}$. Assume first that $\mu > 0$, then $\mu < 0$, $\mu = 0$.

Technically the case when $\mu \in \mathbf{C}$ was not covered in the introductory ODE course. Actually, the difference with the real case is minuscule, but let me proceed carefully in this case (still omitting some technical steps). I will look for the solution in the form of a power series with undetermined coefficients:

$$X(x) = c_0 + c_1 x + c_2 x^2 + \ldots = \sum_{k=0}^{\infty} c_k x^k.$$

Differentiating yields

$$X'(x) = c_1 + 2c_2x + 3c_3x^2 + \dots$$
$$X''(x) = 2c_2 + 3 \cdot 2c_3x + 4 \cdot 3c_4x^2 + \dots.$$

Hence I must have

$$2c_2 + 3 \cdot 2c_3x + 4 \cdot 3c_4x^2 + \dots = \mu c_0 + \mu c_1 x + \mu c_2 x^2 + \dots.$$

Two power series equal if and only if the coefficients at the same powers are equal, that is I have

$$\begin{aligned} 2c_2 &= \mu c_0, \\ 3 \cdot 2c_3 &= \mu c_1, \\ 4 \cdot 3c_4 &= \mu c_2, \\ 5 \cdot 4c_5 &= \mu c_3, \\ &\dots \end{aligned}$$

It is very easy to see the pattern, which I can succinctly put as

$$c_{2m} = \frac{\mu^m c_0}{(2m)!}, \quad c_{2m+1} = \frac{\mu^m c_1}{(2m+1)!},$$

where

$$m! = 1 \cdot 2 \cdot \ldots \cdot (m-1) \cdot m.$$

Therefore, if I know c_0 and c_1 my problem is solved. Recall that the space of solutions of equation (10.5) is two dimensional, which also holds for the ODE with complex-valued coefficients, and also note that $X(0) = c_0$, $X'(0) = c_1$, therefore I can pick any two linearly independent in $\mathbf{C}^2$ vectors (c_0, c_1) to produce two solutions that will form a basis of my solution space. I am going to pick first $c_0 = 1, c_1 = \sqrt{\mu}$, and get that

$$X_1(x) = \sum_{k=0}^{\infty} \frac{(\sqrt{\mu}x)^k}{k!},$$

and then $c_0 = 1, c_1 = -\sqrt{\mu}$ to have the second solution

$$X_2(x) = \sum_{k=0}^{\infty} \frac{(-\sqrt{\mu}x)^k}{k!}.$$

(Here I use the notation $\sqrt{\mu}$ and $-\sqrt{\mu}$ to denote two complex square roots of μ, which always exists unless $\mu = 0$.)

Exercise 2. What happens if I pick $c_0 = 1, c_1 = 0$ and $c_0 = 0, c_1 = 1$, which may look like a more natural choice? *Hint:* If you are completely lost at this point, it may help to read to the end of this lecture and return to this exercise after it.

Now an attentive reader should notice that the power series look similar to the ones that were studied in Calculus under the name of Taylor's series. In particular, these are exactly the series for the exponent, the only problem that the series with complex coefficients were not studied. To put aside this problem I simply (for those who would undertake the study of functions of complex variables in the future this word "simply" should be remembered at some point :)) put forward the following definition.

Definition 10.1. *The exponential function,* $\exp z$ *or* e^z*, of the complex variable* $z \in \mathbf{C}$ *is defined as*

$$\exp z = e^z := 1 + z + \frac{z^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{z^k}{k!}\,.$$

Remark 10.2. No discussion was supplied about *convergence* of these power series. It can be proved (not complicated but takes some time) that this series converges absolutely for any fixed z. For those who want to understand the definitions literally, not defined words "converges absolutely" should be read "makes sense."

Remark 10.3. The definition of the exponential function together with binomial formula yield probably the most important property of e^z:

$$e^{z_1+z_2} = e^{z_1}e^{z_2}.$$

I invite the curious student to prove it.

Using my definition of the exponential function I can write my two linearly independent solutions to (10.5) as $X_1(x) = e^{\sqrt{\mu}x}$ and $X_2(x) = e^{-\sqrt{\mu}x}$, and the general solution as

$$X(x) = AX_1(x) + BX_2(x) = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x},$$

where A and B are two (complex) arbitrary constants. To be precise this formula works only if $\mu \neq 0$, but if $\mu = 0$ then two integrations yield

$$X(x) = A + Bx.$$

Now it is time to look more closely at the boundary conditions (10.6).

First, let $\mu = 0$, then I must have that

$$A + B\pi = A - B\pi,$$

which implies $B = 0$, and the second boundary condition is satisfied automatically. Therefore I found that for $\mu = 0$ I always have a nontrivial solution to my boundary value problem, which is

$$X_0(x) = A,$$

where A is an arbitrary constant.

Now let $\mu \neq 0$. For the future use I will define two new function, hyperbolic sine, sinh, and hyperbolic cosine, cosh:

Definition 10.4.

$$\sinh z = \frac{e^z - e^{-z}}{2}\,, \quad \cosh z = \frac{e^z + e^{-z}}{2}\,.$$

Using the boundary conditions I find

$$Ae^{\sqrt{\mu}\pi} + Be^{-\sqrt{\mu}\pi} = Ae^{-\sqrt{\mu}\pi} + Be^{\sqrt{\mu}\pi},$$
$$\sqrt{\mu}Ae^{\sqrt{\mu}\pi} - \sqrt{\mu}Be^{-\sqrt{\mu}\pi} = \sqrt{\mu}Ae^{-\sqrt{\mu}\pi} - \sqrt{\mu}Be^{\sqrt{\mu}\pi},$$

or, using my definitions,

$$A\sinh\sqrt{\mu}\pi = B\sinh\sqrt{\mu}\pi,$$
$$A\sinh\sqrt{\mu}\pi = -B\sinh\sqrt{\mu}\pi.$$

These equalities can be true only if $A = B = 0$, which gives me a trivial zero solution, of if

$$\sinh\sqrt{\mu}\pi = 0.$$

To proceed I will define yet two more functions.

Definition 10.5.

$$\sin z = \frac{e^{\mathrm{i}z} - e^{-\mathrm{i}z}}{2\mathrm{i}}, \quad \cos z = \frac{e^{\mathrm{i}z} + e^{-\mathrm{i}z}}{2},$$

where i *is the imaginary unit,* $\mathrm{i}^2 = -1$.

Exercise 3. Prove that defined in this way functions of the complex variable sine and cosine coincide with our familiar trigonometric functions for the case when z is a real variable.

Exercise 4. What is $\cos \mathrm{i}$?

Note that the last definition implies immediately *Euler's formula*

$$e^{\mathrm{i}z} = \cos z + \mathrm{i}\sin z.$$

Now I am ready to tackle the solution of the equation $\sinh\sqrt{\mu}\pi = 0$. I rewrite it as

$$e^{2\sqrt{\mu}\pi} = 1$$

and assume that $\sqrt{\mu} = \alpha + \mathrm{i}\beta$, where $\alpha, \beta \in \mathbf{R}$. Using Euler's formula, I have

$$e^{2\alpha\pi}(\cos 2\pi\beta + \mathrm{i}\sin 2\pi\beta) = 1 + \mathrm{i}\cdot 0.$$

Two complex numbers are equal if and only if the real and imaginary parts are equal, hence

$$e^{2\alpha\pi}\cos 2\pi\beta = 1, \quad e^{2\alpha\pi}\sin 2\pi\beta = 0,$$

where now all the constants are real and I can use my knowledge about the trigonometric functions. I immediately conclude that the last two equalities can be true if and only if

$$\alpha = 0, \beta = k, \quad k \in \mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\}.$$

Returning to the original constant μ I get

$$\sqrt{\mu} = \alpha + \mathrm{i}\beta = \mathrm{i}k, \quad k \in \mathbf{Z} \implies \mu = -k^2, \quad k = 1, 2, \ldots$$

In words, my boundary value problem has a nonzero solution only if the values of the constant in (10.5) are $-k^2$, where $k = 0, 1, 2, \ldots$ (note that I included $\mu = 0$ in this counting). The solutions, and I am from now on going to use the index k to emphasize the dependence on k, are

$$X_k(x) = A_k e^{\mathrm{i}kx} + B_k e^{-\mathrm{i}kx}, \quad k = 0, 1, 2, \ldots,$$

where still A_k, B_k are some arbitrary complex constants. Since my solution describes the temperature I am mostly interested in a *real* valued solution. A complex expression M is real if and only if $M = \overline{M}$, where the bar means complex conjugate. To have all $X_k(x)$ real I must have hence that $A_k = \overline{B}_k$. Let $A_k = a_k/2 + \mathrm{i}b_k/2$ for some real a_k, b_k. Then my real solutions are

$$X_k(t) = (a_k/2 + \mathrm{i}b_k/2)e^{\mathrm{i}kx} + (a_k/2 - \mathrm{i}b_k/2)e^{-\mathrm{i}kx} = a_k \cos kx + b_k \sin kx,$$

using Euler's formula.

Now I finally can summarize that my boundary value problem (10.5), (10.6) with the complex variable μ has nontrivial solutions only if $\mu = -k^2$, where $k = 0, 1, 2, \ldots$, and these solutions are given by

$$X_k(x) = a_k \cos kx + b_k \sin kx,$$

where the constants are real. Note that the case $\mu = 0$ is also included in these formulas.

At this point I finally can consider solutions to (10.4):

$$T_k(t) = C_k e^{-k^2\alpha^2 t},$$

and hence *each* function

$$u_k(t,x) = T_k(t)X_k(x) = e^{-\alpha^2 k^2 t}(a_k \cos kx + b_k \sin kx), \quad k = 0, 1, 2, \ldots$$

solves the heat equation (10.1) and satisfies the boundary conditions (10.3) (abusing the notation a little I absorbed arbitrary constants C_k in a_k, b_k).

What about the initial condition? Here I will rely again, as before in Duhamel's principle, on the linearity on the equation, and in particular on the *principle of superposition:* If u_1, u_2 solve (10.1) then any linear combination of these functions also solves it. Hence I can write that an infinite linear combination

$$u(t,x) = \sum_{k=0}^{\infty} u_k(t,x) = a_0 + \sum_{k=1}^{\infty} e^{-\alpha^2 k^2 t}(a_k \cos kx + b_k \sin kx)$$

solves the heat equation. I use the initial condition to find (I replace for some notational reasons a_0 with $a_0^*/2$, the reason for this will be clear in the next lecture)

$$g(x) = \frac{a_0^*}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx).$$

And this was Jean-Baptiste Joseph Fourier, French scholar, who was one of the scientists in the French Army during the Egyptian expedition under Napoleon's lead at the beginning of the 19th century, who in 1822 declared, to a big surprise and disbelief of the mathematical community, that any function can be represented as a linear combination of trigonometric functions. In other words, he meant that given g, he can always find a_k, b_k, and the corresponding series converges ("makes sense"). My next task is to actually figure out how to determine a_k, b_k and also to contemplate a little about the question whether the found series provides us with a legitimate classical solution to the heat equation.

10.2 Test yourself

1. What is the general solution to ODE
$$X'' - 4X = 0?$$
2. What is the general solution to ODE
$$X'' = 0?$$
3. What is the general solution to ODE
$$X'' + 4X = 0?$$

10.3 Solutions to the exercises

Exercise 1. So I need to solve $X'' = \mu X$, $X(-\pi) = X(\pi)$, $X'(-\pi) = X'(\pi)$ for $\mu \in \mathbf{R}$. I assume that the students know that to solve a linear ODE with constant coefficients requires looking at the so-called characteristic equation, and the general solution depends on the particular form (real, complex, multiple) of its roots. I will consider three cases.

Case 1: $\mu = 0$ implies $X(x) = Ax + B$, or, after using boundary conditions, $X(x) = B$ for any constant $B \in \mathbf{R}$. ■

Case 2: $\mu > 0$ implies the characteristic equation to be $r^2 = \mu$, or $r_{1,2} = \pm\sqrt{\mu}$, which implies that the general solution is $X(x) = Ae^{\sqrt{\mu}x} + Be^{-\sqrt{\mu}x}$. This general solution plus boundary conditions yield the system

$$A\left(e^{\sqrt{\mu}\pi} - e^{-\sqrt{\mu}\pi}\right) + B\left(e^{-\sqrt{\mu}\pi} - e^{\sqrt{\mu}\pi}\right) = 0,$$
$$A\left(e^{\sqrt{\mu}\pi} - e^{-\sqrt{\mu}\pi}\right) + B\left(-e^{-\sqrt{\mu}\pi} + e^{\sqrt{\mu}\pi}\right) = 0,$$

which is a system of linear homogeneous equations with respect to unknowns A, B. From Linear Algebra I know that this system has a nontrivial solution if and only if the determinant of its matrix is zero. But this determinant is (after some somewhat tedious computations) simplifies to

$$2\left(e^{\sqrt{\mu}\pi} - e^{-\sqrt{\mu}\pi}\right)^2,$$

which can be equal to zero only if $\mu = 0$, the case which I treated already above (do you see why?). So no nontrivial solutions for $\mu > 0$.

Case 3: $\mu < 0$ implies that the characteristic equation has two purely imaginary roots $r_{1,2} = \pm\omega \mathrm{i}$, where $\omega = \sqrt{-\mu}$. The general solution is $X(x) = A\cos\omega x + B\sin\omega x$, and the boundary conditions yield

$$2B\sin\pi\omega = 0, \quad 2A\sin\pi\omega = 0,$$

which could be true if both $A = B = 0$ (trivial solution), or

$$\sin\omega\pi = 0,$$

which implies that $\omega = k$, $k \in \mathbf{Z}$, or returning to the original variable $\mu = -\omega^2 = -k^2, k = 1, 2, 3, \ldots$, exactly as in the case considered in the main text, of course.

Exercise 2. By picking $c_0 = 1, c_1 = 1$ I get

$$c_{2m} = \frac{\mu^m}{(2m)!} = \frac{\sqrt{\mu}^{2m}}{(2m)!}, \quad c_{2m+1} = 0,$$

and $c_0 = 0, c_1 = 1$ leads to

$$c_{2m} = 0, \quad c_{2m+1} = \frac{\sqrt{\mu}^{2m}}{(2m+1)!},$$

which both should resemble (at least on a formal level) familiar (hopefully) series

$$\cosh\sqrt{\mu}x = 1 + \frac{(\sqrt{\mu}x)^2}{2!} + \frac{(\sqrt{\mu}x)^4}{4!} + \dots$$

and

$$\frac{\sinh\sqrt{\mu}x}{\sqrt{\mu}} = \frac{x + \frac{(\sqrt{\mu}x)^3}{3!} + \frac{(\sqrt{\mu}x)^5}{5!} + \dots}{\sqrt{\mu}}.$$

■

Exercise 3. I will do it only for sine, for cosine the reasoning is exactly the same. Assume that z is real and consider the series for

$$e^{\mathrm{i}z} = 1 + (\mathrm{i}z) + \frac{(\mathrm{i}z)^2}{2!} + \dots = 1 + \mathrm{i}z - \frac{z^2}{2!} - \mathrm{i}\frac{z^3}{3!} + \dots$$

and

$$e^{-\mathrm{i}z} = 1 + (-\mathrm{i}z) + \frac{(-\mathrm{i}z)^2}{2!} + \dots = 1 - \mathrm{i}z - \frac{z^2}{2!} - \mathrm{i}\frac{z^3}{3!} + \dots$$

Now I form, using my definition, $\sin z = \frac{e^{\mathrm{i}z} - e{-\mathrm{i}z}}{2\mathrm{i}}$ and cancel what I can cancel (because the series converge absolutely), ending up with

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots,$$

which is exactly Taylor's series for sine if its argument is real. Hence my definition of the trigonometric functions sine and cosine coincide with the familiar one. ■

Exercise 4. Using not included part of the solution of the previous exercise, I know that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$$

hence

$$\cos \mathrm{i} = 1 + \frac{1}{2!} + \frac{1}{4!} + \dots,$$

which is a real expression and obviously bigger than 1!

Moreover, using Exercise 2 I find that $\cos \mathrm{i} = \cosh 1$. ■