11 Fourier series

In this lecture I will talk about the *trigonometric Fourier's series*:

$$g(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx),\quad -\pi\leq x\leq\pi, \tag{11.1}$$

where a_k, b_k are real constants.

My discussion will be geared towards computational aspects, theoretical discussion can be found elsewhere (among many existing excellent sources I especially recommend the book by Brat Osgood, *Lectures on the Fourier Transform and Its Applications*, AMS, 2019).

There are several relevant questions which need to be addressed:

- If (11.1) is true then how to find a_k, b_k?
- What if my interval is different from $[-\pi,\pi]$?
- In what sense the equality sign in (11.1) must be understood? Note that the right hand side is an infinite series and hence a discussion on *convergence* is relevant.
- Is it possible to replace $\{1,\cos kx,\sin kx\}$ with, say, only $\{\cos kx\}$? or only $\{\sin kx\}$?
- These series are important for us because they represent solutions to the second order PDE and hence in order to be *classical* solutions they should be differentiable. What are the conditions for (11.1) such that I can take a derivative of the right hand side and obtain a Fourier series for g'?

11.1 Formulas for the coefficients of (11.1)

Since I am using sines and cosines the expression (11.1) is called a *trigonometric Fourier's series*. The right hand side of (11.1) means that I represent g as an infinite linear combination of $\{1,\cos kx,\sin kx\}_{k=1}^{\infty}$, which I will call the *trigonometric system of functions*. This system possesses a special property, which makes computations particularly simple. To introduce it, I first introduce an *inner product* on the set of functions defined on $[-\pi,\pi]$.

Definition 11.1. *Let f,g be two real valued functions defined on $[-\pi,\pi]$ and assume that $\int_{-\pi}^{\pi}f^2(x)\,\mathrm{d}x<\infty$ and $\int_{-\pi}^{\pi}g^2(x)\,\mathrm{d}x<\infty$. I will denote the set of all possible such functions as $L^2[-\pi,\pi]$ (in words, these functions are called square integrable on $[-\pi,\pi]$). An inner product of $f,g\in L^2[-\pi,\pi]$ is defined as*

$$\langle f,g\rangle=\int_{-\pi}^{\pi}f(x)g(x)\,\mathrm{d}x.$$

(*and this is only one of many possible inner products*).

Math 483/683: Partial Differential Equations by Artem Novozhilov
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Exercise 1. Show that for any $f, g, h \in L^2[-\pi, \pi]$ (and if you never dealt with subtleties of L^2 space before, assume additionally that f, g, h are continuous on $[-\pi, \pi]$)

$$\begin{aligned}\langle f, f\rangle \geq 0, \quad \langle f, f\rangle = 0 \Leftrightarrow f = 0,\\ \langle af, g\rangle = a\langle f, g\rangle, \quad a \in \mathbf{R},\\ \langle f, g\rangle = \langle g, f\rangle,\\ \langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle.\end{aligned}$$

The key property I talked about above is *orthogonality*.

Definition 11.2. *Two functions $f, g \in L^2[-\pi, \pi]$ are called orthogonal if*

$$\langle f, g\rangle = 0.$$

Lemma 11.3. *The trigonometric system of functions is orthogonal, meaning that each function in this set belongs to $L^2[-\pi, \pi]$ and any pair of them is orthogonal.*

Proof. First,

$$\int_{-\pi}^{\pi} 1 \,\mathrm{d}x = 2\pi < \infty, \quad \int_{-\pi}^{\pi} \cos^2 kx \,\mathrm{d}x = \pi < \infty, \quad \int_{-\pi}^{\pi} \sin^2 kx \,\mathrm{d}x = \pi < \infty,$$

which proves that all my functions are in $L^2[-\pi, \pi]$.

Second, to show orthogonality, I need to calculate

$$\begin{aligned}\langle 1, \cos kx\rangle &= \int_{-\pi}^{\pi} 1 \cdot \cos kx \,\mathrm{d}x = \frac{\sin kx}{k}\Bigg|_{x=-\pi}^{x=\pi} = 0,\\ \langle 1, \sin kx\rangle &= \int_{-\pi}^{\pi} 1 \cdot \sin kx \,\mathrm{d}x = -\frac{\cos kx}{k}\Bigg|_{x=-\pi}^{x=\pi} = 0,\\ \langle \cos kx, \sin mx\rangle &= \int_{-\pi}^{\pi} \cos kx \sin mx \,\mathrm{d}x = 0,\\ \langle \cos kx, \cos mx\rangle &= \int_{-\pi}^{\pi} \cos kx \cos mx \,\mathrm{d}x = \begin{cases} 0, & k \neq m,\\ \pi, & k = m,\end{cases}\\ \langle \sin kx, \sin mx\rangle &= \int_{-\pi}^{\pi} \sin kx \sin mx \,\mathrm{d}x = \begin{cases} 0, & k \neq m,\\ \pi, & k = m,\end{cases}\end{aligned}$$

where $k, m = 1, 2, 3, \ldots$. ■

Remark 11.4. To evaluate the above integrals I used

$$\begin{aligned}\sin\alpha \sin\beta &= \frac{1}{2}\Big(\cos(\alpha-\beta) - \cos(\alpha+\beta)\Big),\\ \cos\alpha \cos\beta &= \frac{1}{2}\Big(\cos(\alpha-\beta) + \cos(\alpha+\beta)\Big),\\ \sin\alpha \cos\beta &= \frac{1}{2}\Big(\sin(\alpha+\beta) + \sin(\alpha-\beta)\Big).\end{aligned}$$

I will use the orthogonality of the trigonometric system of functions to find a_k, b_k in (11.1). Let me take the inner product of the left and right hand sides of (11.1) with 1 (just one remark: I am doing something not exactly legitimate here, because it *does not* follow from the properties of the inner product that I can distribute it in infinite series, but since I am mostly interested in computational aspects, and get the correct results at the end, I will bravely proceed):

$$\langle g, 1\rangle = \langle \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx), 1\rangle\,.$$

I use the properties of my inner product (see Exercise 1 and remark above) to have

$$\langle g, 1\rangle = \frac{a_0}{2}\langle 1, 1\rangle + \sum_{k=1}^{\infty} a_k \langle \cos kx, 1\rangle + b_k \langle \sin kx, 1\rangle\,.$$

Now notice that due to the orthogonality all the terms except $\langle 1, 1\rangle$ are zero:

$$\langle g, 1\rangle = \frac{a_0}{2} 2\pi \implies a_0 = \frac{\langle g, 1\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x.$$

Similarly, taking the inner product with $\cos mx$ and $\sin mx$ respectively I find (switching again m with k)

$$\begin{aligned} a_k &= \frac{\langle g, \cos kx\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos kx\,\mathrm{d}x, \quad k = 0, 1, 2, \ldots \\ b_k &= \frac{\langle g, \sin kx\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin kx\,\mathrm{d}x, \quad k = 1, 2, \ldots, \end{aligned} \tag{11.2}$$

note that I also included the case a_0 in my formulas, and this was the reason to have $a_0/2$ in (11.1).

Equations (11.2) give me the *Fourier coefficients* of the trigonometric Fourier series (11.1).

Often I will need to consider the set of functions $L^2[-l, l]$, where l is some constant, in general different from π. I can simply introduce a new variable y in the way that

$$y = \frac{\pi x}{l}\,,$$

such that I am rescaling my variable, and when $x = \pm l$ then $y = \pm\pi$. Using this change of variable, and replacing back y with x I immediately conclude that

Lemma 11.5. *The system of functions* $\left\{1, \cos\frac{\pi k x}{l}, \sin\frac{\pi k x}{l}\right\}$ *is orthogonal with respect to the inner product*

$$\langle f, g\rangle = \int_{-l}^{l} f(x)g(x)\,\mathrm{d}x.$$

If function $g \in L^2[-l, l]$ *can be represented by its trigonometric Fourier series*

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos\frac{\pi k x}{l} + b_k \sin\frac{\pi k x}{l}\right), \quad -l \le x \le l,$$

then the coefficients are found as

$$\begin{aligned} a_k &= \frac{1}{l}\int_{-l}^{l} g(x)\cos\frac{\pi k x}{l}\,\mathrm{d}x, \quad k=0,1,2,\ldots \\ b_k &= \frac{1}{l}\int_{-l}^{l} g(x)\sin\frac{\pi k x}{l}\,\mathrm{d}x, \quad k=1,2,\ldots. \end{aligned} \tag{11.3}$$

Note that Lemma 11.5 answers two first questions that I posed.

11.2 On the convergence of (11.1)

Example 11.6. To motivate the discussion of the convergence of the Fourier series and practice the formulas (11.3) consider the following example:

$$g(x)=\begin{cases}0, & -1\leq x\leq 0,\\ 1-x, & 0<x\leq 1.\end{cases}$$

I find that (note that I calculate a_0 and $a_k, k=1,2,\ldots$ separately):

$$\begin{aligned} a_0 &= \int_0^1 (1-x)\,\mathrm{d}x = \frac{1}{2}, \\ a_k &= \int_0^1 (1-x)\cos\pi k x\,\mathrm{d}x = \frac{1}{(\pi k)^2}(1-\cos\pi k) = \frac{1-(-1)^k}{(\pi k)^2}, \\ b_k &= \int_0^1 (1-x)\sin\pi k x\,\mathrm{d}x = \frac{1}{\pi k}. \end{aligned}$$

Now I can form the partial sums

$$S_k(x)=\frac{a_0}{2}+\sum_{m=1}^{k}(a_m\cos\pi m x+b_m\sin\pi m x),$$

and compare them with the graph of the initial function. My expectation is that the bigger the value of k I take, the closer the graph of S_k should be to the graph of g. Indeed, my expectations is correct, see Fig. 1.

Moreover, for bigger k the approximation becomes better and better (see Fig. 2).

Before jumping to conclusions let me answer another question: What would happen if I look at the values of x outside of the interval $[-1,1]$? Both my function and the partial sums of Fourier series are obviously defined for them. Let me take, say, $x\in[-2,2]$ (see Fig. 3, left panel).

As it should be expected, the partial sums of Fourier series are periodic function (in my case with period 2), and the original function is not periodic. However, I can always take a *periodic extension* of my function g, such that the new function becomes also periodic with period 2. Instead of writing careful (and mostly useless) definition, just look at Fig. 3, right panel, to see what a periodic extension is.

Finally, my function g, as well as the corresponding periodic extension, are not continuous, whereas the partial sums as the sums of continuous functions are continuous. It actually stays true even in the limit $k\to\infty$, the right hand side of (11.1) is continuous, and if the original function has a jump discontinuity then the corresponding Fourier series converges exactly to the middle of the interval of discontinuity.

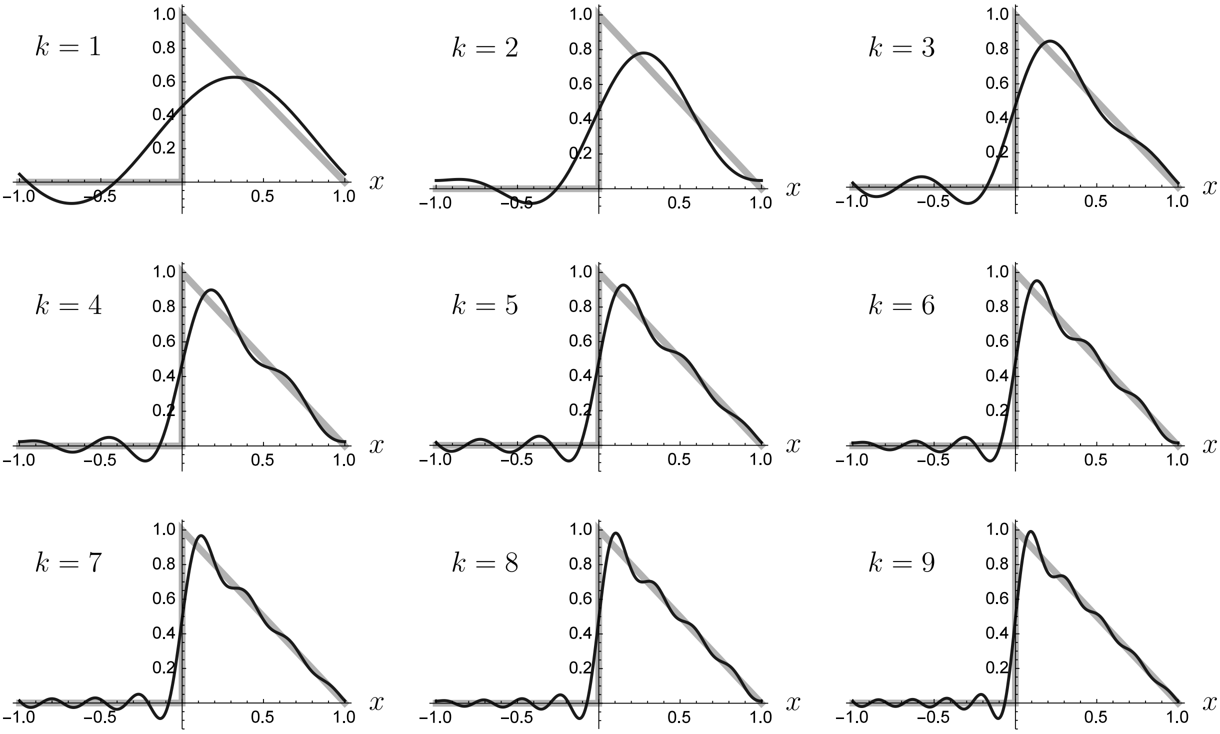

Figure 1: Comparison the graphs of the original function g (light gray) and the partial sums S_k (dark grey) of the corresponding Fourier series.

Let me put together all the information I gained in the previous example. I start with a definition of a piecewise smooth function.

Definition 11.7. *Function $g\colon [a,b] \longrightarrow \mathbf{R}$ is said to be piecewise smooth of the class $\mathcal{C}^{(k)}$ if it belongs to the class $\mathcal{C}^{(k)}$ at any point of the interval $[a,b]$ except, possibly, a finite number of points of discontinuity $x_1 < x_2 < \ldots < x_p$. Moreover, the right and left limits at all these points exist and finite (i.e., all the*

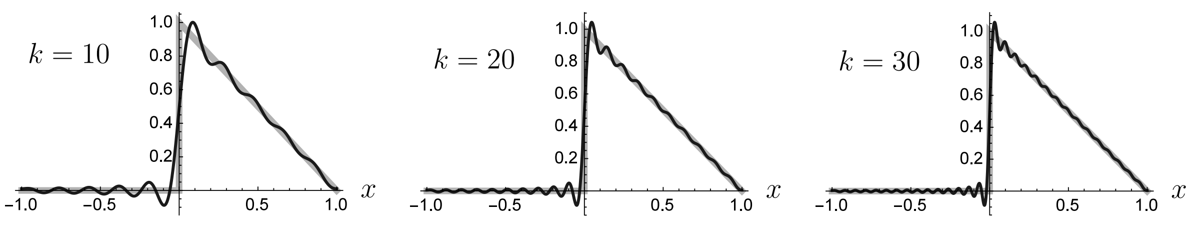

Figure 2: Comparison of the graphs of the original function g (light gray) and the partial sums S_k (dark grey) of the corresponding Fourier series.

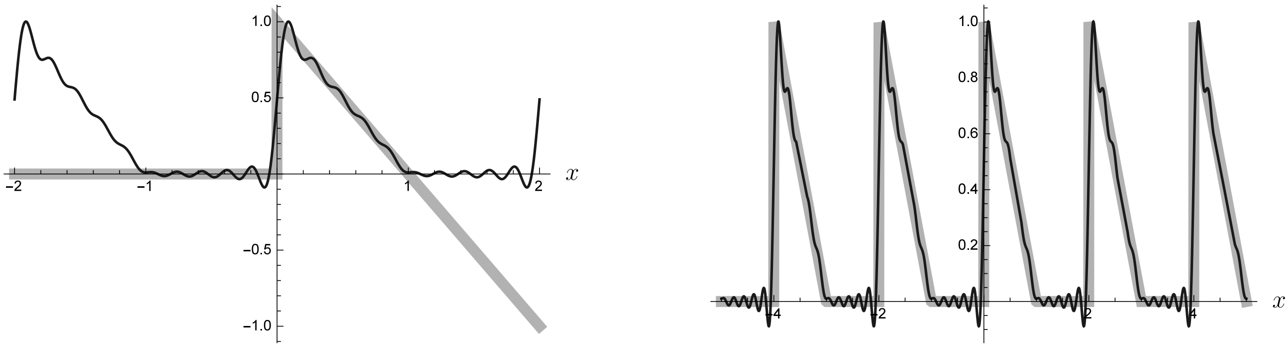

Figure 3: Left: Comparison of the graphs of the original function g (light gray) and the partial sum S_{10} (dark grey) on the interval $[-2, 2]$. Right: Comparison of the graphs of the periodic extension of the function g (light gray) and the partial sum S_{10} (dark grey) on the interval $[-5, 5]$.

discontinuities are of the jump type):

$$\lim_{x \to x_j + 0} g(x) = g(x_j + 0), \quad \lim_{x \to x_j - 0} g(x) = g(x_j - 0), \quad j = 1, \ldots, p.$$

The value $g(x_j + 0) - g(x_j - 0)$ is called the magnitude of the jump.

Now I can state the celebtated

Theorem 11.8 (Dirichlet)**.** *Let $g\colon [-l, l] \longrightarrow \mathbf{R}$ be a piecewise smooth function of the class $\mathcal{C}^{(1)}$, and let $\tilde{g}\colon \mathbf{R} \longrightarrow \mathbf{R}$ be its $2l$-periodic extension. Then, at any point $x \in \mathbf{R}$, its Fourier series converges to the value $\tilde{g}(x)$ if $\tilde{g}$ is continuous at x, or to $(\tilde{g}(x + 0) + \tilde{g}(x - 0))/2$ if x is a point of a jump discontinuity.*

This theorem (partially) answers the third question from my list.

Remark 11.9. Note that the stated theorem stops being true in general for piecewise continuous functions (of the class $\mathcal{C}$). All these subtle points are very important mathematically but fall outside of our PDE course.

11.3 Sine and cosine series. Odd and even extensions

I had only one example of calculating Fourier coefficients so far, but it should be already clear that this is the point that requires sometimes some tedious calculations. It is always nice when we can see immediately answers without calculating integrals. Here is one such trick.

Recall that f is odd if $f(-x) = -f(x)$ for all x, geometrically the graph of an odd function is symmetric with respect to the origin; and f is even if $f(-x) = f(x)$ for all x, the graph of the even function is symmetric with respect to the y-axis. An example of an odd function is sine, and of an even function is cosine.

Exercise 2. Show that the product of two even or two odd functions is even, and the product of even and odd functions is odd.

Now I can use this symmetry to calculate the integrals for Fourier coefficients. Note that if f is odd then the integral with respect to a symmetric interval is zero, and if f is even, then the integral with respect to the interval $[-l, l]$ is double the integral from 0 to l.

Lemma 11.10. *Let g be an odd function. Then*

$$a_k = 0, \quad b_k = \frac{2}{l}\int_0^l g(x)\sin\frac{\pi k x}{l}\,\mathrm{d}x, \quad k = 0, 1, \ldots,$$

Let g be an even function. Then

$$a_k = \frac{2}{l}\int_0^l g(x)\cos\frac{\pi k x}{l}\,\mathrm{d}x, \quad b_k = 0, \quad k = 0, 1, \ldots.$$

In other words, the trigonometric Fourier series of odd functions contain only sines, the trigonometric Fourier series of even functions contain only a constant and cosines. This is a great news for many calculations, however, there is a much deeper consequence of the last lemma. Note first that all the integrals are calculated from 0 to l.

Now assume that I need to find Fourier series of a function defined on $[0, l]$. I can, of course, shift this interval by $l/2$ and use the formulas for the full Fourier series (which is by the way is not necessary, see below). I also can *extend* my function evenly, for example (recall the discussion of the solution of the wave equation on a half-infinite string!), and in this case my Fourier series will consists only of cosines and a constant. Or I can use an odd extension, and in this case my Fourier series will contain only sine terms.

Dirichlet's theorem, together with the discussion above, thus imply that for any piecewise continuous function $g\colon [a, b] \longrightarrow \mathbf{R}$ of class $\mathcal{C}^{(1)}[a, b]$ I can almost equally easy find either the full trigonometric Fourier series, or the sine Fourier series, or the cosine Fourier series, and the convergence of these series follows from that of Dirichlet's theorem.

Example 11.11. Find the sine and cosine Fourier series for

$$g(x) = x, \quad 0 \leq x \leq 1.$$

Let me start with the sine series. The odd extension is (this is an illustration, I never use these extensions in my calculations!)

$$g_{odd}(x) = x, \quad -1 \leq x \leq 1.$$

Since this is an odd function, hence $a_k = 0$ for any k. Using the formulas above

$$b_k = 2\int_0^1 x \sin \pi k x\,\mathrm{d}x = 2\frac{(-1)^{k+1}}{k\pi},$$

the results are shown in Figs. 4 and 5.

Recall that on the whole real line my Fourier series converges to the corresponding periodic extension (see Fig. 5).

Now I consider the even extension:

$$g_{even}(x) = \begin{cases} x, & 0 \leq x \leq 1, \\ -x, & -1 \leq x \leq 0. \end{cases}$$

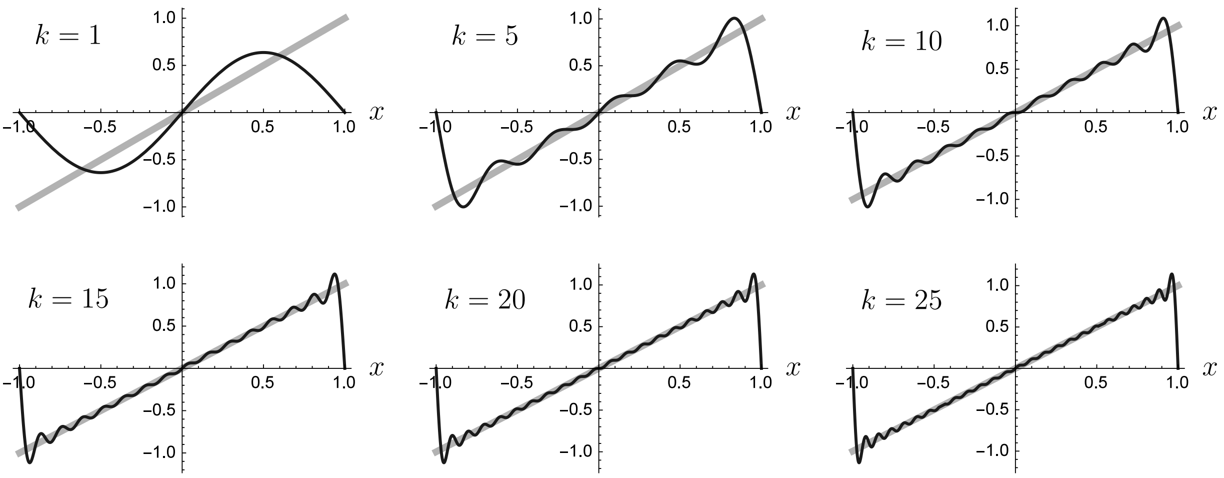

Figure 4: Sine Fourier series for $g(x) = x$ on $[0, 1]$.

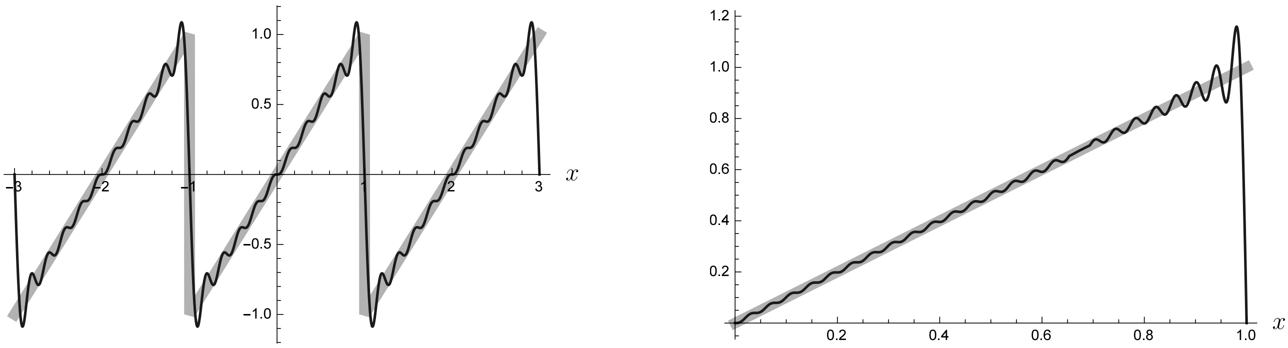

Figure 5: Sine Fourier series for $g(x) = x$ on $[0, 1]$. Left: Periodic extension. Right: Just the interval $[0, 1]$.

I find

$$b_k = 0, \quad a_0 = 1, \quad a_k = \frac{(-1)^k - 1}{\pi^2 k^2}, \quad k = 1, 2, \ldots$$

The results are shown in Figs. 6 and 7.

There is one important thing to notice from this example. The periodic extension of the odd function in my example is discontinuous, and my coefficients have the form C/k for some constant C. The periodic extension of the even function is continuous (see the figures), but not continuously differentiable (it has corners), and my coefficients are of the form C/k^2 for some (different) constant C. Clearly, the latter series converges faster, which can also be seen directly from the figures. This is a general phenomenon, i.e., the smoother the periodic extension of my function, the faster my Fourier series converges.

To finish this example let me find the full trigonometric Fourier series for the same g, defined on $[0, 1]$. I mentioned above that I may first shift it $1/2$ units to the left, to be able to use the formulas

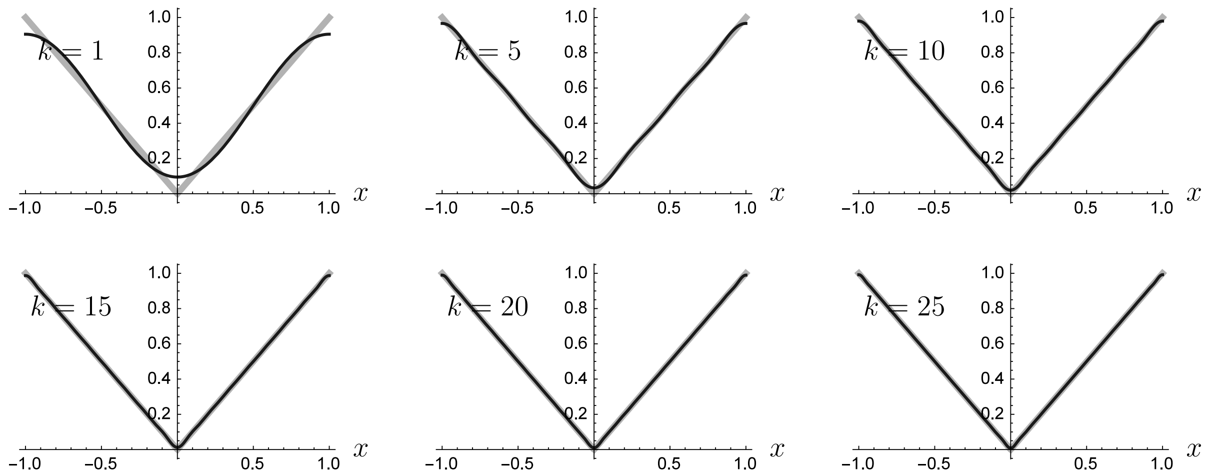

Figure 6: Cosine Fourier series for $g(x) = x$ on $[0, 1]$.

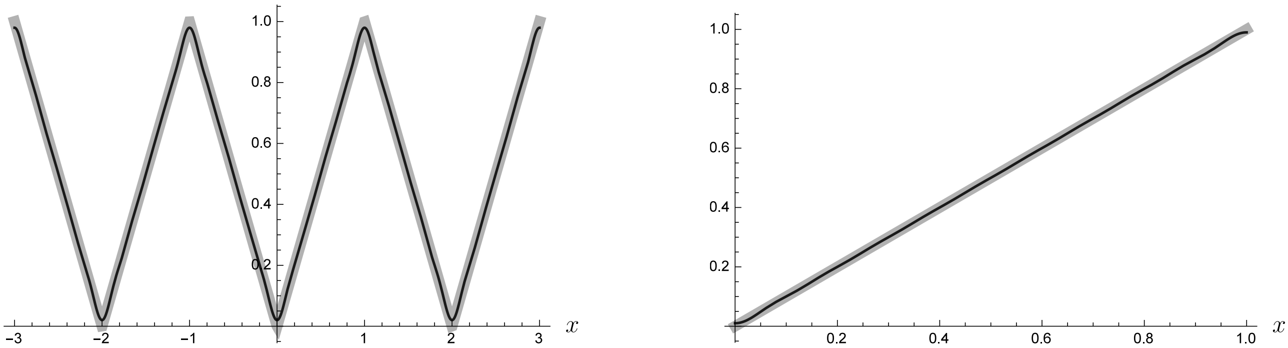

Figure 7: Cosine Fourier series for $g(x) = x$ on $[0, 1]$. Left: Periodic extension. Right: Just the interval $[0, 1]$. Compare this to the corresponding sine Fourier series.

for the full trigonometric series, but this is not necessary, due to the simple and yet important fact that if f is a T-periodic function then

$$\int_{-T/2}^{T/2} f(x)\,\mathrm{d}x = \int_{a-T/2}^{a+T/2} f(x)\,\mathrm{d}x$$

for any constant a. That is, I can use my formulas for Fourier coefficients integrating one period without requiring the limits of integration be symmetric. For my example $T = 1$ (this is my period),

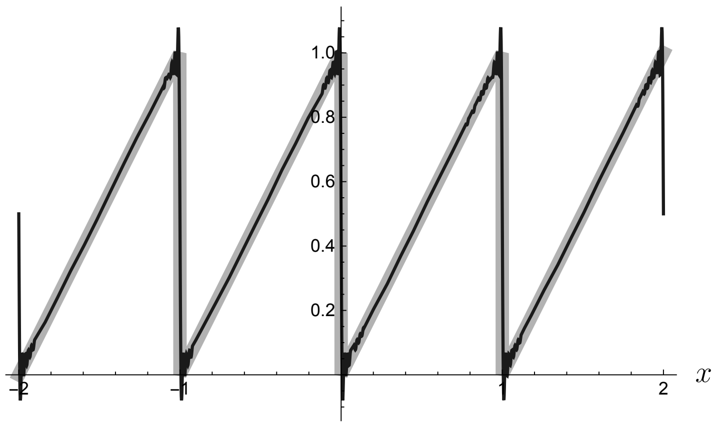

Figure 8: Full Fourier series for $g(x) = x$ on $[0, 1]$. The comparison is with 1-periodic function g and the partial sum of the Fourier series with $k = 50$.

and hence l in the formulas above should be replaced with $l = T/2 = 1/2$ in my case. That is,

$$a_0 = 2\int_0^1 x\,\mathrm{d}x = 2,$$
$$a_k = 2\int_0^1 x\cos 2\pi kx\,\mathrm{d}x = 0, \quad k = 1, 2, \ldots,$$
$$b_k = 2\int_0^1 x\sin 2\pi kx\,\mathrm{d}x = -\frac{1}{k\pi}\,.$$

Hence I end up with the partial sums

$$S_k(x) = 1 + \sum_{m=1}^{k} b_m \sin 2\pi mx$$

as approximations of my original function (see Fig. 8).

Exercise 3. Can you find a cosine Fourier series of function sin on $x \in [0, \pi]$?

Exercise 4. Prove that the integral of T-periodic function over one period does not depend on the start and end points of this period.

11.4 Differentiating and integrating Fourier series

Finally, for the future use I need some information on when I can differentiate Fourier series such that I can talk about *classical* solutions of PDE problems. Without going into any technical details, note that if I differentiate the terms in Fourier series, I will get factor k in front of each term. Hence, to be able to differentiate, I need that my Fourier coefficients would be of the form C/k^{α} for a sufficiently large α, and if the student read the previous material carefully they should understand that it does put some restrictions of the form of the periodic extensions of my functions. The question is not that simple, and instead of formulating dry theorems, let me consider a few examples.

Example 11.12. We already know that for $g(x)=x$ on $[-1,1]$ I have

$$x=\sum_{k=1}^{\infty}\frac{2(-1)^{k+1}}{k\pi}\sin k\pi x=\frac{2}{\pi}\sin \pi x-\frac{2}{2\pi}\sin 2\pi x+\frac{2}{3\pi}\sin 3\pi x-\dots$$

where the equality is understood in the sense of Dirichlet's theorem.

Now, formally differentiating this equality, I would get

$$1=2\sum_{k=1}^{\infty}(-1)^{k+1}\cos k\pi x=2\cos \pi x-2\cos 2\pi x+2\cos 3\pi x-\dots,$$

which is clearly something ridiculous (try plugging $x=0$ in the right hand side), which indicates that I must be careful with my solutions in the form of Fourier series.

On the other hand, if I integrate my Fourier series, I get

$$\frac{x^2}{2}=C+\sum_{k=1}^{\infty}\frac{2(-1)^k}{k^2\pi^2}\cos k\pi x,$$

where the constant C can be found, for instance, by using orthogonality of all the cosines and constant 1, yielding $C=1/6$.

This is exactly the Fourier series for $x^2/2$, if I were to use the formulas for a_k, b_k, which also indicates that operation of integration is less dangerous than differentiation (which is of course not surprising since integration makes functions to be smoother whereas differentiation reduces the smoothness).

Anyway, this simple example shows that anyone should be careful while writing solutions to our mathematical problems in the form of Fourier series. On a more positive note I would like to claim an important from computational point of view fact: If the initial and boundary conditions are "reasonable enough," then the resulting series make perfect sense even (sic!) if one is not capable differentiating them. I will come to this point later in the course (still skipping most of the mathematical rigor in my discussion).

The discussion here is over but I cannot help mentioning that plugging $x=1$ in the last series yields absolutely amazing formula

$$\frac{\pi^2}{6}=\sum_{k=1}^{\infty}\frac{1}{k^2}\,.$$

Similarly, by plugging $x=0$ I will find yet another remarkable result

$$\frac{\pi^2}{12}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\,.$$

11.5 Final remarks and generalizations

Basically, what was shown and stated (mostly without proofs), is that any piecewise continuous function g on $[-\pi,\pi]$ with piecewise continuous derivative can be represented as its Fourier series, using either

$$\{1,\cos kx,\sin kx\},$$

or

$$\{1, \cos kx\},$$

or

$$\{\sin kx\}.$$

Moreover, I found the formulas that can be used to calculate the coefficients of Fourier series. The key property that allowed me to do this is the orthogonality of the first system of functions on $[-\pi, \pi]$ and orthogonality of the second and third systems on $[0, \pi]$.

In general, and quite abstractly, I can consider an arbitrary system of functions

$$\{g_1(x), g_2(x), g_2(x), \dots\},$$

introduce an inner product $\langle g_j, g_k\rangle$ that satisfies some natural properties and define orthogonality.

Let me assume that $\{g_j(x)\}$ is an orthogonal system of functions. Then I can try to represent an arbitrary function as a generalized Fourier series

$$g(x) = c_1 g_1(x) + c_2 g_2(x) + \dots.$$

The orthogonality will immediately let me find my coefficients:

$$c_k = \frac{\langle g, g_k\rangle}{\langle g_k, g_k\rangle}.$$

A very important question, of course, is when I have *enough* functions to represent *any* allowable function as its generalized Fourier series. A quite general answer to this question (without proof) will be given later in the course.

Exercise 5. Consider complex valued functions of the real argument and an inner product of the form

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x.$$

Show that system of functions

$$g_0(x) = 1, \quad g_k(x) = e^{\mathrm{i}kx}, \quad k = \pm 1, \pm 2, \pm 3, \dots$$

is orthogonal on $[-\pi, \pi]$. Find the expression for the coefficients c_k in the complex Fourier's series

$$g(x) = c_0 + \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{i}kx}.$$

Can you see how c_k are connected with a_k, b_k from the trigonometric Fourier series?

11.6 Test yourself

1. What is the Fourier series of $\sin 2x$ on $[-\pi, \pi]$? on $[0, \pi]$?

2. Give an example of an odd function, even function.

3. Let $f(x)$ be defined as follows:

$$f(x)=\begin{cases} x+1, & x\in[-1,0),\\ x-1, & x\in[0,1].\end{cases}$$

Sketch its 2-periodic extension and carefully state to which points the corresponding Fourier series will converge.

11.7 Solutions to the exercises

Exercise 1. All these properties are direct consequences of the properties of Riemann integral, which was studies in Calculus course. Some difficulty can arise in discussing the first property, which I will deal with here. I have

$$\langle f,f\rangle=\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x,$$

which must be non-negative (by, say, definition of the Riemann sums), since $f^2(x)\geq 0$ for all x.

If f is identically zero then my integral is zero. In the opposite direction, assuming that integral is zero, I must show that f must be zero almost everywhere. For this I will recall the fact that Riemann integrable function on a finite interval must be continuous almost everywhere (note that I do not distinguish the functions are are equal to each other *almost everywhere*). Looking for a contradiction, I will assume that $\langle f,f\rangle=0$ but there is a point $x\in(-\pi,\pi)$ such that f is continuous at x and $f(x)\neq 0$. Then, by continuity, there must be an interval $(x-\varepsilon,x+\varepsilon)$ where $f\neq 0$, and hence $\int_{x-\varepsilon}^{x+\varepsilon} f^2(x)\,\mathrm{d}x>0$, contradicting the assumption. Hence $f(x)=0$ for (almost) all points x. ■

Exercise 2. Let f,g be both even, then $h(-x)=f(-x)g(-x)=f(x)g(x)=h(x)$, hence h is even. Let f,g be both odd, then $h(-x)=f(-x)g(-x)=(-1)^2f(x)g(x)=h(x)$, hence h is even. Finally, let f be even and g be odd, then $h(-x)=f(-x)g(-x)=(-1)f(x)g(x)=-h(x)$, hence h is odd. ■

Exercise 3. All I need is to calculate the following integrals

$$a_0=\frac{2}{\pi}\int_0^{\pi}\sin x\,\mathrm{d}x=\frac{4}{\pi}\,.$$

and

$$a_k=\frac{2}{\pi}\int_0^{\pi}\sin x\cos kx\,\mathrm{d}x=\frac{2(1-(-1)^k)}{(k^2-1)\pi}\,.$$

Note that the last formula is not determined for $k=1$, but direct calculations show that $a_1=0$ (as well as all other odd coefficients). Partial sums for my Fourier series are shown in Fig. 9. ■

Exercise 4. The goal is to show that for the function with the property

$$f(x)=f(x+T)$$

for all x and constant $T>0$, I have that

$$\int_a^{a+T} f(x)\,\mathrm{d}x=\int_0^T f(x)\,\mathrm{d}x$$

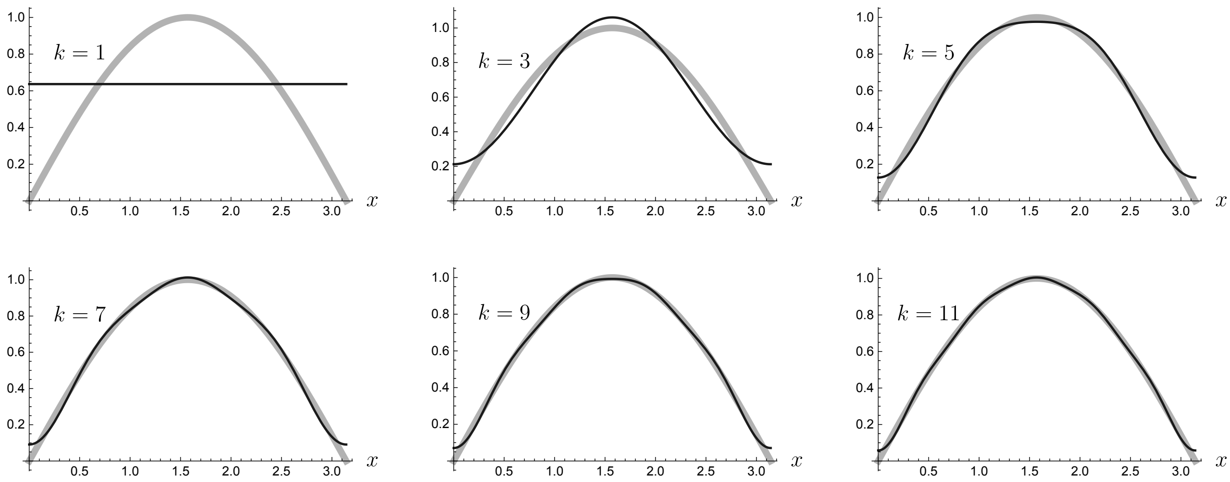

Figure 9: Comparison of the graphs of the original function $\sin x$ (light gray) and the partial sums S_k (dark grey) of its Fourier cosine series for various k.

for all a.

By the properties of the integral I have

$$\int_0^T f(x)\,\mathrm{d}x + \int_T^{T+a} f(s)\,\mathrm{d}s = \int_0^a f(t)\,\mathrm{d}t + \int_a^{T+a} f(x)\,\mathrm{d}x.$$

But

$$\int_T^{T+a} f(s)\,\mathrm{d}s = (\text{substitution } s = t + T) = \int_0^a f(t)\,\mathrm{d}t,$$

and I get what is required.

(There is much slicker proof of the same property, for an interested student: Consider

$$\frac{\mathrm{d}}{\mathrm{d}a} \int_a^{T+a} f(x)\,\mathrm{d}x$$

and show that it is zero, hence the integral does not depend on a.) ■

Exercise 5. I have that for $g_k(x) = e^{\mathrm{i}kx}, k = 0, \pm 1, \pm 2, \ldots$

$$\langle g_k, g_k \rangle = \int_{-\pi}^{\pi} 1\,\mathrm{d}x = 2\pi$$

and for $k \neq m$

$$\langle g_k, g_m \rangle = \int_{-\pi}^{\pi} e^{\mathrm{i}kx} e^{-\mathrm{i}mx}\,\mathrm{d}x = \int_{-\pi}^{\pi} e^{\mathrm{i}x(k-m)} = \frac{2}{\mathrm{i}(k-m)} \sin \pi(k-m) = 0,$$

hence my system of functions is orthogonal. Using this orthogonality I find that

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) e^{-\mathrm{i}kx}\,\mathrm{d}x, \quad k = 0, \pm 1, \pm 2, \ldots$$

Using Euler's formula I have

$$\int_{-\pi}^{\pi} g(x)e^{-\mathrm{i}kx}\,\mathrm{d}x = \int_{-\pi}^{\pi} g(x)\cos kx\,\mathrm{d}x - \mathrm{i}\int_{-\pi}^{\pi} g(x)\sin kx\,\mathrm{d}x,$$

and hence

$$c_k = \frac{1}{2}(a_k - \mathrm{i}b_k), \quad k = 0, 1, 2, \ldots$$

and

$$c_{-k} = \frac{1}{2}(a_k + \mathrm{i}b_k), \quad k = 1, 2, , \ldots,$$

which also yields

$$a_k = c_k + c_{-k}, \quad b_k = \mathrm{i}(c_k - c_{-k}), \quad k = 0, 1, 2, \ldots$$

■