12 Fourier method for the heat equation

Now I am well prepared to work through some simple problems for a one dimensional heat equation on a bounded interval.

12.1 A zoo of examples

Example 12.1. Assume that I need to solve the heat equation

$$u_t = \alpha^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \tag{12.1}$$

with the homogeneous Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t > 0, \tag{12.2}$$

and with the initial condition

$$u(0,x) = g(x), \quad 0 \leq x \leq 1. \tag{12.3}$$

Let me start again with the ansatz

$$u(t,x) = T(t)X(x).$$

The equation (12.1) implies

$$T'X = \alpha^2 TX'',$$

where the primes denote the derivatives with respect to the corresponding variables. Separating the variables implies that

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = -\lambda,$$

where the minus sign I chose for notational reasons. Therefore, I now have two ODE, moreover the second ODE

$$X'' + \lambda X = 0$$

is supplemented with the boundary conditions $X(0) = X(1) = 0$, which follows from (12.2).

Recall that I already analyzed a similar situation with periodic boundary conditions, and I considered all possible complex values of the separation constant. The goal is to determine all possible constants λ such that the corresponding boundary value problem has a nontrivial solution (i.e., different from being identically zero). Here I will consider only real values of λ, a rigorous (and elementary!) general proof that they must be real will be given later.

I start with the case $\lambda = 0$. This means $X'' = 0 \implies X(x) = Ax + B$, where A and B are two real constants. The boundary conditions imply that $A = B = 0$ and hence for $\lambda = 0$ my boundary value problem has no nontrivial solution.

Now assume that $\lambda < 0$. The general solution to my ODE in this case is

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x},$$

Math 483/683: Partial Differential Equations by Artem Novozhilov
e-mail: artem.novozhilov@ndsu.edu. Spring 2023

and the boundary conditions yield

$$A + B = 0, \quad Ae^{\sqrt{-\lambda}} + Be^{-\sqrt{-\lambda}} = 0,$$

or

$$B(e^{2\sqrt{-\lambda}} - 1) = 0,$$

which implies that $A = B = 0$ since $e^{2\sqrt{-\lambda}} \neq 1$ for any real negative λ. Therefore again no nontrivial solutions.

Exercise 1. Reconsider the case $\lambda < 0$ starting with the general solution $X(x) = A \sinh \sqrt{-\lambda} x + B \cosh \sqrt{-\lambda} x$.

Finally, assuming that $\lambda > 0$, I get

$$X(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x,$$

and the boundary conditions imply that

$$A = 0, \quad B \sin \sqrt{\lambda} = 0,$$

which can be true if $B = 0$ (trivial solution), or if

$$\sin \sqrt{\lambda} = 0 \implies \sqrt{\lambda} = \pi k, \quad k \in \mathbf{Z}.$$

Therefore, for any

$$\lambda_k = (\pi k)^2, \quad k = 1, 2, 3, \ldots$$

note that I disregarded 0 and all negative values of k since they do not yield new solutions, I get nontrivial solutions

$$X_k(x) = B_k \sin \pi k x.$$

Now I can return to the ODE $T' = -\alpha^2 \lambda T$ with the solutions for admissible values of lambda in the form

$$T_k(t) = C_k e^{-\alpha^2 \pi^2 k^2 t}.$$

The analysis above implies (I take $b_k = B_k C_k$) that

$$u_k(t, x) = b_k e^{-\alpha^2 \pi^2 k^2 t} \sin \pi k x, \quad k = 1, 2, \ldots$$

solve equation (12.1) and satisfy the boundary conditions (12.2). The remaining part is to satisfy the initial condition (12.3). For this I will use the superposition principle that says that if u_k solve (12.1) then any linear combination is also a solution, i.e.,

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t, x) = \sum_{k=1}^{\infty} b_k e^{-\alpha^2 \pi^2 k^2 t} \sin \pi k x$$

is my solution. I use (12.3) to find that

$$g(x) = \sum_{k=1}^{\infty} u_k(0, x) = \sum_{k=1}^{\infty} b_k \sin \pi k x,$$

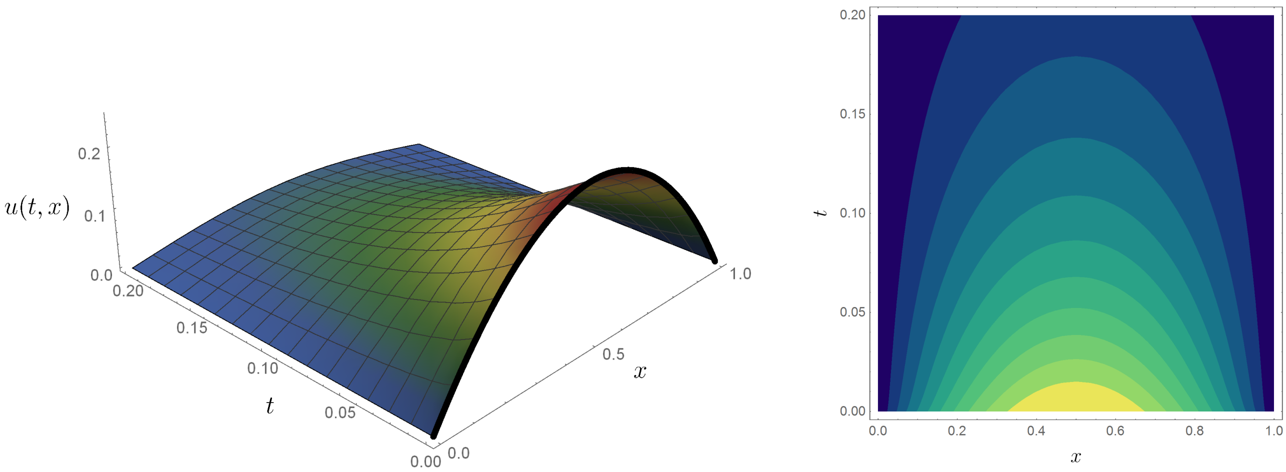


Figure 1: Solution to the heat equation with the homogeneous Dirichlet boundary conditions and the initial condition (bold curve) $g(x) = x - x^2$ Left: Three dimensional plot, right: contour plot.

which is exactly the sine series for my function g, the coefficients of which can be found as

$$b_k = 2\int_0^1 g(x)\sin \pi k x \,\mathrm{d}x, \quad k = 1, 2, \ldots.$$

As a specific example I can take

$$g(x) = x - x^2.$$

Then

$$b_k = \frac{4(1-(-1)^k)}{\pi^3 k^3}.$$

The solutions are graphically represented in Fig. 1. We can see that, as expected, the temperature in the rod approaches zero as time goes to infinity.

What else can be inferred from the representation of our solution as its Fourier series?

First I note that the exponents are responsible for the speed of approaching the equilibrium state, moreover, for sufficiently large t, all the expressions of the form e^{-Ak^2t} are very small compared to the first exponent with index k if $b_k \neq 0$. Therefore, in many practical situations it is possible to concentrate only on the first nonzero term of the Fourier series

$$u(t,x) \approx u_k(t,x) = b_k e^{-\alpha^2 k^2 \pi^2 t}\sin \pi k x, \quad \text{first } b_k \neq 0.$$

The approximation becomes better and better as t grows. In Fig. 2 one can see the difference $u_1(t,x) - \sum_{k=1}^{10} u_k(t,x)$ for my example with $g(x) = x - x^2$.

Second, and more important, I note that the same negative exponents in the representation of the solution by the sine Fourier series will guarantee that *any* derivative of the Fourier series will converge (it does require some proof). This is an important characterization of the solutions to the heat equation: Its solution, irrespective of the initial condition, is an infinitely differentiable function for any $t > 0$.

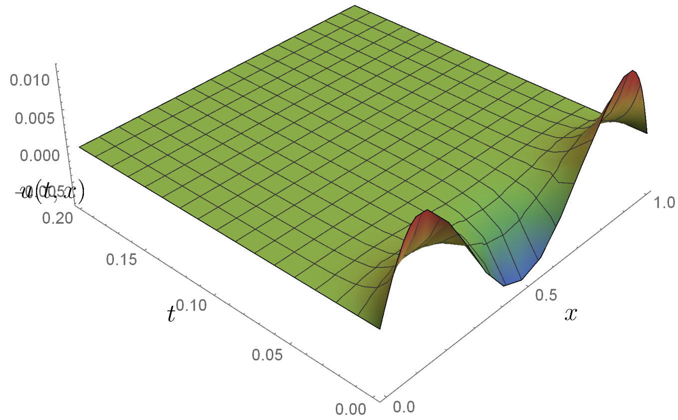


Figure 2: The difference $u_1(t,x) - \sum_{k=1}^{10} u_k(t,x)$ in the example with $g(x) = x - x^2$.

Here is the same problem with

$$g(x) = \begin{cases} 0, & 0 < x < 1/4, \\ 1, & 1/4 < x < 3/4, \\ 0, & 3/4 < x < 1. \end{cases}$$

You can see the smoothing effect of the heat equation on the discontinuous initial condition (see Fig. 3).

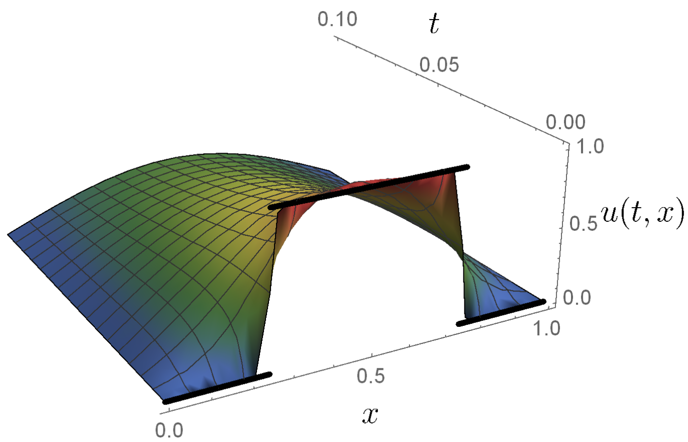


Figure 3: Solution to the heat equation with a discontinuous initial condition. For any $t > 0$ the solution is an infinitely differential function.

I can also note that if we would like to revert the time and look into the past and not to the future, then all the exponent would have the sign plus, which means that in general Fourier series will diverge for any $t < 0$. This is actually a manifestation of the fact that the inverse problem for the heat equation is not *well posed*, the heat equation represents a meaningful mathematical model only for $t > 0$ and the solutions are not reversible. (As a side remark I note that *ill-posed* problems are very important and there are special methods to attack them, including solving the heat equation for $t < 0$, note that this is equivalent to solve for $t > 0$ the equation of the form $u_t = -\alpha^2 u_{xx}$.)

Example 12.2. Consider now the Neumann boundary value problem for the heat equation (recall that homogeneous boundary conditions mean insulated ends, no energy flux):

$$\begin{aligned} u_t =& \alpha^2 u_{xx}, \quad t > 0, \\ u_x(t,0) = u_x(t,1) = 0, \quad t > 0, \\ u(0,x) = g(x), \quad 0 \leq x \leq 1. \end{aligned}$$

Now, after introducing $u(t,x) = T(t)X(x)$ I end up with the boundary value problem for X in the form

$$X'' + \lambda X = 0, \quad X'(0) = X'(1) = 0.$$

I will leave it as an exercise to show that if $\lambda < 0$ then I do not have non-trivial solutions. If, however, $\lambda = 0$, I have that

$$X(x) = A + Bx,$$

and the boundary conditions imply that $B = 0$ leaving me free variable A. Hence I conclude that for $\lambda = 0$ my solution is $X(x) = A$. If $\lambda > 0$ then

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

and the boundary conditions imply that $B = 0$ and

$$A\sin\sqrt{\lambda} = 0,$$

which will be true if $\lambda = \pi^2 k^2$, $k = 1, 2, 3, \ldots$. Putting everything together I found that my ODE boundary value problem has nontrivial solutions only if (note that I include $k = 0$)

$$\lambda_k = \pi^2 k^2, \quad k = 0, 1, 2, \ldots$$

and these solutions are

$$X_k(x) = A_k \cos \pi k x, \quad k = 0, 1, \ldots.$$

From the other ODE I find

$$T_k(t) = C_k e^{-\alpha^2 k^2 \pi^2 t}, \quad k = 0, 1, \ldots$$

and therefore, due to the same superposition principle, I can represent my solution as

$$u(t,x) = \sum_{k=0}^{\infty} T_k(t) X_k(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-\alpha^2 k^2 \pi^2 t} \cos \pi k x.$$

Using the initial condition, I find that

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos \pi k x,$$

which is the cosine Fourier series for g, where

$$a_k = 2\int_0^1 g(x)\cos \pi k x\,\mathrm{d}x.$$

Note that as expected, my solution tends to

$$u(t,x) \to \frac{a_0}{2} = \int_0^1 g(x)\,\mathrm{d}x, \quad t\to\infty,$$

which is a mathematical description of the fact that the energy inside my rod must be conserved. The solution that I found is also, as in the Dirichlet case, infinitely differentiable at any $t>0$, and the problem is ill-posed for $t<0$.

Example 12.3. Recall the problem for the heat equation with periodic boundary conditions:

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \quad t>0,\, -\pi<x\leq \pi,\\ u(t,-\pi) &= u(t,\pi), \quad t>0,\\ u_x(t,-\pi) &= u(t,\pi), \quad t>0,\\ u(0,x) &= g(x), \quad -\pi<x\leq\pi. \end{aligned}$$

We found that the boundary value problem

$$X'' + \lambda X = 0, \quad X(-\pi)=X(\pi),\, X'(-\pi)=X'(\pi)$$

has a non-trivial solution only if

$$\lambda_k = k^2, \quad k=0,1,2,\ldots,$$

and these solutions are

$$X_k(x) = A_k\cos kx + B_k \sin kx.$$

Moreover, the full solution is given by the Fourier series

$$u(t,x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-\alpha^2k^2t}(a_k\cos kx + b_k\sin kx),$$

where a_k, b_k are the coefficients of the trigonometric Fourier series for g on $-\pi\leq x\leq\pi$ (the exact expressions are given in the previous section). Again, since the rod is insulated, I find that as $t\to\infty$

$$u(t,x)\to \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)\,\mathrm{d}x.$$

Example 12.4. Now let me consider a problem for the heat equation with Robin or Type III boundary condition on one end. I need to solve

$$\begin{aligned} u_t &= \alpha^2 u_{xx}, \quad t>0,\, 0<x<1,\\ u(t,0) &= 0, \quad t>0,\\ u_x(t,1) + hu(t,1) &= 0, \quad t>0,\\ u(0,x) &= g(x), \quad 0\leq x\leq 1. \end{aligned}$$

Here I will assume that my constant h is positive.

Again, using the usual method of the separation of variables, I end up with

$$T' = \alpha^2 \lambda T,$$

and

$$X'' + \lambda X = 0, \quad X(0) = 0,\, X(1) + hX(1) = 0.$$

First I consider the latter problem. I will look into only real values of constant λ.

$\lambda = 0$ implies that $X(x) = 0$ and hence of no interest to me. If $\lambda < 0$ then I get the system

$$\begin{aligned} 0 &= A + B, \\ 0 &= A(he^{-\sqrt{-\lambda}} - \sqrt{-\lambda}e^{-\sqrt{-\lambda}}) + B(\sqrt{-\lambda}e^{\sqrt{-\lambda}} + he^{\sqrt{-\lambda}}). \end{aligned}$$

This is a system of linear homogeneous equations with respect to A and B, and it has a nontrivial solution if and only if the corresponding determinant of the system vanishes, which is equivalent, after some simplification, to

$$e^{2\sqrt{-\lambda}} = \frac{h - \sqrt{-\lambda}}{h + \sqrt{-\lambda}}\,.$$

Note that the left hand side as the function of $\sqrt{-\lambda}$ has positive derivative, and the left hand side has negative derivative, moreover they cross at the point $\lambda = 0$. Therefore, for $-\lambda > 0$ it is impossible to have solutions to this equations, which rules out the case $\lambda < 0$.

Finally, if $\lambda > 0$, then I get

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

and hence my boundary conditions imply that $A = 0$ and

$$B(\sqrt{\lambda}\cos\sqrt{\lambda} + h\sin\sqrt{\lambda}x) = 0.$$

The last equality can be true if $B = 0$ (not interesting) or if

$$\tan\sqrt{\lambda} = -\frac{\sqrt{\lambda}}{h}\,.$$

From the geometric considerations (see Fig. 4) it is clear that there is an infinite sequence of $(\lambda_k)_{k=1}^{\infty}$, $0 < \lambda_1 < \lambda_2 < \ldots$, such that $\lambda_k \to \infty$ as $k \to \infty$, and it is quite easy to find these lambdas numerically, but I do not have a convenient formula for them. My solutions are

$$X_k(x) = B_k \sin\sqrt{\lambda_k}x,$$

and hence any function

$$u_k(t, x) = b_k e^{-\alpha^2\lambda_k t}\sin\sqrt{\lambda_k}x$$

solves the PDE and satisfies the boundary conditions. Now I need to satisfy the initial condition. For this I will take the infinite linear combination of u_k and plug $t = 0$. I get

$$g(x) = \sum_{k=1}^{\infty} b_k \sin\mu_k x,$$

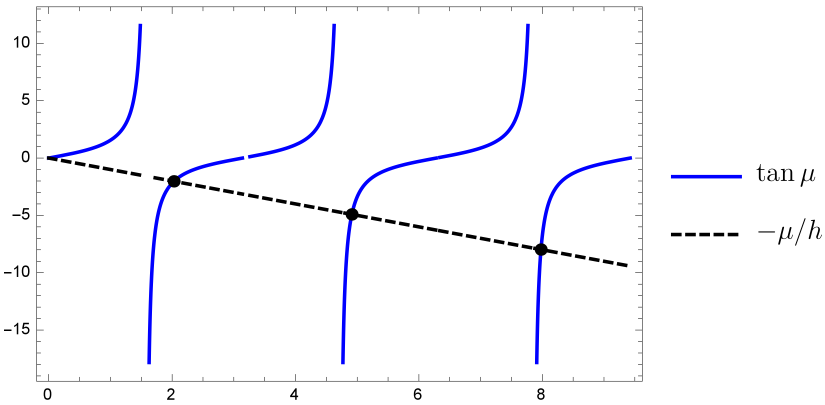


Figure 4: Solutions to the equation $\tan \mu = -\mu/h$.

where $\mu_k = \sqrt{\lambda_k}$. This *looks like* a Fourier sine series, but this is not, because in the classical Fourier sine series I need $\mu_k = \pi k$, which is not true for my example, and hence I cannot use the formulas for the coefficients. Luckily, however, it turns out the the system of functions $\{\sin \mu_k x\}$ is orthogonal on $[0, 1]$, and following the same steps that were done when I derived the coefficients for the trigonometric Fourier series, I can find that

$$b_k = \frac{2\mu_k}{\mu_k - \sin \mu_k \cos \mu_k} \int_0^1 g(x) \sin \mu_k x \, \mathrm{d}x.$$

Now my problem is fully solved because for any piecewise continuous g my time dependent Fourier series is an infinitely differentiable function.

Exercise 2. Confirm that the system $\{\sin \mu_k x\}$ is orthogonal on $[0, 1]$ and also confirm the formula for b_k.

As a specific example let me take

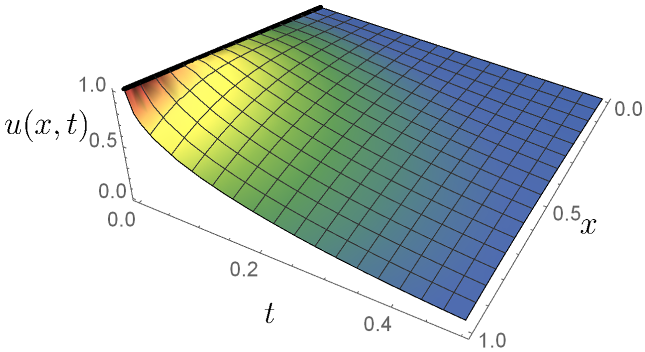
$$g(x) = x.$$

Then the solution has the form as in Fig. 5.

Not surprisingly, the solution approaches the trivial steady state, since the problem can be interpreted as the spread of the heat in an insulated rod with the fixed zero temperature at the left end and the temperature of the surrounding medium around the right end of the rod set to zero. Eventually the temperature evens out.

To emphasize that what I found is not a usual Fourier sine series, I will plot my infinite series for $t = 0$ in the symmetric interval $(-5, 5)$ along with a periodic extension of function x on $(-1, 1)$ (Fig. 6).

In all the examples above the boundary conditions were homogeneous. I emphasize that it must be always true to be able to proceed with the method of the separation of variables (more technically, I always need an ODE *plus* some boundary conditions, but if the original problem has inhomogeneous boundary conditions then I have no way to deduce from them the boundary conditions for my ODE problem). What to do if I am given non-homogeneous boundary conditions? Sometimes we can reduce the problem with inhomogeneous boundary conditions to the problem with homogeneous ones.



]

Figure 5: Solutions to the heat equation with Robin boundary condition and the initial condition $g(x) = x$.

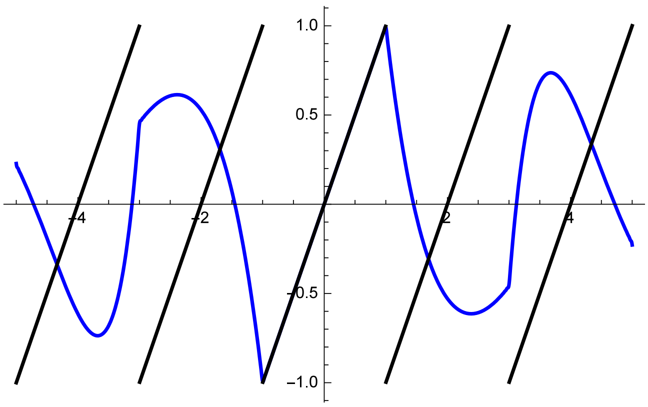


Figure 6: The periodic extension (black) of $g(x) = x$ on $(-1, 1)$ along with its generalized Fourier series (blue) based on the system $\{\sin \mu_k x\}$ on the interval $(-5, 5)$.

Consider as an example the Dirichlet problem for the heat equation with

$$u(t, 0) = k_1, \quad u(t, l) = k_2.$$

Here k_1, k_2 are two given constants. It should be clear (if not, carefully do all the math) that the equilibrium (stationary) temperature in this case is given by

$$u_{eq}(x) = k_1 + \frac{x}{l}(k_2 - k_1).$$

Now consider

$$u(t, x) = u_{eq}(x) + v(t, x).$$

For the function v I will get (check!) the problem

$$v_t = \alpha^2 v_{xx},$$
$$v(t,0) = v(t,l) = 0,$$
$$v(0,x) = g(x) - u_{eq}(x),$$

where g is the initial condition from the original problem for u. Now I have homogeneous boundary conditions and can use the Fourier method. Such approach will usually work when the boundary conditions do not depend on t, otherwise I will end up with a nonhomogeneous heat equation, which still can be solved using the separation of variables technique, but the solution process is slightly more involved.

12.2 Conclusion

In this lecture I considered four examples that have a lot of common features. In particular, all of them involve solving

$$X'' + \lambda X = 0,$$

or, better,

$$-X'' = \lambda X,$$

with some boundary conditions. By solving I mean "identifying those values of the parameter λ such that my boundary value problem for ODE has a nontrivial (nonzero) solution." In all four cases I find an infinite sequence of such lambdas, all of which are real. Even more importantly, in all four cases the corresponding solutions form an orthogonal system of functions, and hence the Fourier series technique can be applied to represent the solution to my original PDE problem in the form of a generalized Fourier series.

Is it a coincidence? Will it be the same for some other boundary conditions? I will answer these questions in the next section.

12.3 Test yourself

1. Consider three BVP for
$$X'' + \lambda X = 0$$
with
$$(a) \quad X(0) = X(l) = 0, \quad (b) \quad X'(0) = X'(l) = 0, \quad (c) \quad X(-l) = X(l),\, X'(-l) = X'(l).$$
All three problems are solved in the text (up to a scaling of the length of the interval). Reproduce these solutions and clearly state for which λ there are nontrivial solutions, and write down these solutions.

2. Solve
$$u_t = \alpha^2 u_{xx}, \quad x \in (0,1), \quad t > 0$$
with $u(t,0) = u(t,1) = 0$ and $u(0,x) = \sin 2\pi x - \frac{1}{2}\sin 5\pi x$.

3. Solve
$$u_t = \alpha^2 u_{xx}, \quad x \in (0,1), \quad t > 0$$
with $u'(t,0) = u'(t,1) = 0$ and $u(0,x) = \cos 2\pi x - \pi \cos 3\pi x$.

12.4 Solutions to the exercises

Exercise 1. The goal of this exercise is to show that in some situations a specific choice of the general solution leads to much simpler calculations. I need to solve $X'' + \lambda X = 0$ with $X(0) = X(1) = 0$ and $\lambda < 0$. My general solution is given by $X(x) = AX_1(x) + BX_2(x)$, where X_1 and X_2 are *any* linearly independent solutions to my ODE. Very often one takes $X_1(x) = e^{\sqrt{-\lambda}x}$ and $X_2(x) = e^{-\sqrt{-\lambda}x}$ but this is certainly not the only possible choice. Another possibility is to take $X_1(x) = \sinh\sqrt{-\lambda}x$ and $X_2(x) = \cosh\sqrt{-\lambda}x$. Hence

$$X(x) = A\sinh\sqrt{-\lambda}x + B\cosh\sqrt{-\lambda}x.$$

Using the boundary conditions and the basic properties of the hyperbolic sine and cosine I immediately get that the first boundary condition implies that $B = 0$. The second one yields hence that

$$A\sinh\sqrt{-\lambda} = 0,$$

which can be true if either $A = 0$ or $\lambda = 0$, and therefore there exists no nontrivial solutions to my boundary value problem with $\lambda < 0$, much simpler than the calculations (mainly skipped!) in the main text.

I repeat that in many situations using this particular choice for the general solution of $X''+\lambda X = 0$ leads to significant simplifications in calculations. ■

Exercise 2. The goal is to show that is μ_k are the positive solutions to $\tan\mu_k = -\mu_k/h$ then the system of functions $\{\sin\mu_k x\}$ is orthogonal on $[0, 1]$. Assume that $\mu_k \neq \mu_m$ are two such roots, and consider

$$\begin{aligned}\int_0^1 \sin\mu_k x\sin\mu_m x\,\mathrm{d}x &= \frac{1}{2}\int_0^1\Big(\cos(\mu_k-\mu_m)x - \cos(\mu_k+\mu_m)\Big)\,\mathrm{d}x\\ &= \frac{1}{2}\frac{(\mu_k+\mu_m)\sin(\mu_k-\mu_m) - (\mu_k-\mu_m)\sin(\mu_k-\mu_m)}{\mu_k^2-\mu_m^2}\\ &= \frac{\mu_m\sin\mu_k\cos\mu_m - \mu_k\sin\mu_m\cos\mu_k}{\mu_k^2-\mu_m^2}.\end{aligned}$$

Now, using the fact that $\tan\mu_k = \sin\mu_k/\cos\mu_k = -\mu_k/h$ implies that $\sin\mu_k = -\mu_k\cos\mu_k/h$, I get, as expected

$$\frac{\mu_m\sin\mu_k\cos\mu_m - \mu_k\sin\mu_m\cos\mu_k}{\mu_k^2-\mu_m^2} = \frac{-\mu_k\mu_m\cos\mu_k\cos\mu_m + \mu_k\mu_m\cos\mu_m\cos\mu_k}{h(\mu_k^2-\mu_m^2)} = 0.$$

To confirm the formula for b_k I need to calculate

$$\int_0^1\sin^2\mu_k x\,\mathrm{d}x = \frac{1}{2}\int_0^1(1-\cos 2\mu_k x)\,\mathrm{d}x = \frac{1}{2} - \frac{1}{4\mu_k}\sin 2\mu_k = \frac{\mu_k - \sin\mu_k\cos\mu_k}{2},$$

compare this with the expression for b_k. ■