13 Sturm–Liouville problems. Eigenvalues and eigenfunctions

In the previous lecture I gave four examples of different boundary value problems for a second order ODE that resulted in a countable number of constants (lambdas) and a countable number of corresponding solutions, which were used afterwards to build a corresponding Fourier series to represent solutions for PDE. Not surprisingly, these four examples can be generalized in a relatively abstract framework, which I discuss in this lecture. In the literature this framework is called Sturm–Liouville problem after two mathematicians who first concentrated on this problem.

I start with a definition of a Sturm–Liouville differential operator $L$ on the interval $x \in [a; b]$:

$$Lu := -(p(x)u')' + q(x)u,$$

where $p, q$ are continuous on $[a; b]$ and $p(x) > 0$ for any $x \in [a, b]$. Note that all four problems from the previous lecture can be written as

$$Lu = \lambda u,$$  \hfill (13.2)

with $p(x) = 1$ and $q(x) = 0$ and some additional boundary conditions. Most of these boundary conditions can be written in the general form

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \alpha_1^2 + \alpha_2^2 > 0,$$

$$\beta_1 u(b) + \beta_2 u'(b) = 0, \quad \beta_1^2 + \beta_2^2 > 0.$$  \hfill (13.3)

Quite natural (recall the definition of eigenvalues and eigenvectors for, e.g., matrices) problem (13.2)–(13.3) is called the Sturm–Liouville eigenvalue problem. Sometimes instead of (13.3) periodic boundary conditions are used:

$$u(a) = u(b), \quad p(a)u'(a) = p(b)u'(b).$$  \hfill (13.4)

To study the properties of the eigenvalues of the Sturm–Liouville problem, I start with a derivation of Lagrange’s identity. Let $u, v$ be two arbitrary $C^{(2)}[a, b]$ functions, then (check the skipped steps)

$$uLv - vLu = -(pu'v + vpu' - quv) = (pvu' - quv)'.'$$

From Lagrange’s identity, by integrating from $a$ to $b$, I get Green’s formula

$$\int_a^b (uLv - vLu) \, dx = p(vu' - uv')|_a^b.$$  

Both Lagrange’s identity and Green’s formula hold for any $u$ and $v$. I claim that if $u$ and $v$ are such that they satisfy (13.3) or (13.4) then Green’s formula becomes

$$\int_a^b (uLv - vLu) \, dx = 0.$$  

Indeed, if $u$ and $v$ satisfy, e.g., (13.3) then I must have that

$$\alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \alpha_1^2 + \alpha_2^2 > 0,$$

$$\alpha_1 v(a) + \alpha_2 v'(a) = 0,$$
and therefore the determinant of the matrix
\[
\begin{bmatrix}
u(a) & u'(a) \\
v(a) & v'(a)
\end{bmatrix}
\]
must be zero, that is
\[u(a)v'(a) - v(a)u'(a) = 0.\]
Similarly,
\[u(b)v'(b) - v(b)u'(b) = 0,
\]
which proves the stated fact for the boundary conditions (13.3). I leave checking (13.4) as an exercise.

Now, using the notation for the inner product:
\[
\langle u; v \rangle = \int_a^b u(x)v(x) \, dx;
\]
I can rewrite Green’s identity as
\[
\langle u, L v \rangle = \langle L u, v \rangle.
\]
An operator that satisfies such condition is called self-adjoint\(^1\) (think about symmetric real matrices, such that \(A = A^\top\)), and hence I proved that the Sturm–Liouville operator defined on the functions that satisfy (13.3) or (13.4) is self-adjoint (note that it is important to add the boundary conditions, without them the self-adjointness does not make any sense).

Now I am ready to prove

**Lemma 13.1.** Let \(L\) be a self-adjoint Sturm–Liouville operator. Then all the eigenvalues of \(L\) are real.

**Proof.** I will prove this lemma by contradiction. Let \(\lambda \in \mathbb{C}\) be my eigenvalue and \(u \neq 0\) a corresponding eigenfunction. By the properties of differential operator and linearity of \(L\) I get that \(\overline{\lambda}\) and \(\mathfrak{m}\) must be another eigenvalue and corresponding eigenfunction. Now consider
\[
\langle Lu, \mathfrak{m} \rangle = \langle \lambda u, \mathfrak{m} \rangle = \lambda \langle u, \mathfrak{m} \rangle.
\]
On the other hand, since \(L\) is self-adjoint,
\[
\langle Lu, \mathfrak{m} \rangle = \langle u, L \mathfrak{m} \rangle = \langle u, \overline{\lambda} \mathfrak{m} \rangle = \overline{\lambda} \langle u, \mathfrak{m} \rangle.
\]
Therefore
\[
(\lambda - \overline{\lambda}) \langle u, \mathfrak{m} \rangle = (\lambda - \overline{\lambda}) \int_a^b |u|^2 \, dx,
\]
and since \(\int_a^b |u|^2 \, dx > 0\) then
\[
\lambda = \overline{\lambda},
\]
which means that \(\lambda\) is real. \(\blacksquare\)

**Lemma 13.2.** Let \(L\) be a self-adjoint operator. Then if \(\lambda_1\) and \(\lambda_2\) are two different eigenvalues and \(u_1, u_2\) are two corresponding eigenfunctions then \(u_1\) and \(u_2\) are orthogonal.

\(^1\)The real life is much more complicated, I can only refer to a proper graduate course to set the matter straight.
Proof. I have
\[ \langle Lu_1, u_2 \rangle = \lambda_1 \langle u_1, u_2 \rangle = \lambda_2 \langle u_1, u_2 \rangle , \]
or
\[ (\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle = 0 \implies \langle u_1, u_2 \rangle = 0 . \]

Two previous lemmas are very nice, however, they are true under the assumption that my operator has any eigenvalues and eigenfunctions at all. A more impressive theorem, whose proof is significantly more involved, and hence omitted here, is as follow.

**Theorem 13.3.** Consider the Sturm–Liouville eigenvalue problem, i.e., (13.2) plus (13.3) or (13.4). Then there exists a countable sequence of eigenvalues
\[ \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots , \]
such that \( \lambda_k \to \infty \) as \( k \to \infty \). The corresponding system of eigenfunctions \( \{u_1, u_2, \ldots \} \) is complete in \( L^2[a, b] \), i.e., any function \( f \in L^2[a, b] \) can be represented as a convergent generalized Fourier series
\[ f(x) = c_1 u_1(x) + c_2 u_2(x) + \ldots , \]
where the coefficients are given by
\[ c_k = \frac{\langle f, u_k \rangle}{\langle u_k, u_k \rangle} . \]


To conclude this section let me collect together all the results for the Sturm–Liouville operator \( Lu = -u'' \) that we got so far (in the previous lecture and in homework problems).

<table>
<thead>
<tr>
<th>Boundary conditions</th>
<th>Eigenvalues</th>
<th>Eigenfunctions</th>
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<tbody>
<tr>
<td>( u(0) = u(1) = 0 )</td>
<td>( \lambda_k = \pi^2 k^2 ), ( k = 1, 2, \ldots )</td>
<td>( u_k(x) = B \sin \pi k x )</td>
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<tr>
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<td>( \lambda_k = \frac{\pi^2 k^2}{L^2} ), ( k = 1, 2, \ldots )</td>
<td>( u_k(x) = B \sin \frac{\pi k x}{L} )</td>
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<tr>
<td>( u'(0) = u'(1) = 0 )</td>
<td>( \lambda_k = \pi^2 k^2 ), ( k = 0, 1, 2, \ldots )</td>
<td>( u_k(x) = A \cos \pi k x )</td>
</tr>
<tr>
<td>( u(-\pi) = u(\pi), u'(-\pi) = u'(\pi) )</td>
<td>( \lambda_k = k^2 ), ( k = 0, 1, 2, \ldots )</td>
<td>( u_k(x) = A \cos k x, v_k(x) = B \sin k x )</td>
</tr>
<tr>
<td>( u(0) = u'(1) + hu(1) = 0 )</td>
<td>Solutions to tan ( \sqrt{\lambda} = -\sqrt{\lambda}/h )</td>
<td>( u_k(x) = B \sin \sqrt{\lambda_k} x )</td>
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