13 Sturm–Liouville problem. Eigenvalues and eigenfunctions

Mathematics is the art of giving the same name to different things.

Henri Poincaré (great French polymath, 1854–1912)

In the previous lecture I gave four examples of different boundary value problems for a second order ODE that resulted in a countable number of constants (lambdas) and a countable number of corresponding solutions, which were used afterwards to build the corresponding Fourier series to represent solutions for the heat equation. Not surprisingly, these four examples can be generalized in a relatively abstract framework, which I discuss in this lecture. In the literature this framework is called *Sturm–Liouville* problem after two mathematicians who first concentrated on it.

13.1 Regular Sturm–Liouville eigenvalue problem

I start with the definition of *Sturm–Liouville differential operator* L on the interval $x \in [a, b]$:

$$Lu := -(p(x)u')' + q(x)u, \tag{13.1}$$

where p, q are continuous on $[a, b]$ and $p(x) > 0$ for any $x \in [a, b]$. Note that all four problems from the previous lecture can be written as

$$Lu = \lambda u, \tag{13.2}$$

with $p(x) = 1$ and $q(x) = 0$ and some additional boundary conditions. Most of these boundary conditions can be written in the general form

$$\begin{aligned} \alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \alpha_1^2 + \alpha_2^2 > 0,\\ \beta_1 u(b) + \beta_2 u'(b) = 0, \quad \beta_1^2 + \beta_2^2 > 0. \end{aligned} \tag{13.3}$$

Quite natural (recall the definition of eigenvalues and eigenvectors for, e.g., matrices) problem (13.2)–(13.3) is called the *regular Sturm–Liouville eigenvalue problem.* Sometimes instead of (13.3) periodic boundary conditions are used:

$$u(a) = u(b), \quad p(a)u'(a) = p(b)u'(b). \tag{13.4}$$

To study the properties of the eigenvalues of the Sturm–Liouville problem, I start with a derivation of *Lagrange's identity.* Let u, v be two arbitrary $\mathcal{C}^{(2)}[a, b]$ functions, then (check the skipped steps)

$$uLv - vLu = -u(pv')' + quv + v(pu')' - quv = v(pu')' - u(pv')' = \big(p(vu' - uv')\big)'.$$

From Lagrange's identity, by integrating from a to b, I get *Green's formula*

$$\int_a^b (uLv - vLu)\,\mathrm{d}x = p(vu' - uv')\big|_a^b.$$

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Both Lagrange's identity and Green's formula hold for any u and v. I claim that if u and v are such that they satisfy (13.3) or (13.4) then Green's formula becomes

$$\int_a^b (uLv - vLu)\,\mathrm{d}x = 0.$$

Indeed, if u and v satisfy, e.g., (13.3) then I must have that

$$\begin{aligned} &\alpha_1 u(a) + \alpha_2 u'(a) = 0, \quad \alpha_1^2 + \alpha_2^2 > 0,\\ &\alpha_1 v(a) + \alpha_2 v'(a) = 0, \end{aligned}$$

and therefore the determinant of the matrix

$$\begin{bmatrix} u(a) & u'(a)\\ v(a) & v'(a) \end{bmatrix}$$

must be zero, that is

$$u(a)v'(a) - v(a)u'(a) = 0.$$

Similarly,

$$u(b)v'(b) - v(b)u'(b) = 0,$$

which proves the stated fact for the boundary conditions (13.3). I leave checking (13.4) as an exercise.

Now, using the notation for the inner product:

$$\langle u, v\rangle = \int_a^b u(x)v(x)\,\mathrm{d}x,$$

I can rewrite Green's formula as

$$\langle u, Lv\rangle = \langle Lu, v\rangle.$$

An operator that satisfies such condition is called *self-adjoint*[1] (think about symmetric real matrices, such that $\boldsymbol{A} = \boldsymbol{A}^\top$), and hence I proved that the Sturm–Liouville operator defined on the functions that satisfy (13.3) or (13.4) is self-adjoint (note that it is important to add the boundary conditions, without them the self-adjointness does not make any sense).

What is so special of eigenvalues and eigenvectors of self-adjoint operator? It turns out that the eigenvalues must be real (as I already mentioned several times) and the eigenvectors (another common term is *eigenfunctions*) corresponding to different eigenvalues must be orthogonal. I will show it in a minute, but first I must pause for a second and introduce a different inner product to make my mathematics rigorous. The point is that if I am about to talk of *complex* eigenvalues (and hence potentially complex eigenvectors), the definition for the inner product, which is perfectly fine for the *real* world will not do. To see this, let me take $u(x) = \mathrm{i}$ and compute

$$\langle u, u\rangle_{\text{real}} = \int_a^b \mathrm{i}^2\,\mathrm{d}x = -(b-a) < 0!$$

That is I am loosing positive definiteness of the inner product of vector with itself. To straiten things out let me now define an inner product for the complex valued functions of real argument x as follows:

$$\langle u, v\rangle = \int_a^b u(x)\overline{v(x)}\,\mathrm{d}x.$$

[1] The real life is much more complicated, I can only refer to a proper graduate course to set the matter straight.

This inner product satisfies the following properties:

$$\begin{aligned}(i)\quad & \langle u, u\rangle \geq 0,\\ (ii)\quad & \langle u, u\rangle = 0 \Leftrightarrow u = 0,\\ (iii)\quad & \langle u, v\rangle = \overline{\langle v, u\rangle},\\ (iv)\quad & \langle \alpha u, v\rangle = \alpha\langle u, v\rangle.\end{aligned}$$

The definition of a self-adjoint operator remains unchanged.

Now I am ready to prove

Lemma 13.1. *Let L be a self-adjoint Sturm–Liouville operator. Then all the eigenvalues of L are real.*

Proof. Indeed, I have, assuming that $Lu = \lambda u$, and $u \neq 0$,

$$\langle Lu, u\rangle = \langle \lambda u, u\rangle = \lambda\langle u, u\rangle.$$

On the other hand, since L is self-adjoint,

$$\langle Lu, u\rangle = \langle u, Lu\rangle = \langle u, \lambda u\rangle = \overline{\lambda}\langle u, u\rangle.$$

Therefore

$$(\lambda - \overline{\lambda})\langle u, u\rangle = 0,$$

and since $\langle u, u\rangle > 0$ then

$$\lambda = \overline{\lambda},$$

which means that λ is real. ■

Lemma 13.2. *Let L be a self-adjoint operator. Then if λ_1 and λ_2 are two different eigenvalues and u_1, u_2 are two corresponding eigenfunctions then u_1 and u_2 are orthogonal.*

Proof. I have

$$\langle Lu_1, u_2\rangle = \lambda_1\langle u_1, u_2\rangle = \lambda_2\langle u_1, u_2\rangle,$$

or

$$(\lambda_1 - \lambda_2)\langle u_1, u_2\rangle = 0 \implies \langle u_1, u_2\rangle = 0.$$

■

Two previous lemmas are very nice, however, they are true only under the assumption that my operator has any eigenvalues and eigenfunctions at all, which I never proved. A significantly more impressive theorem, whose proof is quite more involved, and hence omitted here, is as follow.

Theorem 13.3. *Consider the Sturm–Liouville eigenvalue problem, i.e.,* (13.2) *plus* (13.3) *or* (13.4). *Then there exists a countable sequence of eigenvalues*

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots,$$

such that $\lambda_k \to \infty$ *as* $k \to \infty$*. The corresponding system of eigenfunctions* $\{u_1, u_2, \dots\}$ *is complete in* $L^2[a,b]$*, i.e., any function* $f \in L^2[a,b]$ *can be represented as a convergent generalized Fourier series*

$$f(x) = c_1 u_1(x) + c_2 u_2(x) + \dots,$$

where the coefficients are given by

$$c_k = \frac{\langle f, u_k \rangle}{\langle u_k, u_k \rangle}.$$

Proof. A proof (of an even more general theorem) can be found in, e.g., P. Olver, Introduction to Partial Differential Equations, 2016. For those interested in seeing all the mathematical subtleties and learn more, I would like to mention the now classic book Coddington and Levinson, Theory of Ordinary Differential Equations, 1955. ∎

Clearly, implicitly I already used this theorem when I was assuming that my Fourier series faithfully represent my functions.

Note that the theorem implies that it is possible to have only a finite number of eigenvalues which are negative. Sometimes it is important to guarantee that there are no negative eigenvalues at all. Here is a simple and yet useful result. I call operator L (please note that I *include* the boundary conditions into this notation) *positive definite* (*positive semi-definite*) if for all allowable $u \neq 0$ I have $\langle Lu, u \rangle > 0$ ($\langle Lu, u \rangle \geq 0$).

Lemma 13.4. *Let* L *be a positive definite* (*positive semi-definite*) *Sturm–Liouville operator. Then all the eigenvalues are positive* (*nonnegative*)*.*

Proof. Since

$$\langle Lu, u \rangle = \lambda \langle u, u \rangle,$$

therefore,

$$\lambda = \frac{\langle Lu, u \rangle}{\langle u, u \rangle}.$$

∎

Example 13.5. Let me show something that we already know: I will prove that all the eigenvalues of the problem

$$-X'' = \lambda X, \quad X(0) = X(1) = 0,$$

are positive. This time, however, I will not compute them. Let u be a differentiable function that satisfies $u(0) = u(1) = 0$ and consider

$$\langle Lu, u \rangle = -\int_0^1 u''(x) u(x)\, \mathrm{d}x.$$

I have, using the integration by parts, that

$$-\int_0^1 u''(x)u(x)\,\mathrm{d}x = -u(1)u'(1) + u(0)u'(0) + \int_0^1 \big(u'(x)\big)^2\,\mathrm{d}x = \int_0^1 \big(u'(x)\big)^2\,\mathrm{d}x \geq 0.$$

Now I claim that for $u \neq 0$ and $u(0) = u(1) = 0$ the inequality is actually strict. Indeed, assuming that $\int_0^1 \big(u'(x)\big)^2\,\mathrm{d}x = 0$ implies that $u'(x) = 0$, that is u is a constant, but the boundary conditions imply that this constant must be zero. That is, I showed that my differential operator is positive definite, and hence, by the proven lemma, all the eigenvalues are positive (as I already knew by explicit computation). I will leave it as an exercise to show that the operator in the problem $-X'' = \lambda X$, $X'(0) = X'(1) = 0$ is positive semi-definite.

To conclude this section let me collect together all the results for the Sturm–Liouville operator $Lu = -u''$ that we got so far (in the previous lecture and in the homework problems).

Boundary conditions	Eigenvalues	Eigenfunctions
$u(0) = u(1) = 0$	$\lambda_k = \pi^2 k^2$, $k = 1, 2, \dots$	$u_k(x) = B \sin \pi k x$
$u(0) = u(l) = 0$	$\lambda_k = \frac{\pi^2 k^2}{l^2}$, $k = 1, 2, \dots$	$u_k(x) = B \sin \frac{\pi k}{l} x$
$u'(0) = u'(1) = 0$	$\lambda_k = \pi^2 k^2$, $k = 0, 1, 2, \dots$	$u_k(x) = A \cos \pi k x$
$u(-\pi) = u(\pi)$, $u'(-\pi) = u'(\pi)$	$\lambda_k = k^2$, $k = 0, 1, 2, \dots$	$u_k(x) = A \cos kx$, $v_k(x) = B \sin kx$
$u(0) = u'(1) + hu(1) = 0$	Solutions to $\tan \sqrt{\lambda} = -\sqrt{\lambda}/h$	$u_k(x) = B \sin \sqrt{\lambda_k} x$
$u(0) = u'(1) = 0$	$\lambda_k = (\pi(k - 1/2))^2$, $k = 1, 2, \dots$	$u_k(x) = B \sin \pi(k - 1/2)x$, $k = 1, 2, \dots$
$u'(0) = u(1) = 0$	$\lambda_k = (\pi(k - 1/2))^2$, $k = 1, 2, \dots$	$u_k(x) = B \cos \pi(k - 1/2)x$, $k = 1, 2, \dots$

13.2 Critical patch size

It is rather impressive that given the theory I already covered I am capable of producing important and not-obvious otherwise conclusions about the real world around me. Let me in this section, following two famous research papers[2], show an existence of the so-called *critical patch size*, which is defined as the smallest geometric size of an ecological habitat, in which stable population growth can be observed.

Recall that I derived the diffusion equation as the limit of a simple random walk; as a first approximation it is reasonable to assume that a number of biological populations (e.g., phytoplankton) move randomly, thus resulting in something that closely resembles diffusion in the continuous limit. The key difference here is that, in addition to the random movements, the populations reproduce as well, thus adding a second process in our mathematical model. Let me assume, again for simplicity, that the reproduction process follows the simplest *Malthusian* equation $\dot{N} = rN$, which results in the presence of unlimited resources in the exponential growth $N(t) = N_0 e^{rt}$. Putting together the processes of population growth and diffusion I obtain the problem for the unknown population size $N(t, x)$ at time t at the point x:

$$N_t = rN + \alpha^2 N_{xx}, \quad 0 < x < l, \quad t > 0$$

with the initial condition

$$N(0, x) = N_0(x), \quad 0 < x < l,$$

and, to start with, the homogeneous Dirichlet boundary conditions

$$N(t, 0) = N(t, l) = 0.$$

An ecological interpretation of these conditions is that an individuum that approaches any of the boundaries immediately dies. Thus I have mutually opposite forces of reproduction (that pushes the

[2]Skellam, J. G. (1951). Random dispersal in theoretical populations. Biometrika, 38(1/2), 196–218; Kierstead, H., & Slobodkin, L. B. (1953). The size of water masses containing plankton blooms. J. mar. Res, 12(1), 141-147. I especially recommend the second one to read for an interested student.

population numbers up) and diffusion that makes at least some of the individuals to approach the boundaries and hence do not contribute to the population growth. Here is a very natural question: What is the smallest length l of my interval such that the population described by this equation does not do extinct?

To answer the posed question I would need to solve the formulated problem by using the separation of variables. It is completely doable and I invite the student to go this rout. I will, however, will make the following trick: I will make the change of variables $N(t,x)=e^{rt}u(t,x)$, and obtain

$$e^{rt}u_t+re^{rt}u=e^{rt}u+\alpha^2e^{rt}u_{xx}\implies u_t=\alpha^2u_{xx},\quad 0<x<l,\quad t>0,$$

with the additional conditions $u(0,x)=N_0(x)$ and $u(t,0)=u(t,l)=0$. But this is the problem which I already solved! Indeed, I have

$$u(t,x)=\sum_{k=1}^{\infty}b_ke^{-\frac{\alpha^2\pi^2k^2t}{l^2}}\sin\frac{\pi kx}{l}\,,$$

where b_k can be found as Fourier coefficients of the sine Fourier series for N_0 on $(0,l)$.

Returning to the original variable I get

$$N(t,x)=\sum_{k=1}^{\infty}b_ke^{\left(r-\frac{\alpha^2\pi^2k^2}{l^2}\right)t}\sin\frac{\pi kx}{l}\,,$$

which will be growing if and only if

$$r-\frac{\alpha^2\pi^2}{l^2}>0\implies l>\pi\sqrt{\frac{\alpha^2}{r}}\,.$$

I remark that the quantities α^2 and r are measurable for a given population, and in principle the value l that guarantees non extinction can be estimated.

Exercise 1. Find the critical population size for the problem

$$N_t=rN+\alpha^2N_{xx},\quad 0<x<l,\quad t>0,$$

with Type III boundary conditions:

$$N_x(t,0)=+hN(t,0),\quad N_x(t,l)=-hN(t,l),\quad t>0,\quad h>0.$$

Consider the limiting cases $h\to 0$ and $h\to\infty$.

To finish this short section I note that this particular way to unite the ecological description and diffusion process turned out to be very profitable. In many situations PDE of the form

$$N_t=f(N)+\alpha^2\Delta N$$

are studied, where in general nonlinear function f is responsible for the details of reproduction process, and the Laplace operator describes simple random walk. Not surprisingly such equations are called *reaction–diffusion* equations (see, e.g., Keener, J. P. (2021). Biology in time and space: a partial differential equation modeling approach (Vol. 50). American Mathematical Soc. for more details).

13.3 Analogies from linear algebra

I already mentioned several times that the boundary value problems for differential equations, when seen for the first time, can be profitably considered as generalizations of the standard eigenvalue problem for symmetric matrices. In my experience, however, many students could not recall these basic facts from linear algebra so that I will list them myself in this section. Since my goal is to present the theory in the form suitable for calculations, I will try to avoid abstract definitions and objects as much as possible (e.g., I will talk about matrices and not about linear operators). I emphasize that this section is not intended to be the first encounter with linear algebra, only as a refresher.

Let $\mathbf{R}^d$ be a d-dimensional vector space. In its simplest form it is by definition the set of vectors $\boldsymbol{x} = (x_1, \ldots, x_d)$, i.e., a vector is a row (or frequently better a column) of d real numbers. We are allowed to add two vectors

$$\boldsymbol{x} + \boldsymbol{y} = (x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d)$$

and multiply vectors by (real) scalars

$$\alpha \boldsymbol{x} = \alpha(x_1, \ldots, x_d) = (\alpha x_1, \ldots, \alpha x_d).$$

Together with $\mathbf{R}^d$ consider the set of square $d \times d$ real matrices, which I denote with capital letters, $\boldsymbol{A}, \boldsymbol{B}, \ldots$. Each such matrix defines a function on $\mathbf{R}^d$, for instance $\boldsymbol{A} \colon \mathbf{R}^d \longrightarrow \mathbf{R}^d$ by the standard matrix multiplication:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}, \quad y_i = \sum_{k=1}^{d} a_{ik} x_k, \quad i = 1, \ldots, d.$$

Matrix multiplication is linear, meaning that (check)

$$\boldsymbol{A}(\alpha \boldsymbol{x} + \boldsymbol{y}) = \alpha \boldsymbol{A}\boldsymbol{x} + \boldsymbol{A}\boldsymbol{y}.$$

Very naturally, one can define an inner product on $\mathbf{R}^d$:

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x} \cdot \boldsymbol{y} = \sum_{k=1}^{d} x_k y_k,$$

which satisfies all the same properties as I listed for the inner product for two functions (recall them and check!). Hence I have orthogonality (and angles), for instance the vectors $(1, 0)$ and $(0, 1)$ from $\mathbf{R}^2$ are orthogonal since $(1, 0) \cdot (0, 1) = 0$. In this situation, however, I can use my geometric imagination and intuition that indeed these two vectors are orthogonal (perpendicular) to each other (make a picture). Now, for each square matrix $\boldsymbol{A}$ I have that

$$\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{A}^\top \boldsymbol{y} \rangle,$$

where $\boldsymbol{A}^\top$ is the transpose of matrix $\boldsymbol{A}$, by definition $\boldsymbol{A}^\top = [a_{ij}^\top] = [a_{ji}]$ (I switch the roles of the rows and columns in my matrix).

Exercise 2. Prove that $\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{A}^\top \boldsymbol{y} \rangle$.

For each matrix $\boldsymbol{A}$ an eigenvalue problem can be posed: Find all nonzero vectors $\boldsymbol{x}$ and the corresponding constants λ such that

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x},$$

such vectors are called the eigenvectors, and the corresponding constants are called eigenvalues. Clearly (why?) for a λ to be an eigenvalue, it must be the root of the characteristic polynomial

$$p_d(\lambda) = \det(\boldsymbol{A} - \lambda \boldsymbol{I}),$$

where $\boldsymbol{I}$ is the identity matrix (i.e., the diagonal matrix with ones on the main diagonal and zeroes everywhere else), and det is the function *determinant* which maps the set of square matrices to real numbers (recall how it is computed for 2×2, 3×3 matrices, and in the general $d \times d$ case). The degree of p_d is exactly d, and therefore I immediately know that for matrices I can have *at most d* real eigenvalues. It is completely possible to have no real eigenvalues at all (give an example of such matrix), but if one moves to the complex world, that is consider now the vectors space $\mathbf{C}^d$, where now the vectors are rows (or columns) of *complex* numbers, the fundamental theorem of algebra implies that any (complex) matrix $\boldsymbol{A}$ has exactly d complex eigenvalues, if each eigenvalue counted according to its multiplicity. The general case is quite involved (you may want to look up *the Jordan canonical form*), but in some situations one can say more just looking at a matrix. In particular, the following theorem holds

Theorem 13.6. *Consider $\mathbf{R}^d$ and the set of all $d \times d$ real matrices. Let $\boldsymbol{A} = \boldsymbol{A}^\top$, i.e., matrix $\boldsymbol{A}$ is symmetric, then*

(i) all the eigenvalues of $\boldsymbol{A}$ are real;

(ii) there are exactly d linearly independent eigenvectors of matrix $\boldsymbol{A}$;

(iii) these eigenvectors can be chosen to form an orthogonal system $\{\boldsymbol{x}_j\}_{j=1}^d$, i.e., $\langle \boldsymbol{x}_j, \boldsymbol{x}_i \rangle = 0$ if $i \neq j$;

(iv) these vectors form a basis of $\mathbf{R}^d$, i.e., any other vector $\boldsymbol{x}$ can be uniquely represented as

$$\boldsymbol{x} = \alpha_1 \boldsymbol{x}_1 + \ldots + \alpha_d \boldsymbol{x}_d,$$

where the coefficients can be found as

$$\alpha_j = \frac{\langle \boldsymbol{x}, \boldsymbol{x}_j \rangle}{\langle \boldsymbol{x}_j, \boldsymbol{x}_j \rangle}.$$

Exercise 3. How many statements of this theorem you can prove?

Note that the assumption of a matrix to be symmetric implies that $\langle \boldsymbol{A}\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, \boldsymbol{A}\boldsymbol{y} \rangle$ for any $\boldsymbol{x}, \boldsymbol{y}$. To prove that eigenvalues are real one has to visit the complex world, where the inner product can be defined as

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = x_1 \bar{y}_1 + \ldots + x_d \bar{y}_d.$$

Exercise 4. Take an arbitrary symmetric matrix and check all the statements of Theorem 13.6 by example.

Now I encourage the reader to look again at the statements of Theorem 13.3 and 13.6 and closely compare them, the analogy should be clear (and I will use this analogy later in the course).

While I totally agree that it is useful for a student to use the intuition developed in the linear algebra course to set the matter straight for the Sturm–Liouville problems, I would like to note that historically the subject of *orthogonal functions* and *generalized Fourier series* appeared significantly earlier than the precise results of (finite dimensional) linear algebra, which is an interesting twist on its own; not always the simpler facts are discovered first.

13.4 Test yourself

1. Give a definition of an eigenvector and eigenvalue of a matrix.

2. Give an example of a regular Sturm–Liouville problem with full computation of eigenvalues and eigenfunctions.

3. Consider
$$-X'' = \lambda X$$
with the boundary conditions $X'(0) = X'(l)$. Find the eigenvalues and corresponding eigenfunctions.

4. In all the problems we considered the eigenvalues are nonnegative. Can you come up with a Sturm–Liouville problem, which must have a negative eigenvalue? Physical reasonings are sufficient in this problem.