14 Solving the wave equation by Fourier method

In this lecture I will show how to solve an initial–boundary value problem for one dimensional wave equation:

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < l, \quad t > 0, \tag{14.1}$$

with the initial conditions (recall that we need two of them, since (14.1) is a mathematical formulation of the second Newton's law):

$$\begin{aligned} u(0, x) &= f(x), \quad 0 < x < l, \\ u_t(0, x) &= g(x), \quad 0 < x < l, \end{aligned} \tag{14.2}$$

where f is the initial displacement and g is the initial velocity.

I start with the homogeneous boundary conditions of type I:

$$\begin{aligned} u(t, 0) &= 0, \quad t > 0, \\ u(t, l) &= 0, \quad t > 0, \end{aligned} \tag{14.3}$$

which physically means that I am studying the oscillations of a string of length l with *fixed* ends.

Using the same ansatz

$$u(t, x) = T(t)X(x)$$

I find that

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda,$$

and hence I have two ordinary differential equations

$$T'' + c^2 \lambda T = 0 \tag{14.4}$$

for T, and

$$X'' + \lambda X = 0 \tag{14.5}$$

for X. Using the boundary conditions (14.3) I conclude that equation (14.5) must be supplemented with the boundary conditions

$$X(0) = X(l) = 0. \tag{14.6}$$

Problem (14.5)–(14.6) is a Sturm–Liouville eigenvalue problem, which we already solved several times. In particular, we know that there is an infinite series of eigenvalues

$$\lambda_k = \frac{k^2 \pi^2}{l^2}, \quad k = 1, 2, \ldots$$

and the corresponding eigenfunctions

$$X_k(x) = C_k \sin \sqrt{\lambda_k} x = C_k \sin \frac{\pi k x}{l}, \quad k = 1, 2, \ldots,$$

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moreover all the eigenfunctions are orthogonal on $[0, l]$. Here C_k are some arbitrary real constants. Since I know which lambdas I can use, I now can look at the solutions to (14.4). Since all my $\lambda_k > 0$ then I have the general solution

$$T_k(t) = A_k \cos c\sqrt{\lambda_k}t + B_k \sin c\sqrt{\lambda_k}t = A_k \cos \frac{\pi c k t}{l} + B_k \sin \frac{\pi c k t}{l},$$

and hence each function

$$u_k(t, x) = T_k(t)X_k(x) = \left(a_k \cos \frac{\pi c k t}{l} + b_k \sin \frac{\pi c k t}{l}\right) \sin \frac{\pi k x}{l},$$

solves the wave equation (14.1) and satisfies the boundary conditions (14.3). Here $a_k = A_k C_k$, $b_k = B_k C_k$. Since my PDE is linear I can use *the superposition principle* to form my solution as

$$u(t, x) = \sum_{k=1}^{\infty} u_k(t, x),$$

my task is to determine a_k and b_k. For this I will need the initial conditions. Note that using the first initial conditions implies

$$f(x) = \sum_{k=1}^{\infty} a_k \sin \frac{\pi k x}{l},$$

which means that

$$a_k = \frac{2}{l}\int_0^l f(x) \sin \frac{\pi k x}{l}\, \mathrm{d}x.$$

To use the second boundary condition I first differentiate my series and then plug in $t = 0$:

$$g(x) = \sum_{k=1}^{\infty} \frac{b_k c \pi k}{l} \sin \frac{\pi k x}{l},$$

which gives me

$$b_k = \frac{2}{\pi k c}\int_0^l g(x) \sin \frac{\pi k x}{l}\, \mathrm{d}x.$$

Hence I found a formal solution to my original problem. I am writing "formal" since I must also check that all the series are convergent and can be differentiated twice in t and x to guarantee that what I found is a classical solution. Note that contrary to the heat equation, my series representations of solutions do not have quickly vanishing exponents and hence the question on differentiability is not as simple as before. Putting some additional smoothness requirements on the initial conditions can be used to conclude that my series are classical solutions.

Now let me see what I can infer from the found solution.

The found functions u_k are called the *normal modes*. Using the following trick:

$$\begin{aligned}
A \cos \alpha t + B \sin \alpha t &= \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}} \cos \alpha t + \frac{B}{\sqrt{A^2 + B^2}} \sin \alpha t\right) \\
&= \sqrt{A^2 + B^2}\left(\cos \phi \cos \alpha t + \sin \phi \sin \alpha t\right) \\
&= R \cos(\alpha t - \phi), \quad \phi = \tan^{-1} \frac{B}{A}, R = \sqrt{A^2 + B^2},
\end{aligned}$$

I can rewrite the normal modes in the form

$$u_k(t,x)=R_k\cos\left(\frac{c\pi kt}{l}-\phi_k\right)\sin\frac{\pi kx}{l}\,. \tag{14.7}$$

Recall that f is periodic with period T if

$$f(t+T)=f(t)$$

for any t. The (minimal) period is the smallest $T>0$ in this formula. Clearly if f is T-periodic then

$$f(t+kT)=f(t),\quad k\in\mathbf{Z}.$$

Moreover, if f is T-periodic then $f(at)$ has period T/a (prove it). These basic facts imply that the normal modes are periodic functions with respect to variable t with the periods

$$T_k=\frac{2l}{ck}\,,$$

because cosine is a 2π-periodic function. More importantly all normal modes have period $2l/c$ (not necessarily minimal), which allows me to conclude that the solution to the problem (14.1)–(14.3) is $2l/c$-periodic:

$$T=2l\sqrt{\frac{\rho}{E}}\,.$$

The *angular frequencies* are

$$\omega_k=\frac{2\pi}{T_k}=\frac{\pi ck}{l}\,.$$

The normal modes are also called the *harmonics*. The first harmonic is the normal mode of the lowest frequency, which is called the *fundamental frequency*

$$\omega_1=\frac{\pi c}{l}\,.$$

And now I finally can make my first big conclusion here: all harmonics in the solution to the initial-boundary value problem for the wave equation have frequencies that are multiple of the fundamental frequency ω_1, and this is the mathematical explanation why we like the sound of musical instruments whose geometry is one-dimensional: violin, guitar, flute, etc.

Geometrically harmonics represent *standing waves* (see Fig. 1). Using the introduced terminology I can conclude that the solution to the wave equation is a sum of standing waves, which is the product of cosines and sines. However, we also know that if the wave equation has no boundary conditions then the solution to the wave equation is a sum of traveling waves, moreover, the same traveling waves and the reflection principle can be used to built a solution on a bounded interval, and, at least in principle, the solution can be expressed through the *arbitrary* initial conditions. Can these two observations be reconciled? The answer is certainly "yes," but it is interesting to note that it took quite some time, and the minds of Euler, D'Alembert, D. Bernoulli, Lagrange, and finally Fourier to resolve this strange contradiction[1].

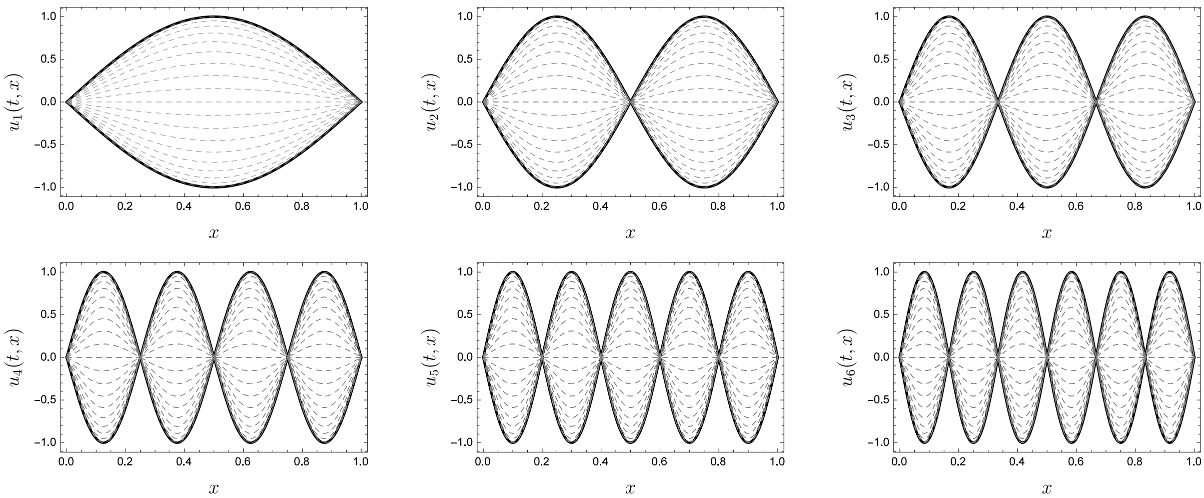


Figure 1: Standing waves for first six normal modes $u_k(t,x) = \cos \pi k t \sin \pi k x$. The bold lines represent the time moments when $\cos \pi k t = \pm 1$, the dotted lines are the graphs of u_k at intermediate time moments. For an observer the standing wave represent a periodic vibration of the string.

Example 14.1. To give a specific example, I assume that the initial displacement has the form shown in Fig. 2, and the initial velocity is zero. I find that $b_k = 0$ for all k and

$$a_k = \frac{2\left(2\sin\left(\frac{\pi k}{4}\right) - \sin\left(\frac{\pi k}{2}\right)\right)}{\pi k^2}.$$

To illustrate the time dependent behavior of my solution I take first 50 terms of my Fourier series and plot them at different time moments (see Fig. 3), you can observe the traveling waves and reflections from the boundaries for my example. A three dimensional graph of the same solution is given in Fig. 4.

[1]The historical details are given in Riemann, B. (1854). On the representation of a function by a trigonometric series, whose translation is given in Collected papers by Riemann, published by Kendrick Press in 2004.

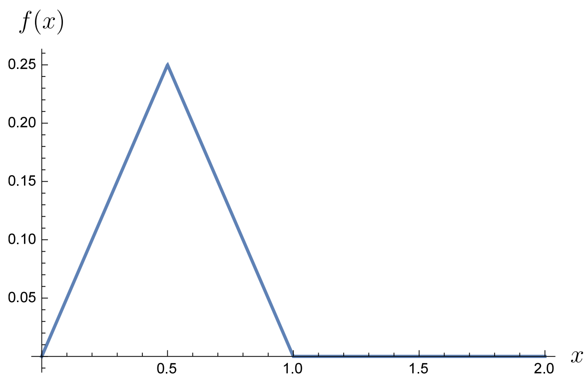


Figure 2: The initial displacement f.

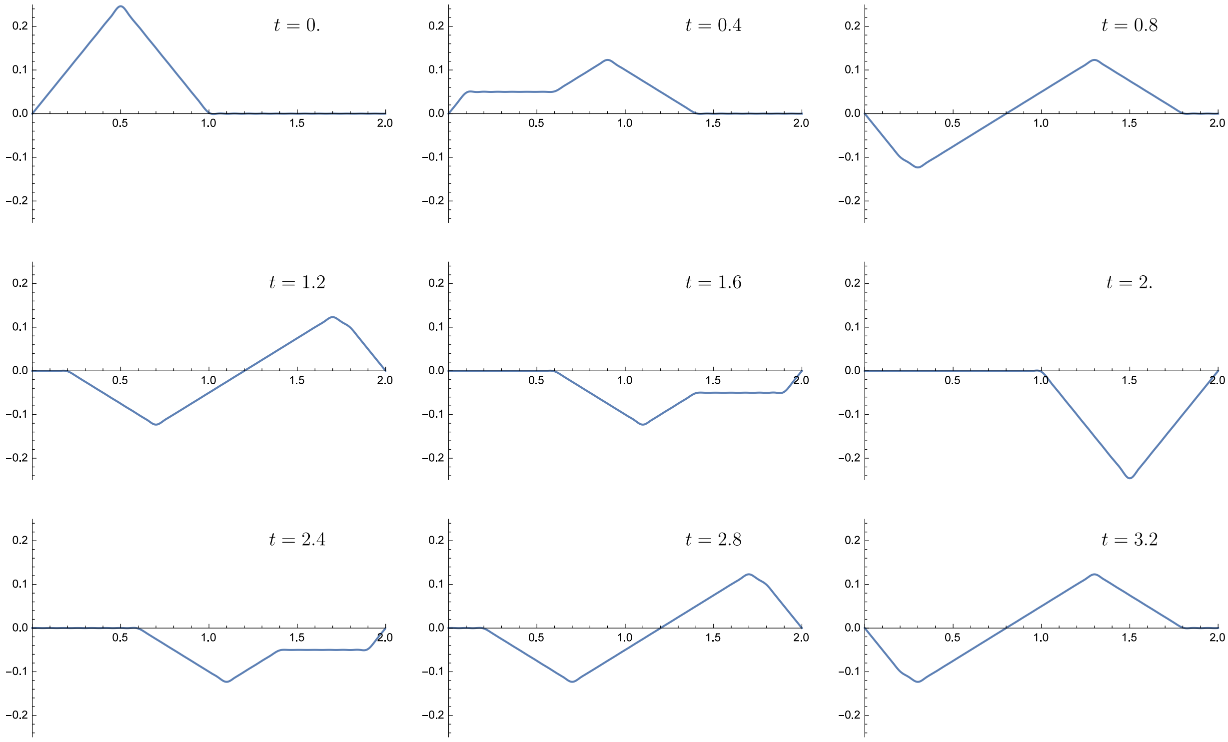


Figure 3: Solutions to the problem (14.1)–(14.3) with the initial displacement as in Fig. 2 and initial velocity $g(x) = 0$ at different time moments. First 50 terms of the Fourier series are used to represent the solution.

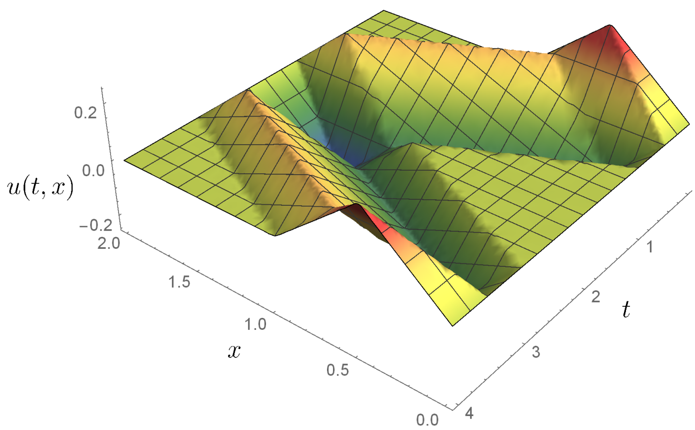
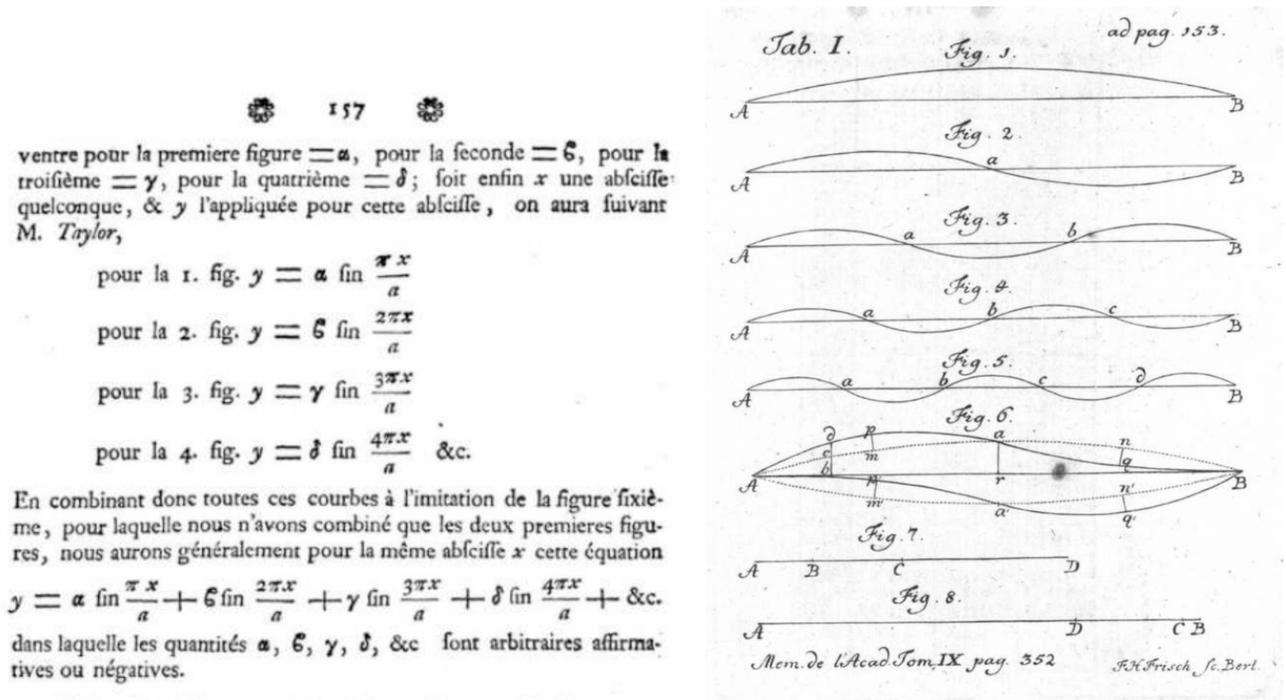


Figure 4: Solution to the problem (14.1)–(14.3) with the initial displacement as in Fig. 2 and initial velocity $g(x) = 0$ in $t, x, u(t, x)$ coordinates. First 50 terms of the Fourier series are used to represent the solution.



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ventre pour la premiere figure $= \alpha$, pour la ſeconde $= \beta$, pour la troiſième $= \gamma$, pour la quatrième $= \delta$; ſoit enfin x une abſciſſe quelconque, & y l'appliquée pour cette abſciſſe, on aura ſuivant M. *Taylor*,

pour la 1. fig. $y = \alpha \sin \frac{\pi x}{a}$

pour la 2. fig. $y = \beta \sin \frac{2\pi x}{a}$

pour la 3. fig. $y = \gamma \sin \frac{3\pi x}{a}$

pour la 4. fig. $y = \delta \sin \frac{4\pi x}{a}$ &c.

En combinant donc toutes ces courbes à l'imitation de la figure ſixième, pour laquelle nous n'avons combiné que les deux premieres figures, nous aurons généralement pour la même abſciſſe x cette équation

$$y = \alpha \sin \frac{\pi x}{a} + \beta \sin \frac{2\pi x}{a} + \gamma \sin \frac{3\pi x}{a} + \delta \sin \frac{4\pi x}{a} + \&c.$$

dans laquelle les quantités α, β, γ, δ, &c ſont arbitraires affirmatives ou négatives.

Figure 5: Two extracts from D. Bernoulli's 1753 paper.

The standing wave solutions allow me to make a guess how the scientists first decided to use separation of variables technique. The fact is that Joseph Fourier was not the first person to assume that. Before him exactly the same guess was used by Lagrange to obtain an analytical solution to the system of masses on the springs (the one I used to derive in the limit the wave equation). Lagrange did not have to deal with the questions of convergence because his "Fourier series" consisted of a finite number of terms. Even earlier, in 1753, Daniel Bernoulli, a famous mathematician and physicist, used "Fourier series" to represent solutions to the wave equation[2]. You can see his "Fourier series" in the left panel in Fig. 5. He actually did not calculate the coefficients of the series, leaving them in undetermined form. A possible motivation for these products comes from a very careful drawing of his father, Johann Bernoulli (one of the early developers of Calculus), which can be seen in Fig. 5, right panel. These are the drawings of the observed string oscillations, and since the graphs of standing waves look the same and can be analytically described as a product of two trigonometric functions, one depends only on time and the other one only on the spatial variable, hence (we can only guess at this point) it was the original motivation to use the form

$$u(t, x) = T(t)X(x)$$

to solve partial differential equations.

[2] Bernoulli, D., 1753. Réflexions et éclaircissemens sur les nouvelles vibrations des cordes. Les Mémoires de l'Académie Royale des Sciences et des Belles-Lettres de Berlin de 1747 et, 1748, pp.47–72.