15 Solving the Laplace equation by Fourier method

I already introduced two or three dimensional heat equation, when I derived it, recall that it takes the form

$$u_t = \alpha^2 \Delta u + F, \tag{15.1}$$

where $u\colon [0,\infty) \times D \longrightarrow \mathbf{R}$, $D \subseteq \mathbf{R}^k$ is the domain in which we consider the equation, α^2 is the diffusion coefficient, $F\colon [0,\infty) \times D \longrightarrow \mathbf{R}$ is the function that describes the sources ($F > 0$) or sinks ($F < 0$) of thermal energy, and Δ is the Laplace operator, which in Cartesian coordinates takes the form

$$\Delta u = u_{xx} + u_{yy}, \quad D \subseteq \mathbf{R}^2,$$

or

$$\Delta u = u_{xx} + u_{yy} + x_{zz}, \quad D \subseteq \mathbf{R}^3,$$

if the processes are studied in three dimensional space. Of course we need also the boundary conditions on ∂D and the initial conditions inside D.

In a similar vein it can be proved that the wave equation in two or three dimensions can be written as

$$u_{tt} = c^2 \Delta u + F, \tag{15.2}$$

where now c is the wave velocity, and F is an external force. We also will need boundary and initial conditions.

Very often the processes described by the heat or wave equation approach some equilibrium if $t \to \infty$. This means that the solution does not change with time and in particular u_t or u_{tt} tend to zero as $t \to \infty$. Therefore equations (15.1) and (15.2) turn into

$$\Delta u = -f, \tag{15.3}$$

where $f = F/\alpha^2$ for the heat equation and $f = F/c^2$ for the wave equation. Equation (15.3) is called *Poisson equation*, and, in case if $f = 0$,

$$\Delta u = 0, \tag{15.4}$$

Laplace equation, one of the most important equations in mathematics (and physics).

Since I am talking about the equilibrium (stationary) problems (15.3) and (15.4) only the boundary conditions are relevant, in the equilibrium state the system "forgets" about the initial conditions (it can be rigorously proved that initial value problem for either Poisson or Laplace equations is ill posed).

In the following I will use the separation of variables to solve the Laplace equation (15.4), and some properties of (15.3) will be discussed in the forthcoming lectures.

Only for some special plane geometries of the domain D it is possible to use the separation of variables. First of all, in Cartesian coordinates, these are various rectangles, I will leave this case for homework problems (see the textbook). It is also possible to use separation of variables in "circular"-based domains, such as interior of the disk, exterior of the disk, sector, annulus, and part of an annulus. To do this I first need to rewrite the Laplace operator in polar coordinates.

Recall that Cartesian coordinates (x, y) and polar coordinates (r, θ) are connected as

$$x = r\cos\theta, \quad y = r\sin\theta,$$

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or

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}\,.$$

I have

$$u(x,y) = u(r\cos\theta, r\sin\theta) = u(r,\theta) = v\left(\sqrt{x^2+y^2}, \arctan\frac{y}{x}\right).$$

For the following I will need

$$\begin{aligned}
r_x &= \frac{x}{\sqrt{x^2+y^2}} = \cos\theta,\\
r_y &= \frac{y}{\sqrt{x^2+y^2}} = \sin\theta,\\
\theta_x &= -\frac{1}{1+(y/x)^2}\frac{y}{x^2} = -\frac{y}{x^2+y^2} = -\frac{\sin\theta}{r},\\
\theta_y &= \frac{1}{1+(y/x)^2}\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{\cos\theta}{r}.
\end{aligned}$$

Now I start calculating the partial derivatives using the usual chain rule

$$\begin{aligned}
u_x &= v_r r_x + v_\theta \theta_x = v_r\cos\theta + v_\theta\left(-\frac{\sin\theta}{r}\right),\\
u_y &= v_r r_y + v_\theta\theta_y = v_r\sin\theta + v_\theta\frac{\cos\theta}{r},\\
u_{xx} &= (u_x)_r r_x + (u_x)_\theta\theta_x =\\
&= \left(v_{rr}\cos\theta - v_{\theta r}\frac{\sin\theta}{r} + \frac{\sin\theta}{r^2}v_\theta\right)\cos\theta + \left(v_{r\theta}\cos\theta - v_r\sin\theta - \frac{\sin\theta}{r}v_{\theta\theta} - \frac{\cos\theta}{r}v_\theta\right)\left(-\frac{\sin\theta}{r}\right),\\
u_{yy} &= (u_y)_r r_y + (u_y)_\theta\theta_y =\\
&= \left(v_{rr}\sin\theta + v_{\theta r}\frac{\cos\theta}{r} - \frac{\cos\theta}{r^2}v_\theta\right)\sin\theta + \left(v_{r\theta}\sin\theta + v_r\cos\theta + \frac{\cos\theta}{r}v_{\theta\theta} - \frac{\sin\theta}{r}v_\theta\right)\frac{\cos\theta}{r}\,.
\end{aligned}$$

Now I add two last lines to find the Laplace operator in polar coordinates (replacing v with u, as it is often and slightly confusing done in many textbooks)

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

I note that in cylindrical coordinates $x = r\cos\theta$, $y = r\sin\theta$, z the Laplace operator is

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} = 0,$$

whereas in the spherical coordinates

$$\begin{aligned}
x &= r\sin\varphi\cos\theta,\\
y &= r\sin\varphi\sin\theta,\\
z &= r\cos\varphi,
\end{aligned}$$

the Laplace operator is

$$\Delta u = u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} + \frac{\cot\theta}{r^2}u_\varphi + \frac{1}{r^2\sin^2\varphi}u_{\theta\theta}.$$

Example 15.1. To show how the separation of variables works for the Laplace equation in polar coordinates, consider the following boundary value problem

$$\Delta u = 0,$$
$$u(r_1, \theta) = g_1(\theta),$$
$$u(r_2, \theta) = g_2(\theta),$$

that is, consider the problem inside the annulus $r_1 < r < r_2$, and on both boundaries Type I non-homogeneous boundary conditions are given. I start with a usual assumption that

$$u(r, \theta) = R(r)\Theta(\theta).$$

Since my equation in polar coordinates takes the form

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0,$$

I get, denoting with the prime the corresponding derivatives,

$$r^2 R''\Theta + rR'\Theta + R\Theta'' = 0,$$

or

$$-\frac{r^2 R'' + rR'}{R} = \frac{\Theta''}{\Theta}.$$

Since the left hand side depends only on r and the right hand side depends only on θ hence both sides must be equal to a constant, which I will denote $-\lambda$. Using this constant I end up with two ODE

$$r^2 R'' + rR' - \lambda R = 0, \tag{15.5}$$

and

$$\Theta'' + \lambda\Theta = 0. \tag{15.6}$$

At this point I must add some boundary conditions to one of these problems so that in the end I get a Sturm–Liouville problem, whose eigenfunctions I can use as building blocks for my generalized Fourier series. The original boundary conditions for u are of no help here since they are non-homogeneous. There should be something else to the problem. And indeed, after some though, it is possible to guess that my solution must be periodic function of θ and, moreover, the solution must be continuously differentiable, which implies that

$$u(r, \theta - \pi) = u(r, \theta + \pi), \quad u_\theta(r, \theta - \pi) = u_\theta(r, \theta + \pi).$$

This implies that my second equation (15.6) must be supplemented with

$$\Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi).$$

Now we know that these periodic boundary conditions plus (15.6) is an eigenvalue Sturm–Liouville problem with the eigenvalues $\lambda_k = k^2$, $k = 0, 1, \ldots$ and eigenfunctions $\Theta_k(\theta) = A_k \cos k\theta + B_k \sin k\theta$. Now I can return to (15.5), which can be written as

$$r^2 R'' + rR' - k^2 R = 0.$$

I start with the case $k = 0$. Then the equation

$$r^2 R'' + rR' = 0$$

can be solved by substitution $R'(r) = S(r)$:

$$r^2 S' + rS = 0 \implies S(r) = \frac{B}{r},$$

which finally gives me

$$R_0(r) = a_0 + b_0 \log r.$$

(I do not use the absolute value since $r > 0$.)

If $k = 1, 2, \ldots$ I have the so-called Cauchy–Euler differential equation, which can be solved by first using the ansatz (educated guess) $R(r) = r^\mu$. I get

$$\mu(\mu - 1) + \mu - k^2 = 0 \implies \mu_{1,2} = \pm k,$$

and hence the general solution is given by

$$R_k(r) = c_k r^k + d_k r^{-k}, \quad k = 1, 2, \ldots$$

What I did in words is the following: I proved that any function of the form

$$u_k(r, \theta) = R_k(r)\Theta_k(\theta), \quad k = 0, 1, 2, \ldots$$

solves the Laplace equation $\Delta u = 0$ (such functions are called *harmonic*) and satisfies the periodic boundary conditions. Since the Laplace equation is *linear*, I will use the principle of superposition to argue that the function

$$u(r, \theta) = A + B \log r + \sum_{k=1}^{\infty} \Big[(C_k r^k + D_k r^{-k}) \cos k\theta + (E_k r^k + G_k r^{-k}) \sin k\theta\Big]$$

solves the Laplace equation and satisfies the periodic conditions. It seems that I have a lot of arbitrary constants to determine from the remaining two boundary conditions, but careful analysis shows that I have enough. To wit, let my boundary conditions have the following Fourier series (notice that I do not divide by 2 the first coefficient)

$$g_1(\theta) = a_0^{(1)} + \sum_{k=1}^{\infty} a_k^{(1)} \cos k\theta + b_k^{(1)} \sin k\theta,$$

$$g_2(\theta) = a_0^{(2)} + \sum_{k=1}^{\infty} a_k^{(2)} \cos k\theta + b_k^{(2)} \sin k\theta.$$

Now, comparing these series with the solution in the form of the series and invoking the boundary conditions, I get

$$\begin{cases} A + B \log r_1 = a_0^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_1(\theta)\, \mathrm{d}\theta, \\ A + B \log r_2 = a_0^{(2)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_2(\theta)\, \mathrm{d}\theta, \end{cases}$$

$$\begin{cases} C_k r_1^k + D_k r_1^{-k} = a_k^{(1)} = \frac{1}{\pi} \int_{-\pi}^{\pi} g_1(\theta) \cos k\theta\, \mathrm{d}\theta, \\ C_k r_2^k + D_k r_2^{-k} = a_k^{(2)} = \frac{1}{\pi} \int_{-\pi}^{\pi} g_2(\theta) \cos k\theta\, \mathrm{d}\theta, \end{cases} \qquad k = 1, 2, \ldots$$

$$\begin{cases} E_k r_1^k + G_k r_1^{-k} = b_k^{(1)} = \frac{1}{\pi} \int_{-\pi}^{\pi} g_1(\theta) \sin k\theta\, \mathrm{d}\theta, \\ E_k r_2^k + G_k r_2^{-k} = b_k^{(2)} = \frac{1}{\pi} \int_{-\pi}^{\pi} g_2(\theta) \sin k\theta\, \mathrm{d}\theta, \end{cases} \qquad k = 1, 2, \ldots$$

and each system for each k is a system of two equations with two unknowns, which can be always (except for some degenerate cases) solved.

For example, assuming that $g_1(\theta) = a$, $g_2(\theta) = b$ implies

$$B = \frac{b-a}{\log r_2/r_1}, \quad A = a - \frac{(b-a)\log r_1}{\log r_2/r_1},$$

and all other constants are zero. Hence the solution is

$$u(r,\theta) = A + B\log r.$$

If I assume that $g_1(\theta) = a\cos\theta$, $g_2(\theta) = b\cos\theta$ then I end up with the system

$$C_1 r_1 + D_1 r_1^{-1} = a,$$
$$C_1 r_2 + D_1 r_2^{-1} = b,$$

which is easy to solve. The solution to the boundary value problem for the Laplace equation is hence

$$u(r,\theta) = (C_1 r + D_1 r^{-1})\cos\theta.$$

Example 15.2 (Interior Dirichlet problem for the Laplace equation and Poisson's formula)**.** Consider now the problem

$$\Delta u = 0, \quad 0 \leq r < 1,$$
$$u(1,\theta) = g(\theta), \quad 0 \leq \theta < 2\pi.$$

To solve it I will do exactly the same steps as in the previous example (assume that the solution can be presented as a product, get two ODE, use the periodic boundary conditions on θ, end up with the same eigenvalues and eigenfunctions, solve the ODE for R) first. Then I note that a significant part of the solutions to the ODE for R has no meaning since I am dealing also with the point $r = 0$ and hence neither $\log r$ nor r^{-k} make sence at this point. Since these solutions have no physical meaning I drop them to end up with the function

$$u(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k r^k \cos k\theta + b_k r^k \sin k\theta.$$

By using the given Type I or Dirichlet boundary condition I immediately find that (note that I conveniently assumed that disk has radius 1, make sure that you can solve the case for an arbitrary radius)

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi)\cos k\phi \,\mathrm{d}\phi, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi)\sin k\phi \,\mathrm{d}\phi.$$

I solved my problem, but it turns out that I can rewrite this solution in a closed neat form. Interchanging the integrals and sums in my solution I get

$$\begin{aligned} u(r,\theta) &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi)\left(\frac{1}{2} + \sum_{k=1}^{\infty} r^k(\cos k\phi\cos k\theta + \sin k\phi \sin k\theta)\right)\mathrm{d}\phi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(\phi)\left(\frac{1}{2} + \sum_{k=1}^{\infty} r^k \cos k(\theta-\phi)\right)\mathrm{d}\phi. \end{aligned}$$

Now

$$
\begin{aligned}
\frac{1}{2}+\sum_{k=1}^{\infty} r^k \cos k\theta &= \operatorname{Re}\left(\frac{1}{2}+\sum_{k=1}^{\infty} z^k\right) \\
&= \operatorname{Re}\left(\frac{1}{2}+\frac{z}{1-z}\right) = \operatorname{Re}\left(\frac{1+z}{2(1-z)}\right) \\
&= \operatorname{Re}\left(\frac{(1+z)(1-\overline{z})}{2|1-z|^2}\right) = \operatorname{Re}\left(\frac{1-|z|^2+z-\overline{z}}{2|1-z|^2}\right) \\
&= \frac{1-r^2}{2(1+r^2-2r\cos\theta)}\,.
\end{aligned}
$$

Therefore, finally, I can conclude that my solution is given by

$$
u(r,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)\frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)}\,\mathrm{d}\phi, \tag{15.7}
$$

which is called *the Poisson integral formula*, and the expression

$$
\frac{1-r^2}{2(1+r^2-2r\cos(\theta-\phi))}
$$

is called the *Poisson kernel.* To emphasize, the Poisson integral formula gives a closed form solution for the Dirichlet boundary problem for the Laplace equation in a disk.

There are several immediate consequences of the Poisson formula.

- The value of the solution at the center of the disk is given by the average of its boundary value.

  $$
  u(0,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi} g(\phi)\,\mathrm{d}\phi.
  $$

  This is a particular case of the following general fact: Let u be harmonic (i.e., solves the Laplace equation) inside the disk of radius a with the center (x_0,y_0), then

  $$
  u(x_0,y_0)=\frac{1}{2\pi a}\oint_{\gamma} u\,\mathrm{d}s,
  $$

  where γ is the boundary of the disk. This immediately follows from the Poisson formula after shift and rescaling.

- If u is a nonconstant harmonic function defined in D then it cannot achieve its local maximum or local minimum at any interior point in D. This is true since the average of a continuous real function lies strictly between its minimal and maximal values, and hence, due to the previous point, I cannot have a local minimum or local maximum at an interior point.

- Immediately from the previous I have that if u is harmonic in D and if m and M are minimal and maximal values of u on the boundary of D, then

  $$
  m\leq u(x,y)\leq M
  $$

  anywhere in D. This statement is called the *maximum principle* for the Laplace equation.

- If u_1 and u_2 solve the same Poisson equation $-\Delta u = f$ on D with the same boundary conditions then $u_1 = u_2$ within D, that is, the solution to the Dirichlet boundary value problem for the Poisson equation is unique. This follows from the linearity of the equation and the maximum principle. Indeed, by linearity function $v = u_1 - u_2$ solves the Laplace equation $\Delta u = 0$ with homogeneous boundary conditions $v = 0$ on ∂D. By the maximum principle this implies that $v(x, y) = 0$ for all $(x, y) \in D$ and hence $u_1 = u_2$.

15.1 Other boundary conditions

15.2 Some historical remarks

15.3 Probabilistic interpretation of the Laplace equation