18 Green’s function for the Poisson equation

Now we have some experience working with Green’s functions in dimension 1, therefore, we are ready
to see how Green’s functions can be obtained in dimensions 2 and 3. That is, I am looking to solve
\[ -\Delta u = f, \quad x \in D \subseteq \mathbb{R}^m, \quad m = 2, 3, \quad (18.1) \]
with the boundary conditions
\[ u|_{x \in \partial D} = 0. \quad (18.2) \]
To solve problem (18.1), (18.2) I need to find the Green function \( G(x; \xi) \), i.e., the solution to
\[ -\Delta G(x; \xi) = \delta(x - \xi), \quad x, \xi \in D \subseteq \mathbb{R}^m, \quad m = 2, 3 \]
\[ G(x; \xi) = 0, \quad x \in \partial D. \quad (18.3) \]
But before attacking problem (18.3), I will into the problem without the boundary conditions.

18.1 Fundamental solution to the Laplace equation

Definition 18.1. The solution \( G_0 \) to the problem
\[ -\Delta G_0(x; \xi) = \delta(x - \xi), \quad x, \xi \in \mathbb{R}^m \quad (18.4) \]
is called the fundamental solution to the Laplace equation (or free space Green’s function).

Planar case \( m = 2 \)

To find \( G_0 \) I will appeal to the physical interpretation of my equation. Physically to solve (18.4) means
to find a potential of the gravitational (or electrostatic) field, caused by the unit mass (unit charge)
positioned at \( \xi \). The field itself is found as the gradient of \( G_0 \). Since I do not expect to have for
my gravitation field any preferred directions, I conclude that my potential should only depend on the
distance \( r = |x - \xi| \) between the points \( x \) and \( \xi \) and not on any angle. Next, I will use the fact that
\( G_0 \) satisfies the Laplace equation \( \Delta G = 0 \) at any point except \( \xi \). Using the polar form of the Laplace
operator and the fact that my potential depends only on \( r \), I get
\[ rG_0'' + G_0' = 0 \]
I solve this equation when I used the separation of variables for the Laplace equation in polar coordinates. The general solution is given by
\[ G_0(r) = A \log r + B. \]
Now I note that constant \( B \) will not contribute to the delta function, since it is infinitely differentiable,

Therefore, my fundamental solution has the form \( A \log r \), and I need only to determine constant \( A \). For
this I will use the characteristic property of the delta function that
\[ \int_{\mathbb{R}^2} \delta(x - \xi) \, d\xi = 1 \]
and the divergence theorem that says that for a nice domain $D$ and smooth vector field $F$

$$\int_D \nabla \cdot F \, dx = \int_{\partial D} F \cdot \mathbf{n} \, dS,$$

where $\mathbf{n}$ is the outward normal to $D$.

Consider a disk $D_\epsilon$ or radius $\epsilon$ around $\xi$. Then I have

$$1 = \int_{\mathbb{R}^2} \delta(x - \xi) \, dx = \int_{D_\epsilon} \delta(x - \xi) \, dx =$$

[due to (18.4)] $\quad -A \int_{D_\epsilon} \Delta \log r \, dx = -A \int_{D_\epsilon} \nabla \cdot \nabla \log r \, dx =$

[due to the divergence theorem] $\quad -A \int_{\partial D_\epsilon} \nabla \log r \cdot \mathbf{n} \, dS =$

[why?] $\quad -A \int_{\partial D_\epsilon} \frac{d \log r}{dr} \, dS = -A \int_0^{2\pi} \frac{1}{r} r \, d\varphi = -A 2\pi,$

hence

$$A = -\frac{1}{2\pi},$$

and therefore

$$G_0(x; \xi) = -\frac{1}{2\pi} \log |x - \xi| = -\frac{1}{4\pi} \log \left( (x - \xi)^2 + (y - \eta)^2 \right)$$

is the fundamental solution to the planar Laplace equation or, physically, is the potential of the gravitation (or electrostatic) field induced by the unit mass (charge). Note that for the field itself

$$\frac{dG_0}{dr} = \frac{1}{r},$$

that is the force is inversely proportional to the distance between the points.

Case $m = 3$

Very briefly, and invoking exactly the same reasonings, I find that my fundamental solution must depend only on $r = |x - \xi|$ and solve everywhere except point $\xi$ the equation

$$r G_0'' + 2 G_0' = 0$$

(see the expression of the Laplace operator in spherical coordinates). The general solution to this equation is

$$\frac{A}{r} + B,$$

and therefore it is reasonable to assume that

$$G_0(r) = \frac{A}{r}.$$
Again, using the properties of delta function and the divergence theorem I get for a sphere $D_\epsilon$ with the center at $\xi$

\[ 1 = \int_{\mathbb{R}^3} \delta(x - \xi) \, dx = \int_{D_\epsilon} \delta(x - \xi) \, dx = -A \int_{D_\epsilon} \nabla \cdot \nabla \frac{1}{r} \, dx = -A \int_{\partial D_\epsilon} \nabla \frac{1}{r} \cdot \hat{n} \, dS = A \int_{\partial D_\epsilon} \frac{1}{r^2} \, dS = \frac{A}{\epsilon^2} \int_{\partial D_\epsilon} \, dS = \frac{A}{\epsilon^2} 4\pi \epsilon^2 = 4\pi A, \]

since the area of the sphere of radius $\epsilon$ is $4\pi \epsilon^2$. Therefore, my fundamental solution is

\[ G_0(r) = \frac{1}{4\pi r}, \]

and the gravitational (or electrostatic) field exerts the force that is inversely proportional to the square of the distance, as we all remember from our physics classes.

**Exercise 1.** Find the fundamental solution to the Laplace equation for any dimension $m$.

### 18.2 Green’s function for a disk by the method of images

Now, having at my disposal the fundamental solution to the Laplace equation, namely,

\[ G_0(x; \xi) = -\frac{1}{2\pi} \log |x - \xi|, \]

I am in the position to solve the Poisson equation in a disk of radius $a$. That is, I consider the problem

\[ -\Delta u = f, \quad x \in D \subseteq \mathbb{R}^2, \quad D = \{(x, y): x^2 + y^2 < a^2\} \tag{18.5} \]

with the homogeneous Dirichlet or Type I boundary conditions

\[ u|_{x \in \partial D} = 0. \tag{18.6} \]

I know that to be able to write the solution to my problem, I need the Green function that solves

\[ -\Delta G(x; \xi) = \delta(x - \xi), \quad x, \xi \in D \subseteq \mathbb{R}^2, \quad G(x; \xi) = 0, \quad x \in \partial D. \tag{18.7} \]

If I am able to figure out the solution to (18.7), then (18.5), (18.6), by the principle of superposition, has the solution

\[ u(x) = \int_D f(\xi) G(x; \xi) \, d\xi. \]

The key idea is to replace the problem (18.7) with another problem on the whole plane $\mathbb{R}^2$, with an additional source (or sources) outside of $D$, such that the boundary condition (18.6) would be satisfied automatically.
I replace my problem (18.7) with the following
\[-\Delta G(x; \xi) = \delta(x - \xi) - \delta(x - \xi^*), \quad x \in \mathbb{R}^2, \ |\xi| < a, \ |\xi^*| > a. \tag{18.8}\]

Since I require the coordinates of my second source be outside of the disk, hence within the disk, due to the properties of the delta function, (18.8) coincides with the equation (18.7). If I am capable to determine the coordinates of my second source as a function of the coordinates of the source inside the disk, such that for \(|x| = a\) my solution vanishes, then it means that I solved my problem. In other words, I am looking for the coordinates \(\xi^*\) of the image of the point \(\xi\), and this explains the name of the method.

So let me try to achieve my goal. I know that solution, again by the superposition principle, to (18.8) is given by
\[G(x; \xi) = -\frac{1}{2\pi} \log |x - \xi| + \frac{1}{2\pi} \log |x - \xi^*| + c = \frac{1}{4\pi} \log \frac{|x - \xi^*|^2}{|x - \xi|^2} + c.\]

Hence, for \(|x| = a\), I must have, due to (18.7),
\[|x - \xi|^2 = k|x - \xi^*|^2, \quad k = e^{4\pi c}.\]

To see whether the last equality must be true, I consider
\[
\begin{align*}
|x - \xi|^2 &= (x - \xi) \cdot (x - \xi) = |x|^2 + |\xi|^2 - 2x \cdot \xi = a^2 + r_0^2 - 2ar_0 \cos \theta, \\
|x - \xi^*|^2 &= (x - \xi^*) \cdot (x - \xi^*) = |x|^2 + |\xi^*|^2 - 2x \cdot \xi^* = a^2 + \gamma r_0^2 - 2\gamma ar_0 \cos \theta,
\end{align*}
\]

where I assumed, to reduce the number of free parameters, that the angle \(\theta\) between \(x\) and \(\xi\) and \(x\) and \(\xi^*\) is the same, that is \(\xi^* = \gamma \xi\).

To get the required equality I must have
\[
a^2 + r_0^2 = ka^2 + k\gamma^2 r_0^2, \quad ar_0 = k\gamma ar_0,
\]
from the second of which \(k\gamma = 1\) and hence from the first
\[\gamma = \frac{a^2}{r_0^2}.\]

Problem solved! You can see geometrically that my point \(\xi^*\) is one of the vertices of the triangle \(0x\xi^*\), which is similar by construction to the triangle \(0x\xi\), see the figure. Now I can write, using the polar coordinates of the point \(x\) as \((r, \phi)\) and of \(\xi\) as \((r_0, \phi_0)\), then my solution to (18.6), (18.7) has the form
\[G(r, \phi; r_0, \phi_0) = \frac{1}{4\pi} \log \frac{a^2(r^2 + r_0^2 - 2rr_0 \cos(\phi - \phi_0))}{r_0^2r^2 + a^4 - 2a^2rr_0 \cos(\phi - \phi_0)}.\]

**Exercise 2.** Find the Green function for the unit sphere.

Similar approach works for some other domains (see the homework problems), but the list of such domains is quite limited. There are other methods to infer the Green function, but they are outside of the scope of this introductory course. Probably still the best reference for a prepared reader to read about various methods to find Green’s functions is the first volume of Courant and Hilbert *Methods of mathematical physics.*
Figure 1: The construction of the image of the source with coordinates $\xi$ for the disk