19 Fourier transform

19.1 A first look at the Fourier transform

In an introductory ODE class you all should have studied the Laplace transform, which was used to turn an ordinary differential equation into an algebraic one. There are a number of other integral transforms, which are useful in one or another situation. Arguably, the most ubiquitous and general is the so-called *Fourier transform*, which generalizes the technique of Fourier series on non-periodic functions. For the motivation of the Fourier transform I recommend reading the textbook or other possible references (see below), in these notes I start with a bare and dry definition.

Definition 19.1. *The Fourier transform of the real valued function* $f\colon \mathbf{R}\longrightarrow\mathbf{R}$ *of the real argument* x *is the complex valued function* $\hat f\colon \mathbf{R}\longrightarrow\mathbf{C}$ *of the real argument* k *defined as*

$$\hat f(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{19.1}$$

The inverse Fourier transform, which allows to recover f *if* $\hat f$ *is known, is given by*

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat f(k)e^{\mathrm{i}kx}\,\mathrm{d}k. \tag{19.2}$$

Remark 19.2.

1. Since the Fourier transform plays a somewhat auxiliary role in my course, I will not dwell on it for very long. There are a lot of available textbooks on Fourier Analysis; I would like to mention two additional sources: the lecture notes by Brad Osgood *The Fourier transform and its applications* (they are freely available on the web) for a thorough introduction to the subject, including careful coverage of the relations of the delta function and Fourier transform, and Körner, T. W. *Fourier analysis.* Cambridge University Press, 1989 for the already initiated.

2. There are different definitions of the Fourier transform. You can also find in the literature

$$\hat f(k)=\frac{1}{A}\int_{-\infty}^{\infty} e^{\mathrm{i}Bkx}\,\mathrm{d}x,$$

   where the following choices are possible:

$$\begin{aligned}&A=\sqrt{2\pi},\quad B=\pm 1,\\ &A=1,\quad B=\pm 2\pi,\\ &A=1,\quad B=\pm 1.\end{aligned}$$

   The only difference in the computations is the factor which appears (or does not appear) in front of the formulas. Be careful if you use some other results from different sources.

3. Very often, together with the hat notation, the operator notation is used

$$\hat f(k)=\mathcal{F}f(x),\quad f(x)=\mathcal{F}^{-1}\hat f(k).$$

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(Strictly speaking something like the following should be used: $\hat{f}(k) = \mathcal{F}\{f(x)\}(k)$, but such notations very quickly becomes unreadable.)

Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are linear operators, since for them, from the properties of the integral, it holds that

$$\mathcal{F}(\alpha f(x) + g(x)) = \alpha \mathcal{F} f(x) + \mathcal{F} g(x),$$

and a similar expression is true for $\mathcal{F}^{-1}$.

To emphasize that the pair f and $\hat{f}$ are related sometimes something like

$$f(x) \rightleftharpoons \hat{f}(k)$$

is used.

4. In the definition of the Fourier transform I have an improper integral, which means that I have to bother about the convergence. Moreover, note that the complex exponent, by Euler's formula, is a linear combination of sine and cosine, and hence my transform should be defined only for those f that tend sufficiently fast to zero. This is actually a very nontrivial question, what the space of functions is on which the Fourier transform is naturally defined, but this will not bother me in this course. Moreover, there will be some examples, which would definitely contradict the classical understanding of the Fourier transform. I will treat them in a heuristic way remembering that the rigorous justification can be made within the theory of generalized functions.

5. If you still insist on some intuitive understanding of the Fourier transform, recall the formula for the Fourier series coefficients in the exponential form (one of the remarks in the Fourier series section). Specifically, if one works on the interval $(-\pi, \pi)$, then

$$c_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) e^{-\mathrm{i}kx} \, \mathrm{d}x, \quad k \in \mathbf{Z}.$$

   The similarity with the definition of Fourier transform should be obvious. That is, intuitively, Fourier transform is nothing else other than the generalization of the formula for the Fourier coefficients on the whole real line $\mathbf{R}$. In this respect the "miraculous" fact that the inverse Fourier transform produces the original function, stops being miraculous, since this is exactly the reason why we introduced the Fourier coefficients — to represent the original function f.

Consider several examples to get a feeling about the Fourier transform.

Example 19.3 (Fourier transform of the rect (for rectangle) function)**.** Let

$$\Pi_a(x) = \begin{cases} 1, & |x| < a, \\ 0, & \text{otherwise.} \end{cases}$$

By the definition (19.1) I have

$$\hat{\Pi}_a(k) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} e^{-\mathrm{i}kx} \, \mathrm{d}x = \frac{e^{\mathrm{i}ka} - e^{-\mathrm{i}ka}}{\sqrt{2\pi}\, \mathrm{i}k} = \sqrt{\frac{2}{\pi}} \frac{\sin ak}{k},$$

using the fact that

$$\sin at = \frac{e^{\mathrm{i}at} - e^{-\mathrm{i}at}}{2\mathrm{i}}.$$

It follows from the definition of the inverse Fourier transform that

$$\Pi_a(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin ak}{k} e^{\mathrm{i}kx} \, \mathrm{d}k.$$

The last integral is quite difficult to evaluate directly, and the given definitions of the Fourier transform are often a direct way for some nontrivial integrals. Note that the Fourier transform of an even function is even (and real valued), which is not a coincidence.

Example 19.4 (The exponential decay)**.** Let

$$f_r(x) = \begin{cases} 0, & x \leq 0, \\ e^{-ax}, & x > 0, \end{cases}$$

where a is a positive constant. Then

$$\hat{f}_r(k) = \frac{1}{\sqrt{2\pi}(a + \mathrm{i}k)},$$

which is complex valued even if k is real. Similarly, for

$$f_l(x) = \begin{cases} e^{ax}, & x \leq 0, \\ 0, & x > 0, \end{cases}$$

I find

$$\hat{f}_l(k) = \frac{1}{\sqrt{2\pi}(a - \mathrm{i}k)}.$$

Now I can easily calculate the Fourier transform for

$$f(x) = e^{-a|x|} = f_l(x) + f_r(x),$$

using the linearity of $\mathcal{F}$:

$$\hat{f}(k) = \hat{f}_r(k) + \hat{f}_l(k) = \sqrt{\frac{2}{\pi}} \frac{a}{k^2 + a^2},$$

which is an even function if k is real.

Example 19.5 (Duality principle)**.** Let

$$f(x) = \frac{1}{x^2 + a^2}, \quad a > 0.$$

I have

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\mathrm{i}kx}}{x^2 + a^2} \, \mathrm{d}x.$$

This integral is difficult to evaluate using only the basic knowledge from Calculus (actually, impossible to the best of my knowledge, let me know if I am wrong). However, from the previous, I know that

$$e^{-a|x|} = \mathcal{F}^{-1}\left(\sqrt{\frac{2}{\pi}}\,\frac{a}{k^2+a^2}\right) = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{\mathrm{i}kx}}{k^2+a^2}\,\mathrm{d}k.$$

If I replace k with x and x with $-k$ I will get the integral (up to a multiplicative factor), that I know how to find. Therefore I conclude that

$$\hat{f}(k) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-a|k|}}{a}\,.$$

This is actually a consequence of the striking resemblance of the Fourier and inverse Fourier transforms, the difference being just an extra minus sign. I would like to formulate this important fact as a theorem.

Theorem 19.6. *If the Fourier transform of $f(x)$ is $\hat{f}(k)$, then the Fourier transform of $\hat{f}(x)$ is $f(-k)$.*

This theorem helps reducing the table of Fourier transforms in half, since if I know the Fourier transform $\hat{f}$ of f this immediately means that I know the Fourier transform $\hat{g}$ of the function $g = \hat{f}$. To practice this theorem convince yourself that

$$\mathcal{F}\left(\frac{\sin ax}{x}\right) = \sqrt{\frac{\pi}{2}}\,\Pi_a(k).$$

Example 19.7 (Fourier transform of the delta function)**.** Let $f(x) = \delta(x)$. Then

$$\hat{f}(k) = \hat{\delta}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(x)e^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\,.$$

Hence the Fourier transform of the delta function is a constant function. From here I can immediately obtain, invoking the duality principle, that the Fourier transform of the constant, say 1, is

$$\mathcal{F}(1) = \sqrt{2\pi}\,\delta(k),$$

is the delta function! But stop, if I'd like to use my definition (19.1) then the integral

$$\int_{-\infty}^{\infty} 1\cdot e^{-\mathrm{i}kx}\,\mathrm{d}k,$$

strictly speaking, does not exist, at least in the sense we discuss it in Calc II! Well, the exact meaning to this integral can be given within the framework of the generalized functions, but this will not bother us here.

19.2 Properties of the Fourier transform. Convolution

Direct evaluation of the Fourier transform becomes very often quite tedious. A list of the properties of the Fourier transform helps evaluating it in many special cases.

1. *Shift theorem.* If $f(x)$ has Fourier transform $\hat{f}(k)$ then the Fourier transform of $f(x-\xi)$ is $e^{-\mathrm{i}k\xi}\hat{f}(k)$. A very particular example of this property is

$$\mathcal{F}\delta(x-\xi)=\frac{1}{\sqrt{2\pi}}e^{-\mathrm{i}k\xi}\,.$$

Similarly (remember the duality), the Fourier transform of $e^{\mathrm{i}\eta x}f(x)$ is $\hat{f}(k-\eta)$.

2. *Dilation theorem.* If $f(x)$ has Fourier transform $\hat{f}(k)$ then the Fourier transform of $f(cx)$, $c\neq 0$ is

$$\frac{1}{|c|}\hat{f}\left(\frac{k}{c}\right).$$

Example 19.8. To practice this theorem let me find the Fourier transform of very important e^{-ax^2}. For the following calculations I recall one of the most remarkable integrals

$$\int_{\mathbf{R}}e^{-x^2}\,\mathrm{d}x=\sqrt{\pi},\quad\text{or}\quad\int_{\mathbf{R}}e^{-x^2/2}\,\mathrm{d}x=\sqrt{2\pi}.$$

To find the required transform I start with specific example:

$$I(k)=\mathcal{F}\left(e^{-\frac{x^2}{2}}\right)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{-\frac{x^2}{2}}e^{-\mathrm{i}kx}\,\mathrm{d}x.$$

I recall that $e^{-\mathrm{i}kx}=\cos kx+\mathrm{i}\sin kx$ by Euler's formula, sin is an odd function and hence contributes nothing to the integral, and therefore

$$I(k)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{-\frac{x^2}{2}}\cos kx\,\mathrm{d}x.$$

Now, direct calculations imply that

$$I'(k)=\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}\left(-xe^{-\frac{x^2}{2}}\right)\sin kx\,\mathrm{d}x=-k\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}e^{-\frac{x^2}{2}}\cos kx\,\mathrm{d}x=-I(k),$$

where I used the integration by parts and the fact that $(e^{-x^2/2})'=-xe^{-x^2/2}$. That is, I now reduced my problem of calculating the integral to the following ODE problem

$$I'(k)=-kI(k),\quad I(0)=1,$$

which implies that

$$I(k)=e^{-\frac{k^2}{2}}\,,$$

that is my exponent coincides with my Fourier transform! In other words, the exponent is an eigenfunction of operator $\mathcal{F}$ with eigenvalue 1 (this may not sound like a big deal, but the fact is that Fourier transform has only four eigenvalues, and I just found one of them makes it sound a little more important).

Finally, using the dilation theorem, taking $c=\sqrt{2a}$, I find

$$\mathcal{F}\left(e^{-ax^2}\right)=\frac{1}{\sqrt{2a}}e^{-\frac{k^2}{4a}}\,.$$

3. *Derivatives and Fourier transform.* If $f(x)$ has Fourier transform $\hat{f}(k)$ then the Fourier transform of $f'(x)$ is $\mathrm{i}k\hat{f}(k)$. Hence the Fourier transform turns the differentiation into an algebraic operation of multiplication by $\mathrm{i}k$. Immediate corollary is that the Fourier transform of $f^{(n)}(x)$ is $(\mathrm{i}k)^n\hat{f}(k)$. By the duality principle or by a direct proof, the Fourier transform of $xf(x)$ is $\mathrm{i}\frac{\mathrm{d}\hat{f}}{\mathrm{d}k}$.

4. *Integration and Fourier transform.* If $f(x)$ has Fourier transform $\hat{f}(k)$ then the Fourier transform of its integral $g(x) = \int_{-\infty}^{x} f(s)\,\mathrm{d}s$ is

$$\hat{g}(k) = -\frac{\mathrm{i}}{k}\hat{f}(k) + \pi\hat{f}(0)\delta(k).$$

Using this property I can immediately find Fourier transform of the Heaviside function $\chi(x)$:

$$\mathcal{F}\chi(x) = -\frac{\mathrm{i}}{k}\frac{1}{\sqrt{2\pi}} + \sqrt{\frac{\pi}{2}}\delta(k).$$

Since

$$\operatorname{sgn} x = \chi(x) - \chi(-x),$$

then

$$\mathcal{F}\operatorname{sgn} x = -\mathrm{i}\sqrt{\frac{2}{\pi}}\frac{1}{k}.$$

Exercise 1. Prove all four properties of the Fourier transform.

5. *Convolution.* Now let me ask the following question: if I know that $f(x) \rightleftharpoons \hat{f}(k)$ and $g(x) \rightleftharpoons \hat{g}(k)$ then which function has the Fourier transform $\hat{f}(k)\hat{g}(k)$?

I have

$$\begin{aligned}
\hat{f}(k)\hat{g}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-\mathrm{i}kx}\,\mathrm{d}x \int_{-\infty}^{\infty} g(y)e^{-\mathrm{i}ky}\,\mathrm{d}y = \\
&= \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(x)g(y)e^{-\mathrm{i}k(x+y)}\,\mathrm{d}x\,\mathrm{d}y = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(y)e^{-\mathrm{i}k(x+y)}\,\mathrm{d}y\right) f(x)\,\mathrm{d}x = [x+y=s] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} g(s-x)e^{-\mathrm{i}ks}\,\mathrm{d}s\right) f(x)\,\mathrm{d}x = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\mathrm{i}ks}\left(\int_{-\infty}^{\infty} g(s-x)f(x)\,\mathrm{d}x\right)\mathrm{d}s.
\end{aligned}$$

So, if I introduce a new function

$$h(s) = \int_{-\infty}^{\infty} g(s-x)f(x)\,\mathrm{d}x,$$

then I showed that

$$\sqrt{2\pi}\hat{f}(k)\hat{g}(k) = \hat{h}(k).$$

Now I can formally state my result.

Definition 19.9. *The convolution of two functions f and g is a function $h = f * g$ defined as*

$$h(s) = f * g = \int_{-\infty}^{\infty} f(s-x)g(x)\,\mathrm{d}x.$$

Basically, by the above reasoning I proved

Theorem 19.10. *Let $h = f * g$. Then*

$$\hat{h}(k) = \sqrt{2\pi}\hat{f}(k)\hat{g}(k).$$

In the opposite direction, the Fourier transform of the product of two functions $u(x) = f(x)g(x)$ is

$$\hat{u} = \frac{1}{\sqrt{2\pi}}\hat{f} * \hat{g}.$$

The second part of the theorem can be proved in a similar way or using the duality principle.

It is instructive to prove that the convolution is commutative ($f * g = g * f$), bilinear $f * (ag + bh) = af * g + bf * h$, and associative $f * (g * h) = (f * g) * h$.

19.3 Prove of the inversion formula

19.4 Test yourself

19.5 Solutions to the exercises