# 20 Applications of Fourier transform to differential equations

Now I did all the preparatory work to be able to apply the Fourier transform to differential equations. The key property that is at use here is the fact that the Fourier transform turns the differentiation into multiplication by $\mathrm{i}k$.

## 20.1 Space-free Green's function for ODE

I start with an ordinary differential equation and consider the problem

$$-u'' + \omega^2 u = h(x)$$

in an infinite interval $x \in \mathbf{R}$. $\omega$ here is a real parameter, and $h$ is a given function. To solve this problem I start with a space-free Green's function, that, as you recall from one of the previous lectures, must satisfy

$$-G_0'' + \omega^2 G_0 = \delta(x - \xi), \quad -\infty < x, \xi < \infty.$$

Let $\hat{G}_0$ be the Fourier transform of $G_0$. Then, using the properties of the Fourier transform, I have that $\hat{G}_0$ must satisfy

$$k^2 \hat{G}_0 + \omega^2 \hat{G}_0 = \frac{e^{-\mathrm{i}k\xi}}{\sqrt{2\pi}}\,,$$

or

$$\hat{G}_0(k) = \frac{e^{-\mathrm{i}k\xi}}{\sqrt{2\pi}(k^2 + \omega^2)}\,.$$

Using the table of the inverse Fourier transform, I find that

$$G_0(x;\xi) = \frac{1}{2\omega} e^{-\omega|x-\xi|}$$

is my free space Green's function. Now, invoking again the superposition principle, the solution to the original problem can be written as

$$u(x) = \int_{-\infty}^{\infty} h(\xi) G_0(x;\xi)\,\mathrm{d}\xi = \frac{1}{2\omega}\int_{-\infty}^{\infty} h(\xi) e^{-\omega|x-\xi|}\,\mathrm{d}\xi.$$

I actually never gave a proof of this formula and appealed only to the linearity and our intuition about linearity. Let me get this answer from scratch, without using any delta-function.

Applying the Fourier transform to the original problem I get

$$\hat{u}(k)(k^2 + \omega^2) = \hat{h}(k) \implies \hat{u}(k) = \hat{h}(k)\frac{1}{k^2 + \omega^2}\,.$$

On the right hand side I have a product of two Fourier transforms. To use my convolution formula I need to account for the factor $\sqrt{2\pi}$ in front. I have that the inverse Fourier transform of $\hat{h}(k)$ is $h(x)$ and for $1/(k^2 + \omega^2)$ is $\sqrt{\pi/2}e^{-\omega|x|}/\omega$, which all put together gives again the same

$$u(x) = \frac{1}{2\omega}\int_{-\infty}^{\infty} h(\xi) e^{-\omega|x-\xi|}\,\mathrm{d}\xi.$$

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## 20.2 General solution to the wave equation

Next, I will show what has to be modified in my method if I'd like to apply Fourier transform to a PDE. For this I consider the initial value problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0, \\ u(0,x) &= f(x), \quad -\infty < x < \infty, \\ u_t(0,x) &= 0. \end{aligned}$$

Since my variable $x$ in this problem takes the values from the whole real line, I will apply my Fourier transform to the function $u$ of two real variables $t$ and $x$ with respect to the variable $x$:

$$\mathcal{F}\left[u(t,x)\right] = \hat{u}(t,k).$$

Applying my Fourier transform to the equation I get

$$\frac{\mathrm{d}^2\hat{u}}{\mathrm{d}t^2} + c^2k^2\hat{u} = 0, \quad \hat{u}(0) = \hat{f}(k), \quad \hat{u}'(0) = 0,$$

note that I use the ordinary derivatives since only the derivatives with respect to $t$ are involved, and the variable $k$ can be simply considered as a parameter. That is, instead of PDE I end up with an ODE, which is easy to solve. In particular, I have that my general solution is

$$\hat{u}(t,k) = C_1(k)\cos ckt + C_2(k)\sin ckt,$$

where $C_1, C_2$ are two arbitrary functions of $k$. Using the initial conditions, I find

$$\hat{u}(t,k) = \hat{f}(k)\cos ckt.$$

To get an inverse Fourier transform, I note that

$$\hat{u}(t,k) = \frac{1}{2}\hat{f}(k)\left(e^{\mathrm{i}ckt} + e^{-\mathrm{i}ckt}\right),$$

and hence the inverse Fourier transform yields the familiar formula

$$u(t,x) = \mathcal{F}^{-1}\left[\hat{u}(t,k)\right] = \frac{f(x-ct) + f(f+ct)}{2},$$

which represents two traveling waves, one is going to the left and another one going to the right.

## 20.3 Laplace's equation in a half-plane

Consider the problem for the Green's function

$$G_{xx} + G_{yy} = 0, \quad y > 0, \, -\infty < x < \infty$$

with the boundary condition

$$G(x,0) = \delta(x).$$

Since the $x$ variable runs from $-\infty$ to $\infty$ I will use the Fourier transform with respect to this variable:

$$-k^2\hat{G} + G''_{yy} = 0, \quad y > 0, \quad G(k,0) = 1/\sqrt{2\pi}.$$

The solution to my problem is

$$\hat{G}(k,y) = C_1(y)e^{ky} + C_2(y)e^{-ky},$$

and I must also have that my Fourier transform would be bounded for $k \to \pm\infty$, therefore I choose my solution as

$$\hat{G}(k,y) = \frac{1}{\sqrt{2\pi}}e^{-|k|y},$$

which both satisfies the equation and the boundary condition. Taking the inverse Fourier transform, I find

$$G(x,y) = \frac{y}{\pi(x^2+y^2)}\,.$$

This Green's function can be used immediately to solve the general Dirichlet problem for the Laplace equation on the half-plane.

## 20.4 Fundamental solution to the heat equation

Solution to the problem

$$u_t = \alpha^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0$$

with the initial condition

$$u(0,x) = \delta(x)$$

is called a *fundamental solution* to the heat equation. The solution is almost immediate using the Fourier transform. Applying the Fourier transform with respect to $x$, I find

$$\hat{u}'_t = -\alpha^2 k^2 \hat{u}, \quad \hat{u}(0,k) = \frac{1}{\sqrt{2\pi}}\,,$$

which implies after an integration

$$\hat{u}(t,k) = \frac{1}{\sqrt{2\pi}}e^{-\alpha^2 t k^2},$$

which is the Gaussian function. Recall that the inverse Fourier transform of the Gaussian function is the Gaussian again:

$$\mathcal{F}\left[e^{-ax^2}\right] = \frac{1}{\sqrt{2a}}e^{-k^2/(4a)}.$$

Carefully using the dilation theorem I find that

$$u(t,x) = \Phi(t,x) = \mathcal{F}^{-1}\hat{u}(t,k) = \frac{1}{2\alpha\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha^2 t}}\,.$$

The graphs of this function are shown in the figure below. You can convince yourself that the integral

$$\int_{\mathbf{R}} \Phi(t,x)\,\mathrm{d}x = 1$$

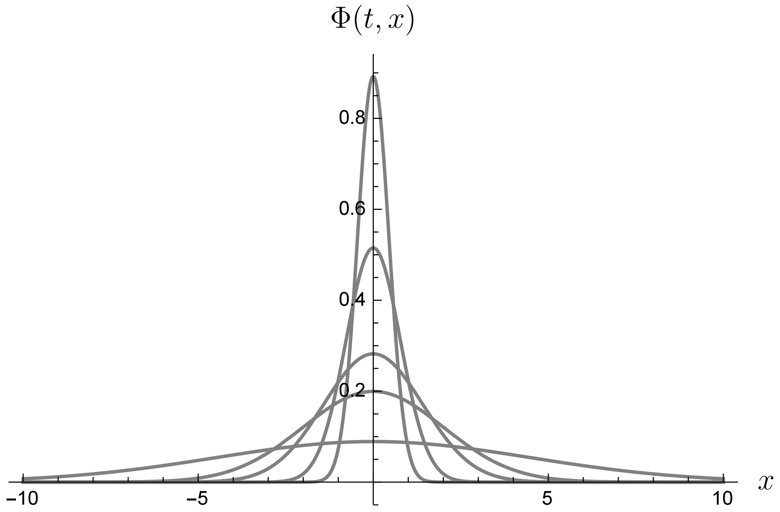


Figure 1: Fundamental solution to the heat equation at different time moments. Note that if $t \to 0^+$ then $\Phi(t,x) \to \delta(x)$

for any time $t$, and since $\Phi(t,x) > 0$ for all $x$ and $t > 0$ then $\Phi$ is, in terms of probability theory, is a *probability density function*. This is actually a probability density function with the mean zero and the standard deviation $\sigma = \sqrt{2t}\alpha$, which connects the random walk model that leads to the diffusion equation with the solution to this equation.

One of the very important consequences of this solution is that it shows that in our model of the heat spread the velocity of the movement of the thermal energy is infinite. Indeed, the initial condition says that $u(0,x) = 0$ in any point except $x = 0$, and at the same time the solution shows that $u(t,x) > 0$ at any point $x$ and any time $t$, which is equivalent to saying that the speed of spread of the heat is infinite. This by no means implies that the actual velocity of the spread of the heat is infinite! This just shows that drawbacks of our model; and if we have a problem at hands in which it is important the velocity of the heat spread, we have to replace the model.

If I change my initial condition for $u(0,x) = \delta(x-\xi)$ then I find

$$u(t,x) = \frac{1}{2\alpha\sqrt{\pi t}} e^{-\frac{(x-\xi)^2}{4\alpha^2 t}},$$

and hence a solution to the initial value problem for the heat equation with the initial condition

$$u(0,x) = f(x)$$

can be written as, by the principle of superposition,

$$u(t,x) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-\xi)^2}{4\alpha^2 t}} f(\xi)\,\mathrm{d}\xi.$$

It is instructive to obtain a proof of this formula by the Fourier transform method.

Unfortunately, it is quite difficult to evaluate the last integral for an arbitrary $f$. This can always be done, however, when $f(x) = e^{-(ax^2+bx+c)}$ by completing the squares. As a (tedious) exercise I ask you to prove that if $f(x)=e^{-x^2}$ then my solution to the initial value problem for the heat equation is (setting $\alpha=2$ for simplicity)

$$u(t,x)=\frac{1}{\sqrt{16t+1}}e^{-\frac{x^2}{16t+1}}.$$

## 20.5 Duhamel's principle revisited

Since I have a solution to the IVP for the heat equation, now I can solve the non-homogeneous problem

$$u_t=\alpha^2u_{xx}+h(t,x),\quad -\infty<x<\infty,\quad t>0,$$

with the initial condition

$$u(0,x)=f(x).$$

Let me write the solution to the homogeneous problem

$$v_t=\alpha^2v_{xx},\quad v(0,x)=f(x)$$

as

$$v(t,x)=\int_{\mathbf{R}}G(t,x;\xi)f(\xi)\,\mathrm{d}\xi,$$

where

$$G(t,x;\xi)=\frac{1}{2\alpha\sqrt{\pi t}}\,e^{-\frac{(x-\xi)^2}{4\alpha^2 t}}.$$

If I now consider a non-homogeneous problem with the zero initial condition:

$$w_t=\alpha^2w_{xx}+h(t,x),\quad w(0,x)=0,$$

then it is a simple exercise to show that $v+w$ give me solution to the original problem. To solve problem for $w$ I now recall Duhamel's principle, which we already used for solving a non-homogeneous wave equation. This principle boils down to

1. Construct a family of solutions of homogeneous Cauchy problems with variable initial time $\tau>0$ and initial data $h(\tau,x)$.
2. Integrate the above with respect to $\tau$ over $(0,t)$.

I will leave it as an exercise to prove the validity of this principle in this particular case. According to this principle the solution to

$$q_t=\alpha^2q_{xx},\quad q(\tau,x)=h(\tau,x)$$

can be used to find

$$w(t,x)=\int_0^t q(t,x;\tau)\,\mathrm{d}\tau,$$

which finally gives me the following general solution

$$u(t,x)=\int_{\mathbf{R}}G(t,x;\xi)f(\xi)\,\mathrm{d}\xi+\int_0^t\int_{\mathbf{R}}G(t-\tau,x;\xi)h(\tau,\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau.$$