22 Heat and wave equations on the plane. Separation of variables revisited

Up till now we studied mostly the equations which have only two independent variables. Now it is time to take a look at the case when there are at least three independent variables. With such assumptions the heat and the wave equations describe the heat transfer or wave processes in planar medium because one variable is time and the other two define a point on a plane. I will consider the heat equations, but basically no change must be made to solve the wave equation.

22.1 Solving the Dirichlet problem for the heat equation in a rectangle

Consider the following problem:

$$u_t = \alpha^2 (u_{xx} + u_{yy}) = \alpha^2 \Delta u, \quad t > 0, \quad (x, y) \in D, \tag{22.1}$$

where α is a given real constant, and D is a rectangle, as in the figure. I supplement my equation with the initial condition

$$u(0, x, y) = f(x, y), \quad (x, y) \in D, \tag{22.2}$$

and Type I or Dirichlet boundary conditions

$$u(t, x, y) = 0, \quad (x, y) \in \partial D, \quad t > 0, \tag{22.3}$$

where ∂D is the boundary of D.

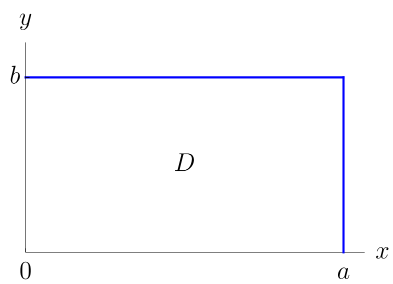

Figure 1: The rectangular domain D

To attack this problem I first assume that my solution can be represented as

$$u(t, x, y) = T(t)V(x, y).$$

Plugging this into (22.1) yields that

$$T'V = \alpha^2 T \Delta V,$$

or, after rearranging,

$$\frac{T'}{\alpha^2 T} = \frac{\Delta V}{V} = -\lambda,$$

Math 483/683: Partial Differential Equations by Artem Novozhilov
e-mail: artem.novozhilov@ndsu.edu. Spring 2016

and since the left hand side depend only on t and the right hand side depend on (x, y) both fractions must be equal to the same constant, which I, for notational reasons, denote as $-\lambda$. Now, after the separation of variables, I end up with two differential equations, one is ordinary,

$$T' = -\lambda\alpha^2 T, \tag{22.4}$$

and the other one is still the partial differential equation

$$-\Delta V = \lambda V. \tag{22.5}$$

Equation (22.5) is called the *Helmholtz equation*. The boundary conditions (22.3) imply that I must supplement problem (22.5) with the boundary conditions

$$V(x, y) = 0, \quad (x, y) \in D,$$

which can be explicitly written, thanks to the simple geometry of D, as

$$V(0, y) = V(a, y) = V(x, 0) = V(x, b) = 0.$$

Helmholtz equation plus the boundary conditions constitute an *eigenvalue problem* for the Laplace operator Δ, that is, I am required to find such values of the constant λ that this problem has a nonzero solution.

To solve this eigenvalue problem I, one more time, will use the assumptions that I can separate the variables:

$$V(x, y) = X(x)Y(y).$$

My equation becomes

$$X''Y + XY'' = -\lambda XY,$$

or, after rearranging ,

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = -\mu,$$

where I used another constant $-\mu$ because the left hand side depend only on x and the right hand side depend only on y. Hence (and using the boundary conditions), my eigenvalue problem for the Laplace equation can be written as two boundary value problems for ordinary differential equations:

$$X'' + \mu X = 0, \quad X(0) = X(a) = 0, \tag{22.6}$$

and

$$Y'' + \eta Y = 0, \quad Y(0) = Y(b) = 0, \quad \eta = \lambda - \mu. \tag{22.7}$$

These are two Sturm–Liouville problems that we already solved in the previous lectures, the solutions are

$$X_k(x) = A_k \sin\sqrt{\mu_k}x\,, \quad \mu_k = \left(\frac{\pi k}{a}\right)^2, \quad k = 1, 2, \dots$$
$$Y_m(x) = B_k \sin\sqrt{\eta_m}y\,, \quad \eta_m = \left(\frac{\pi m}{b}\right)^2, \quad m = 1, 2, \dots.$$

Therefore, my eigenvalue problem for the Laplace operator has the eigenvalues

$$\lambda_{k,m} = \eta_m + \mu_k = \left(\frac{\pi k}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2, \quad k, m = 1, 2, \ldots$$

and the eigenfunctions

$$V_{k,m}(x, y) = \sin\frac{\pi k x}{a} \sin\frac{\pi m y}{b}, \quad k, m = 1, 2, \ldots$$

Now I can return to (22.4):

$$T_{k,m}(t) = a_{k,m} e^{-\lambda_{k,m}\alpha^2 t},$$

and

$$u_{k,m}(t, x, y) = T_{k,m}(t) V_{k,m}(x, y)$$

solves by construction equation (22.1) and satisfies the boundary conditions (22.3).

By the linearity of the original equation and the principle of superposition the double infinite series

$$u(t, x, y) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_{k,m} e^{-\lambda_{k,m}\alpha^2 t} \sin\frac{\pi k x}{a} \sin\frac{\pi m y}{b},$$

satisfies my equation and the boundary conditions. Using the initial conditions we can uniquely identify constants $a_{k,m}$, because, as before in the case of the Sturm–Liouville problem, the eigenfunctions are orthogonal.

I introduce the following *inner product* of two functions defined on D as

$$\langle f, g\rangle = \iint_D f(x, y) g(x, y)\, \mathrm{d}x\, \mathrm{d}y.$$

Then it can be shown (remember that in the case of the rectangle the double integral is just a repeated integral) that

$$\langle V_{k,m}, V_{p,q}\rangle = 0, \quad k \neq p, \text{ or } m \neq q,$$

and

$$\langle V_{k,m}, V_{k,m}\rangle = \frac{ab}{4}.$$

Therefore, using the initial condition (22.2), I get that

$$f(x, y) = \sum_{k=1}^{\infty}\sum_{m=1}^{\infty} a_{k,m} V_{k,m}(x, y),$$

and hence, by orthogonality,

$$a_{k,m} = \frac{\langle f, V_{k,m}\rangle}{\langle V_{k,m}, V_{k,m}\rangle} = \frac{4}{ab}\iint_D f(x, y) \sin\frac{\pi k x}{a} \sin\frac{\pi m y}{b}\, \mathrm{d}x\, \mathrm{d}y, \quad k, m = 1, 2, \ldots$$

Moreover, if $a_{1,1} \neq 0$ then

$$r = \left(\frac{1}{a^2} + \frac{1}{b^2}\right)\pi^2\alpha^2$$

is the rate with which the rectangular plate will be approaching the obvious equilibrium $u(t, x, y) = 0$. Additionally, after some time

$$u(t, x, y) \approx a_{1,1} e^{-rt} V_{1,1}(x, y).$$

We solved our problem. Can we do is a similar way for some other D? Not really, as we will see below.

22.2 Problem for a general domain

Consider now the same problem

$$u_t = \alpha^2 u, \quad t > 0, \quad (x, y) \in D$$

with the same initial condition

$$u(0, x, y) = f(x, y),$$

and the homogeneous Dirichlet boundary condition

$$u(t, x, y) = 0, \quad (x, y) \in \partial D,$$

for some "nice" but arbitrary domain D. (By "nice" you can think of a bounded, simply connected, with piecewise-smooth boundary.) Using the same separation of variables I again end up with an eigenvalue problem

$$-\Delta V = \lambda V, \quad V(x, y) = 0, \quad (x, y) \in \partial D$$

for the Laplace operator. A lot can be proved about the eigenvalues and eigenfunctions of this problem without explicitly computing them. In particular,

- There are infinitely many positive eigenvalues

  $$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots$$

  which tend to infinity as the index grows.

- The corresponding eigenfunctions are orthogonal, that is

  $$\langle V_k, V_m \rangle = \iint_D V_k(x, y) V_m(x, y) \, \mathrm{d}x \, \mathrm{d}y = 0, \quad k \ne m.$$

- The list of eigenfunctions form a system, that is, any sufficiently "nice" function can be represented as a convergent Fourier series

  $$f = \sum_k c_k V_k, \quad c_k = \frac{\langle f, V_k \rangle}{\langle V_k, V_k \rangle}.$$

  Proof of these facts is well beyond the scope of the present course.

22.3 The planar heat equation on a disc

The question still remans: Can we solve our equation explicitly in a domain that is different from a rectangle? The answer is positive, but this will require some additional work. Here I will show where the problems start.

So, consider now the same problem (22.1)-(22.2) with the difference that $D = \{(x, y) : x^2 + y^2 < 1\}$, i.e., it is a unit disk. After separating the variables I will have the same problem for T and the following Helmholtz equation

$$-\Delta V = \lambda V,$$

with the homogeneous boundary condition

$$V(x,y)=0,\quad x^2+y^2=1.$$

I know that I must have infinitely many eigenvalues and eigenfunctions, the latter can be used to build a series solution to the original problem, but how to find them analytically? A natural way to attack this problem is to rewrite my equation in the polar coordinate:

$$v_{rr}+\frac{1}{r}v_r+\frac{1}{r^2}v_{\theta\theta}+\lambda v=0,\quad v(1,\theta)=0,$$

and look for a solution in the form

$$v(r,\theta)=R(r)\Theta(\theta).$$

I get

$$R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''+\lambda R\Theta=0,$$

or, after rearranging,

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\lambda r^2=-\frac{\Theta''}{\Theta}=\mu,$$

where I introduced another constant of separation because the left hand side depends only on r and the right hand side depends only on θ.

I get now two ODE problems. The first one is

$$\Theta''+\mu\Theta=0,\quad \Theta(-\pi)=\Theta(\pi),\quad \Theta'(-\pi)=\Theta'(\pi),$$

where the boundary conditions are from the periodicity requirement. This is our familiar Sturm–Liouville problem with periodic boundary conditions, for which we know that

$$\mu_m=m^2,\quad \Theta_m(\theta)=A\cos m\theta+B\sin m\theta,\quad m=0,1,2,\ldots$$

Therefore for R I get

$$r^2R''+rR'+(\lambda r^2-m^2)R=0,\quad R(1)=0,\quad 0\leq r<1.$$

I also supplement my boundary condition with the "physical" that $R(r)$ must be bounded for all r, in particular at $r=0$: $|R(0)|<\infty$. To slightly simplify my problem I will introduce a new variable

$$z=\sqrt{\lambda}r,$$

and new function

$$h(z)=h(\sqrt{\lambda}r)=R(r).$$

By the chain rule I have that

$$z^2h''+zh'+(z^2-m^2)h=0,$$

where now the derivatives are taken with respect to the new variable z. And now I am stuck since there is *no way* to express a solution to this *linear second order ordinary differential equation with variable coefficients* through the pool of elementary functions. I will need something else, and this will require some facts from the so-called analytic theory of ODE.