23 Elements of analytic ODE theory. Bessel's functions

Recall (I am changing the variables) that we need to solve the so-called *Bessel's equation*

$$x^2u'' + xu' + (x^2 - m^2)u = 0, \quad m = 0, 1, 2, \ldots$$

23.1 Elements of analytic ODE theory

Let

$$p(x)u'' + q(x)u' + r(x)u = 0 \tag{23.1}$$

be a second order linear homogeneous ODE with non-constant coefficients. Recall that function f is called *analytic* at x_0 if it can be represented in some neighborhood of x_0 by a convergent power series:

$$f(x) = u_0 + u_1(x - x_0) + u_2(x - x_0)^2 + u_3(x - x_0)^3 + \ldots$$

The coefficients can be found by

$$u_k = \frac{f^{(k)}(x_0)}{k!},$$

where $k! = 2 \cdot 2 \cdot \ldots \cdot k$.

From the calculus course, for example, we know that exponent, sine and cosine are analytic anywhere in $\mathbf{R}$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots,$$
$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots,$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

Theorem 23.1. *Consider problem* (23.1) *and assume p, q, r are analytic at x_0 and $p(x_0) \neq 0$. Then problem* (23.1) *with the initial conditions $u(x_0) = u_0$, $u'(x_0) = u_1$ has a unique analytic solution.*

This theorem actually gives us a way to work through the problem. All we need to do is to look for the solution in the form

$$u(x) = u_0 + u_1(x - x_0) + u_2(x - x_0)^2 + \ldots$$

and determine $u_2, u_3, \ldots$.

Since the general solution to (23.1) is given by

$$u(x) = A\hat{u}(x) + B\check{u}(x),$$

where A, B are two arbitrary constants and $\hat{u}$ and $\check{u}$ are two linearly independent solutions, we can always use our power series method with two different (and linearly independent) initial conditions, e.g., we can take

$$\hat{u}(x_0) = 1, \quad \hat{u}'(x_0) = 0,$$

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and

$$\check{u}(x_0) = 0, \quad \check{u}'(x_0) = 1.$$

In one of the previous lectures we already saw this method applied to the equation $u'' + \omega u = 0$. Consider another example.

Example 23.2 (Airy equation)**.** Consider

$$u'' - xu = 0,$$

and take the initial conditions

$$u(0) = 1 = u_0, \quad u'(0) = 0 = u_1.$$

The stated theorem obviously works in this case since p is constant and $r(x) = x$. I take

$$u(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 + u_4 x^4 + \dots$$

and hence

$$u''(x) = 2u_2 + 3 \cdot 2u_3 x + 4 \cdot 3u_4 x^2 + \dots.$$

Plugging the obtained expressions into my equation I find

$$2u_2 + 3 \cdot 2u_3 x + 4 \cdot 3u_4 x^2 + 5 \cdot 4u_3 x^3 \dots = u_0 x + u_1 x^2 + u_2 x^3 + u_3 x^4 + u_4 x^5 + \dots$$

Two convergent power series equal only if the coefficients at the same powers are equal, that is

$$\begin{aligned} 2u_2 &= 0, \\ 6u_3 &= u_0, \\ 12u_4 &= u_1, \\ 20u_5 &= u_2, \\ 30u_6 &= u_3, \\ &\dots \\ (n+1)(n+2)u_{n+2} &= u_{n-1}, \\ &\dots \end{aligned}$$

Using the initial conditions I find

$$u_{3k} = \frac{u_{3k-3}}{3k(3k-1)}, \quad k = 1, 2, 3, \dots$$

and all other u_i are zero. The last expression is enough to write that my first linearly independent solution to the Airy equation is

$$\hat{u}(x) = \sum_{k=0}^{\infty} u_{3k} x^{3k},$$

which, as can be proved, converges for any $x \in \mathbf{R}$. I will leave it as an exercise to show that for the initial conditions $u(0) = 0, u'(0) = 1$ the solution is

$$\check{u}(x) = \sum_{k=0}^{\infty} u_{3k+1} x^{3k+1}, \quad u_{3k+1} = \frac{u_{3k-2}}{(3k+1)3k}.$$

23.2 Solving Bessel's equation

Unfortunately, the same method will not work for Bessel's equation, if I'd like to build a power series solution around 0. The reason is that $x_0 = 0$ is not a *regular point*, meaning that $p(0) = 0$. For Bessel's equation $x_0 = 0$ is a *singular point*, but fortunately for as a *regular singular point* (a point x_0 is called a regular singular point if the equation can be written as $(x - x_0)^2 a(x)u'' + q(x)u' + r(x)u = 0$, which is obviously holds for our equation with $x_0 = 0$ and $a(x) = 1$). In this case it turns out the *Frobenius method* will work. Frobenius method says that in this case a solution can be sought in the form

$$u(x) = (x - x_0)^\nu \sum_{n=0}^{\infty} u_n (x - x_0)^n,$$

where ν does not to be integer or positive.

I have, assuming that $u_0 = 1$,

$$\begin{aligned}
u(x) &= x^\nu + u_1 x^{\nu+1} + u_2 x^{\nu+2} + \ldots \\
xu(x) &= x^{\nu+1} + u_1 x^{\nu+2} + u_2 x^{\nu+3} + \ldots \\
x^2 u(x) &= x^{\nu+2} + u_1 x^{\nu+3} + u_2 x^{\nu+4} + \ldots \\
u'(x) &= \nu x^{\nu-1} + (\nu+1) u_1 x^\nu + (\nu+2) u_2 x^{\nu+1} + \ldots \\
u''(x) &= \nu(\nu-1) x^{\nu-2} + (\nu+1)\nu u_1 x^{\nu-1} + (\nu+2)(\nu+1) u_2 x^\nu + \ldots
\end{aligned}$$

The coefficient at x^ν must be

$$\nu(\nu - 1) + \nu - m^2 = 0,$$

which is true only if $\nu = \pm m$, if $m \neq 0$.

For the $\nu + n$ degree I have, replacing m^2 with ν^2,

$$x^{\nu+n}\colon \left[(\nu+n)^2 - \nu^2\right] u_n + u_{n-2} \implies u_n = -\frac{1}{n(2\nu+n)} u_{n-2}, \quad n = 2, 3, 4, \ldots$$

Starting with $u_0 = 1, u_1 = 1$ I get that all the odd indices are zero, whereas for even $n = 2k$

$$u_{2k} = -\frac{u_{2k-2}}{4k(k+\nu)} = \ldots = \frac{(-1)^k}{2^{2k} k! (\nu+k)(\nu+k-1)\ldots(\nu+1)}$$

and hence my solution is

$$u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{\nu+2k}}{2^{2k} k! (\nu+k)(\nu+k-1)\ldots(\nu+1)}.$$

In general this is not necessary, but in our case m is an integer, and if $\nu = -m$ then the denominator in the series above vanishes. Hence we only found one solution to Bessel's equation:

$$u(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{m+2k}}{2^{2k} k! (m+k)(m+k-1)\ldots(m+1)}.$$

I am allowed to multiply my solution by any constant, and I choose, by convention, to multiply the series above by $1/(2^m m!)$, in this case I have

$$J_m(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{m+2k}}{2^{2k+m}k!(m+k)!}\,,$$

Bessel's function of the first kind of the m-th order. An application of the ratio test yields that the series converges for any $x \in \mathbf{R}$ and hence J_m is analytic anywhere in $\mathbf{R}$ (or even in $\mathbf{C}$).

Just to get first idea on the Bessel's functions, note that

$$J_0(0)=1,\quad J_m(0)=0,\quad m=1,2,\ldots$$

So, I found one independent solution and in need of another one. Abel's formula (see Math 266) tells me that for two linearly independent solutions to $u''+a(x)u'+b(x)u=0$ I must have

$$\det\begin{bmatrix}u_1 & u_2\\ u_1' & u_2'\end{bmatrix}=Ce^{-\int a(x)\,\mathrm{d}x}.$$

In my case $a(x)=1/x$ and hence I end up with

$$u_1u_2'-u_1'u_2=\frac{C}{x}\,,$$

or, after some rearranging

$$\left(\frac{u_2}{u_1}\right)'=\frac{C}{xu_1^2}\,.$$

Using $u_1(x)=J_m(x)$ I have

$$\frac{u_2(x)}{J_m(x)}=\int\frac{C}{J_m^2(x)}\,\mathrm{d}x.$$

Therefore,

$$u_2(x)=J_m(x)\int\frac{C}{J_m^2(x)}\,\mathrm{d}x$$

is the second linearly independent solution, which is actually called *Neumann's function* of the second kind of the m-th order.

23.3 Some facts about solutions to Bessel's equation

The full analysis of the solutions to Bessel's equation is beyond the scope of this course. I, however, would like to show how at least some of the important results can be obtained and proved.

I will start with Bessel's function of the first kind of order zero:

$$J_0(x)=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\frac{x^6}{2^2\cdot 4^2\cdot 6^2}+\frac{x^8}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}-\ldots$$

Since the ratio of two consecutive terms is

$$-\frac{x^2}{(2k)^2}\,,$$

which approaches zero as $k \to \infty$ for any fixed x then this series converges absolutely and uniformly, and hence J_0 and all its derivatives are continuous.

For J_1 I have

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \dots,$$

which immediately implies that

$$\frac{\mathrm{d}J_0}{\mathrm{d}x}(x) = -J_1(x)$$

(cf. with $\cos' x = -\sin x$). Analogously,

$$\frac{\mathrm{d}}{\mathrm{d}x}(xJ_1(x)) = xJ_0(x).$$

Let me use my formula for the second independent solution to find Neumann's function of the second kind of zero order.

I have

$$N_0(x) = J_0(x) \int \frac{\mathrm{d}x}{xJ_0^2(x)}.$$

Using the fact that (prove it)

$$\frac{1}{xJ_0^2(x)} = \frac{1}{x} + \frac{x}{2} + \frac{5x^3}{32} + \dots$$

and integrating by terms I find

$$N_0(x) = J_0(x) \left(\log x + \frac{x^2}{4} - \frac{3x^4}{128} + \dots \right).$$

The most important fact here is that N_0 is not defined at zero and approaches $-\infty$ (it behaves like log for small x). Using Neumann's function of the second kind I can define *Bessel's function of the second kind* of zero order as a special linear combination of J_0 and N_0:

$$Y_0(x) = \frac{2}{\pi} \left(N_0(x) - (\log 2 - \gamma) J_0(x) \right),$$

where γ is the Euler constant, $\gamma = \lim_{n\to\infty} \left((\sum_{i=1}^{n}) \frac{1}{n} - \log n \right) \approx 0.5772$. Hence the general solution to Bessel's equation of zero order can be written (this is the most standard form) as

$$u(x) = AJ_0(x) + BY_0(x).$$

Recall that we are mostly interested in solutions to

$$R'' + \frac{1}{r}R' + \left(\lambda - \frac{m^2}{r^2} \right) R = 0.$$

By above and generalizing I showed that the general solution to this equation is given by

$$R(r) = AJ_m(\sqrt{\lambda}r) + BY_m(\sqrt{\lambda}r).$$

Now let me denote $v(x) = J_0(\alpha x)$ and $w(x) = J_0(\beta x)$, where α and β are some constants. Due to the above I have that v and w solve

$$xv'' + v' + \alpha^2 xv = 0,$$
$$xw'' + w' + \beta^2 xw = 0.$$

If I multiply the first equation by w, second by v and subtract then I get, after simplifications,

$$\big(x(v'w - vw')\big)' = (\beta^2 - \alpha^2)xvw.$$

By integrating from 0 to 1 I proved that

$$(\beta^2 - \alpha^2)\int_0^1 xJ_0(\alpha x)J_0(\beta x)\,\mathrm{d}x = \alpha J_0'(\alpha)J_0(\beta) - \beta J_0'(\beta)J_0(\alpha).$$

Similarly, by multiplying the equation for v by $2xv'$ I can show (exercise) that

$$\int_0^1 xJ_0^2(\alpha x)\,\mathrm{d}x = \frac{1}{2}(J_0^2(\alpha) + J_1^2(\alpha)).$$

An important corollary is as follows: if α and β are two roots of J_0 then

$$\int_0^1 xJ_0(\alpha x)J_0(\beta x)\,\mathrm{d}x = 0$$

and

$$\int_0^1 xJ_0^2(\alpha x)\,\mathrm{d}x = \frac{1}{2}J_1^2(\alpha).$$

The question is, of course, do we have any roots at all? To see that there are always infinitely many roots, let me make the change of variables

$$v(x) = u(x)\sqrt{x}$$

in Bessel's equation of the order zero. Then, after straightforward manipulations, I find that

$$v'' = -\left(1 + \frac{1}{4x^2}\right)v,$$

that is, when x is large, then the equation is approximately $v'' + v = 0$, and hence for large x Bessel's equation has an approximate solution

$$u(x) = \frac{A\cos(x - \phi)}{\sqrt{x}},$$

for some constants A and ϕ, which indicates that Bessel's functions approach zero as $x \to \infty$ and that Bessel's functions have infinitely many real positive roots.

Let me introduce the inner product

$$\langle f, g\rangle_B = \int_0^1 xf(x)g(x)\,\mathrm{d}x.$$

Note that the functions $J_0(\zeta_k x)$, $k = 1, 2, 3, \dots$ are *orthogonal* on $[0, 1]$ with respect to this inner product. Here ζ_k is the k-th root of $J_0(x)$. It can be proved that any "nice" function f can be represented as a convergent series

$$f(x) = c_1 J_0(\zeta_1 x) + c_2 J_0(\zeta_2 x) + c_3 J_0(\zeta_3 x) + \dots.$$

This expansion is called the *Fourier–Bessel expansion*, and the coefficients can be found, from the proved formulas above, as

$$c_k = \frac{\langle f, J_0(\zeta_k x)\rangle_B}{\langle J_0(\zeta_k x), J_0(\zeta_k x)\rangle_B} = \frac{2}{J_1^2(\zeta_k)} \int_0^1 x f(x) J_0(\zeta_k x)\,\mathrm{d}x.$$

23.4 Summary about Bessel's functions

In a way similar to the above one can show that the following theorem holds.

Theorem 23.3. *Consider Bessel's ODE of the order m, where $m = 0, 1, 2, \dots$:*

$$x^2 u'' + x u' + (x^2 - m^2) u = 0.$$

The general solution to this equation can be written as

$$u(x) = A J_m(x) + B Y_m(x),$$

where J_m is Bessel's function of the first kind of order m and Y_m is Bessel's function of the second kind of order m. $J_0(0) = 1, J_m(0) = 0, m = 1, 2, \dots$. Bessel's functions of the second kind have a singularity at $x = 0$ for any m. In particular, $\lim_{x\to 0^+} Y_m(x) = -\infty$.

Both J_m and Y_m approach zero as $x \to \infty$, both J_m and Y_m have infinitely many positive roots.

Let $\zeta_{k,m}$ denote the k-th root of J_m. Then

$$\int_0^1 x J_m(\zeta_{k,m} x) J_m(\zeta_{l,m} x)\,\mathrm{d}x = \begin{cases} 0, & l \neq k, \\ \frac{1}{2} J_{m+1}^2(\zeta_{k,m}), & l = k. \end{cases}$$

Any sufficiently nice function f can be represented as Fourier-Bessel series

$$f(x) = \sum_{k=1}^{\infty} c_k J_m(\zeta_{k,m} x),$$

where the explicit form of the coefficients can be inferred from the relation above.

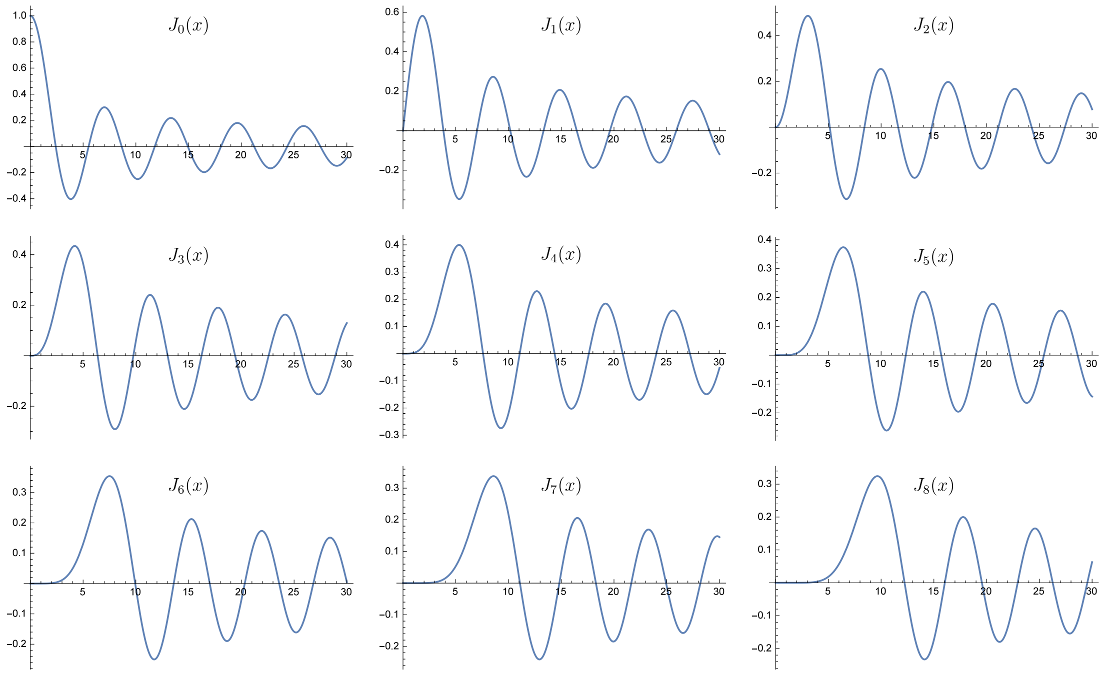

Figure 1: Graphs of several Bessel's functions of the the first kind

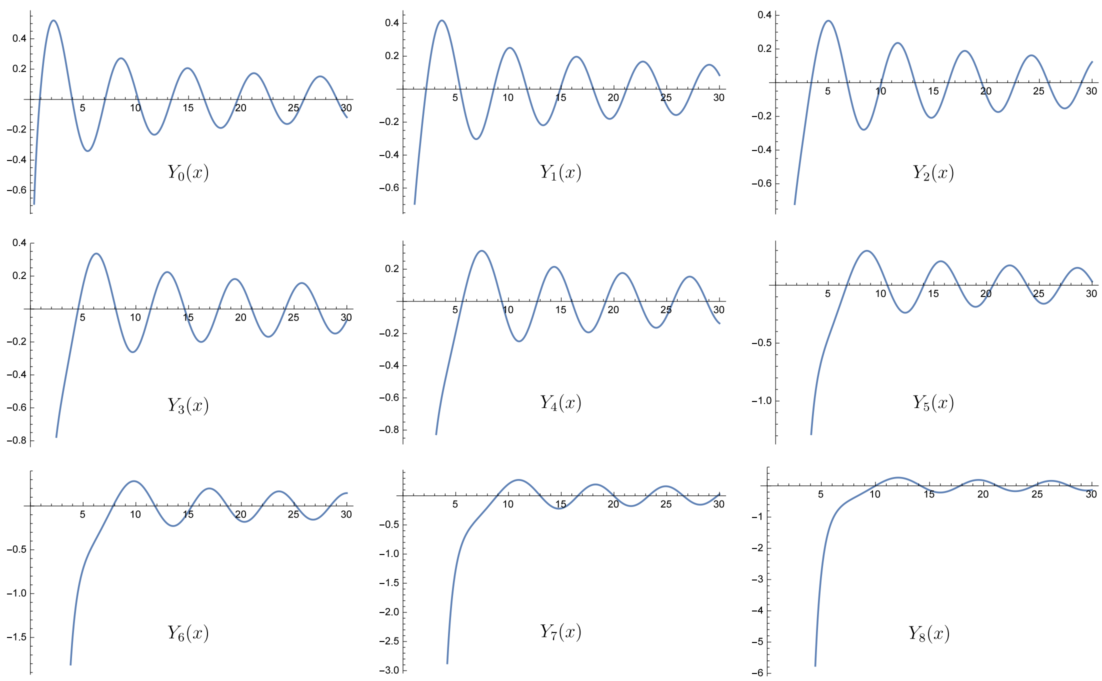

Figure 2: Graphs of several Bessel's functions of the the second kind