24 Solving planar heat and wave equations in polar coordinates

Now that all the preparations are done, I can return to solving the planar heat and wave equations in domains with rotational symmetry.

24.1 Heat equation

Recall that we are solving

$$\begin{aligned} u_t &= \alpha^2 \Delta u, \quad t > 0, \quad x^2 + y^2 < 1, \\ u(0, x, y) &= f(x, y), \quad x^2 + y^2 < 1, \\ u(t, x, y) &= 0, \quad x^2 + y^2 = 1. \end{aligned}$$

We found, by separating the variables $u(t, x, y) = T(t)V(x, y)$ that

$$T' = -\lambda \alpha^2 T,$$

and

$$-\Delta V = \lambda V, \quad x^2 + y^2 < 1, \quad V(x, y) = 0, \quad x^2 + y^2 = 1.$$

For the latter problem we again assumed that variables separate in polar coordinates $v(r, \theta) = R(r)\Theta(\theta)$. For Θ we got

$$\Theta'' + \mu\Theta = 0, \quad \Theta(-\pi) = \Theta(\pi), \quad \Theta'(-\pi) = \Theta'(\pi),$$

which implies that

$$\mu_m = m^2, \quad m = 0, 1, 2, \ldots \quad \Theta(\theta) = A_m \cos m\theta + B_m \sin m\theta,$$

and for R we got Bessel's equation

$$r^2 R'' + rR + (\lambda r^2 - m^2)R = 0, \quad m = 0, 1, \ldots, \quad R(1) = 0, \quad |R(r)| < \infty.$$

Now finally we use the material from the previous section and state that the general solution is given by

$$R(r) = AJ_m(\sqrt{\lambda} r) + BY_m(\sqrt{\lambda} r).$$

From the condition that my solution must be bounded I have that $B = 0$ since Y_m is not bounded close to zero, from the boundary condition I have

$$J_m(\sqrt{\lambda}) = 0,$$

that is my $\sqrt{\lambda}$ must be the root of the corresponding equation. Hence,

$$\lambda_{k,m} = \zeta_{k,m}^2,$$

where $\zeta_{k,m}$ is the k-th positive root of J_m.

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To summarize, the eigenvalues of the eigenvalues problem for the Laplace operator in the unit disk with type I or Dirichlet boundary conditions are

$$\lambda_{k,m} = \zeta_{k,m}^2,$$

with the eigenfunctions

$$\begin{aligned} v_{k,0}(r,\theta) &= J_0(\zeta_{k,0} r), \quad k = 1,2,\ldots, \\ v_{k,m}(r,\theta) &= J_m(\zeta_{k,m} r)\cos m\theta, \ldots, \quad k,m = 1,2,3,\ldots, \\ \tilde{v}_{k,m}(r,\theta) &= J_m(\zeta_{k,m} r)\sin m\theta, \ldots, \quad k,m = 1,2,3,\ldots \end{aligned}$$

Using the standard inner product in Cartesian coordinates

$$\langle f, g\rangle = \iint_D f(x,y)g(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

we know, thanks for the general theory, that all these eigenfunctions are orthogonal with respect to this inner product. However, in this specific case we actually proved this fact. Indeed, in the polar coordinates this inner product reads

$$\langle f, g\rangle = \int_0^1 \int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) g(r\cos\theta, r\sin\theta) r\,\mathrm{d}\theta\,\mathrm{d}r.$$

Hence if we take two eigenfunctions with different m then we get zero due to the orthogonality of the trigonometric system on $[-\pi,\pi)$. If, however, m are the same, but k are different, then we get zeroes due to the orthogonality of Bessel's functions, note the necessary multiple r in our integral. Finally, we need to calculate

$$\begin{aligned} \langle v_{k,0}, v_{k,0}\rangle &= \pi J_1^2(\zeta_{k,0}), \quad k = 1,2,\ldots \\ \langle v_{k,m}, v_{k,m}\rangle &= \frac{\pi}{2} J_{m+1}^2(\zeta_{k,m}), \quad k,m = 1,2,\ldots \\ \langle \tilde{v}_{k,m}, \tilde{v}_{k,m}\rangle &= \frac{\pi}{2} J_{m+1}^2(\zeta_{k,m}), \quad k,m = 1,2,\ldots. \end{aligned}$$

Therefore, putting everything together, my solution is given by

$$u(t, r\cos\theta, r\sin\theta) = \sum_{k=1}^{\infty} a_{k,0} e^{-\alpha^2 \zeta_{k,0}^2 t} J_0(\zeta_{k,0} r) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} e^{-\alpha^2\zeta_{k,m}^2 t} J_m(\zeta_{k,m} r)\left(a_{k,m}\cos m\theta + b_{k,m}\sin m\theta\right).$$

The coefficients are found using the initial conditions and orthogonality of the corresponding eigenfunctions. To be precise,

$$\begin{aligned} a_{k,0} &= \frac{\langle f, v_{k,0}\rangle}{\langle v_{k,0}, v_{k,0}\rangle} = \frac{\int_0^1\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) J_0(\zeta_{k,0} r) r\,\mathrm{d}\theta\,\mathrm{d}r}{\pi J_1(\zeta_{k,0})}, \quad k = 1,2,\ldots, \\ a_{k,m} &= \frac{\langle f, v_{k,m}\rangle}{\langle v_{k,m}, v_{k,m}\rangle} = \frac{2\int_0^1\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) J_m(\zeta_{k,m} r) r\cos m\theta\,\mathrm{d}\theta\,\mathrm{d}r}{\pi J_1(\zeta_{k,0})}, \quad k,m = 1,2,\ldots, \\ b_{k,m} &= \frac{\langle f, \tilde{v}_{k,m}\rangle}{\langle \tilde{v}_{k,m}, \tilde{v}_{k,m}\rangle} = \frac{2\int_0^1\int_{-\pi}^{\pi} f(r\cos\theta, r\sin\theta) J_m(\zeta_{k,m} r) r\sin m\theta\,\mathrm{d}\theta\,\mathrm{d}r}{\pi J_1(\zeta_{k,0})}, \quad k,m = 1,2,\ldots. \end{aligned}$$

In particular, for t large one has that

$$u(t, r\cos\theta, r\sin\theta) \approx a_{1,0} e^{-\alpha^2 \zeta_{1,0}^2} J_0(\zeta_{1,0} r),$$

since it can be proved that $\zeta_{1,0}$ is the largest root among all other roots of J_m.

The explicit examples will be given when I consider the wave equation below.

24.2 Wave equation

Consider now the wave equation

$$\begin{aligned} u_{tt} &= c^2 \Delta u, \quad t > 0, \quad (x, y) \in D, \\ u(0, x, y) &= f(x, y), \quad (x, y) \in D, \\ u_t(0, x, y) &= g(x, y), \quad (x, y) \in D, \\ u(t, x, y) &= 0, \quad (x, y) \in \partial D, \end{aligned}$$

where $D \subset \mathbf{R}^2$ some domain.

Similarly to the heat equation, the separation of variable is possible only for some special domains. For example you saw how to solve this problem when $D = \{0 < x < a, 0 < y < b\}$ in your homework problems.

The key difference from the heat equation is that for T one has

$$T'' + c^2 \lambda_{k,m} T = 0,$$

which has the general solution

$$T_{k,m}(t) = A_{k,m} \cos(\omega_{k,m} t - \phi_{k,m}),$$

where

$$\omega_{k,m} = c\sqrt{\lambda_{k,m}}$$

are the *frequencies of vibration*, and $\lambda_{k,m}$ are the corresponding eigenvalues of the eigenvalues problem for the Laplace operator, which is exactly the same as for the heat equation. The *fundamental frequency* is the smallest one. Recall that for the rectangle we found that

$$\lambda_{k,m} = \left(\frac{k^2}{a^2} + \frac{m^2}{b^2}\right)\pi^2,$$

and hence

$$\omega_{k,m} = c\pi\sqrt{\frac{k^2}{a^2} + \frac{m^2}{b^2}}\,.$$

The solution in the form of Fourier series reads

$$u(t, x, y) = \sum_{k,m} T_{k,m}(t) V_{k,m}(x, y),$$

where $V_{k,m}$ are the eigenfunctions of the Laplace operator with type I boundary conditions, therefore we immediately have an important conclusion: contrary to one dimensional case, when the solution to the wave equation is a periodic function of t, the solution to the planar wave equation is *not* periodic because the ratio

$$\frac{\omega_{k,m}}{\omega_{1,1}}$$

is not a rational number, which is required for the solution to be periodic (compare with the one dimensional case).

Now let me consider again $D = \{(x,y)\colon x^2+y^2<1\}$. Following exactly the same steps that I did for the heat equation, I will end up in general with the following solution:

$$u(t, r\cos\theta, r\sin\theta) = \sum_{k=1}^{\infty} T_{k,0}(t)v_{k,0}(r,\theta) + \sum_{m=1}^{\infty}\sum_{k=1}^{\infty} T_{k,m}(t)v_{k,m} + \tilde{T}_{k,m}(t)\tilde{v}_{k,m},$$

where all the constants that are necessary to be determined are inside T and $\tilde{T}$. By using the initial conditions and considering that $T(t) = A\cos c\sqrt{\lambda}t + B\sin c\sqrt{\lambda}t$ we can always find the explicit expressions for these coefficients. I will consider several examples without treating the most general case.

Example 24.1. Let me consider first the case

$$u(t, r\cos\theta, r\sin\theta) = f(r), \quad u_t(t, r\cos\theta, r\sin\theta) = 0,$$

that is the initial condition does not depend on the angle θ. From the symmetry arguments (or this can be easily proved rigorously) the solution will not also depend on θ, and hence will have the form

$$U(t,r) = \sum_{k=1}^{\infty} a_k \cos(c\zeta_{k,0}t)J_0(\zeta_{k,0}r), \quad k = 1, 2, \ldots$$

and I do not have sine in my solution since the initial velocity is zero. The coefficients of my Fourier–Bessel series are found as

$$a_k = \frac{2}{J_1(\zeta_{k,0})}\int_0^1 rf(r)J_0(\zeta_{k,0}r)\,\mathrm{d}r.$$

To get what is actually happening with the solution, consider the graphs of $J_0(\zeta_{k,0}r)$. They have exactly $k-1$ roots on the interval $(0,1)$ and represent the building blocks out of which the whole solution is built. Each term of the form

$$J_0(\zeta_{k,0}r)\cos(\zeta_{k,0}t)$$

represent a *standing wave*, such that the whole solution is a linear combination of these standing waves, see another figure.

Example 24.2. As a second example consider the problem with the initial condition is given by

$$u(0, r\cos\theta, r\sin\theta) = Av_{k,m}(r,\theta).$$

with again the initial velocity equal to zero. The functions $v_{k,m}$ represent *fundamental vibrations* with the frequency

$$w_{k,m} = c\zeta_{k,m},$$

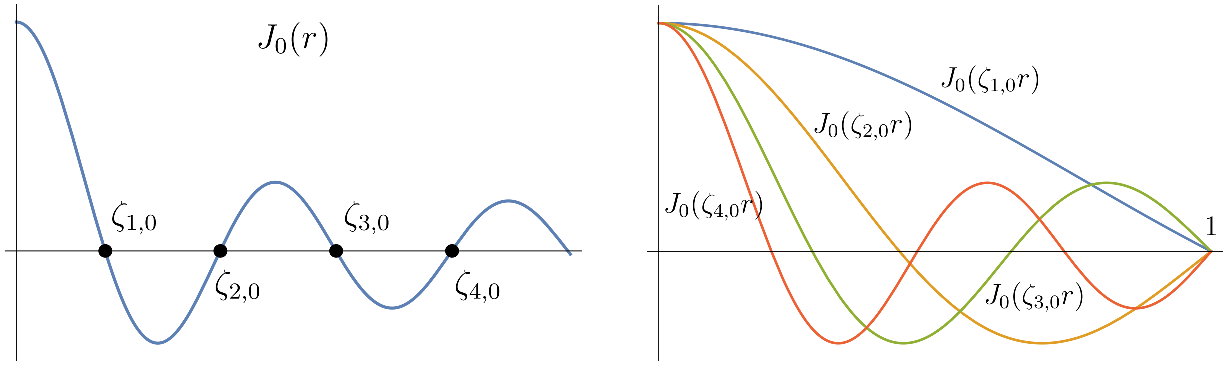

Figure 1: Graphs of $J_0(r)$ and $J_0(\zeta_{k,0}r)$

since the solution to this problem, due to orthogonality of $v_{k,m}$, is given by

$$U(t,r,\theta) = Av_{k,m}(r,\theta)\cos(c\zeta_{k,m}t),$$

and these are the only solutions to my problem that are periodic. Here are the graphs of several fundamental vibrations:

Similarly to the standing waves considered above, these fundamental vibrations will generate the

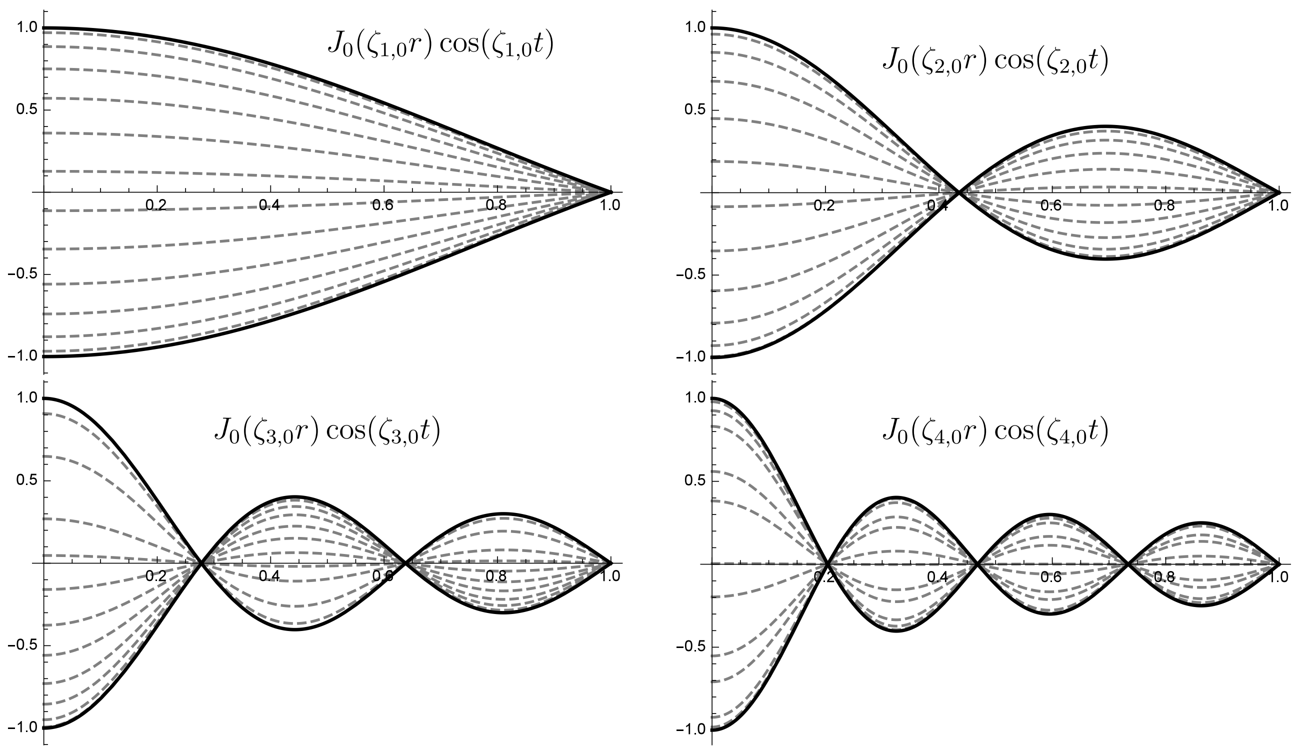

Figure 2: Standing waves. The thick curves are for $\cos(\zeta_{k,0}t) = \pm 1$ and the dashed curves are for time moments between those

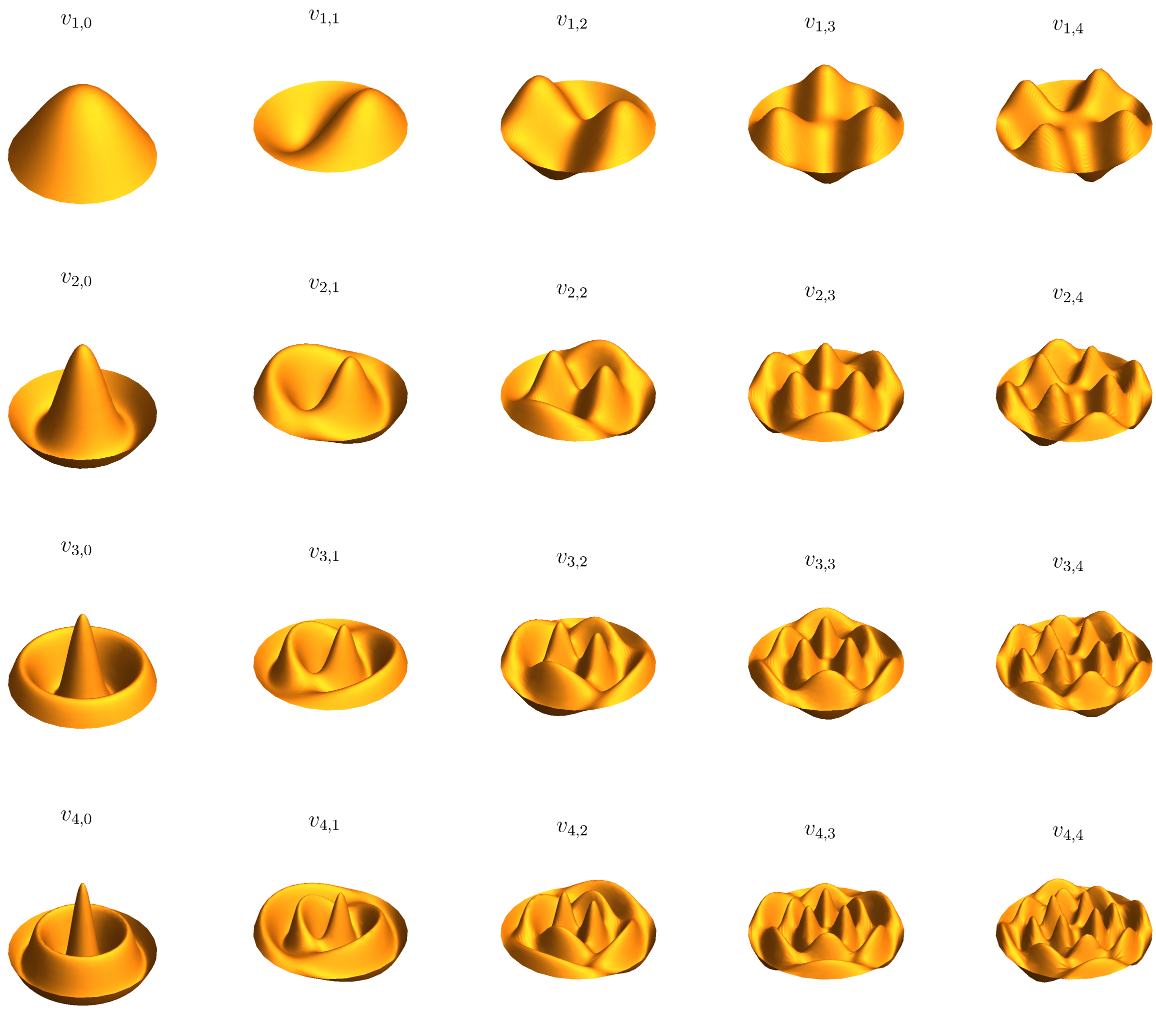

Figure 3: Fundamental vibrations (note that I am using the second index to denote the order of Bessel's function, and the first index to denote the k-th root, which is opposite to what is used in the textbook)

two dimensional standing waves. Note that those sets of points for which

$$v_{k,m}(r, \theta) = 0$$

will stay always zero. It can be probed that these sets are composed of *nodal curves*, that divide the circular drum into several nodal regions.

Example 24.3. In general, the solution to the wave equation on a unit disk can be represented as a linear combination of standing waves, each of which is generated by a fundamental vibration with the corresponding frequency. It can be proved (originally was proved in 1929 by Siegel) that the ration of these frequencies is never a rational number, and hence the solution to the wave equation with

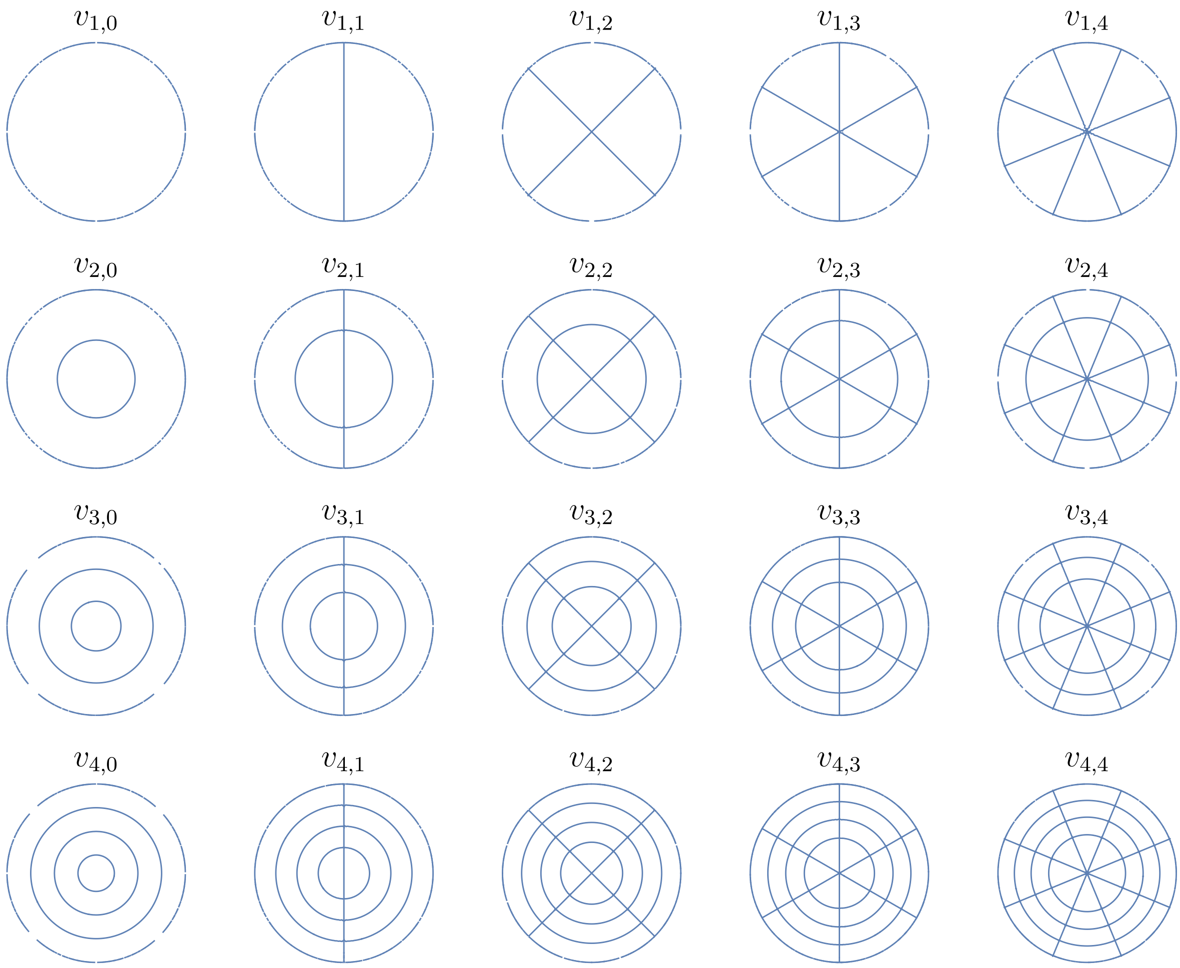

Figure 4: Nodal curves

general initial conditions is *not* a periodic function, the same as for the rectangle plate. In the musical language I can rephrase that higher vibrations for the drumhead are not the pure overtones of the basic frequency $\omega_{1,0}$, which gives a mathematical explanation for the fact that human ear much prefer the sound of one dimensional instruments, such as guitar or violin to two dimensional such as a drum.