25 Solving the wave equation in 2D and 3D space

> 'No,' replied Margarita, 'what really puzzles me is where you have found the space for all this.' With a wave of her hand Margarita emphasized the vastness of the hall they were in. Koroviev smiled sweetly, wrinkling his nose. 'Easy!' he replied. 'For anyone who knows how to handle the fifth dimension it's no problem to expand any place to whatever size you please.'
>
> Mikhail Bulgakov, *The Master and Margarita*

The goal of this concluding section is to find the solution to the initial value problem for the wave equation

$$\begin{aligned} u_{tt} &= c^2\Delta u, \quad \boldsymbol{x}\in\mathbf{R}^k,\\ u(0,\boldsymbol{x}) &= g(\boldsymbol{x}),\\ u_t(0,\boldsymbol{x}) &= h(\boldsymbol{x}), \end{aligned} \tag{25.1}$$

with $k=2,3$. Recall that in the case $k=1$ we already know that the solution is given by *d'Alembert's formula*

$$u(t,x)=\frac{g(x-ct)+g(x+ct)}{2}+\frac{1}{2c}\int_{x-ct}^{x+ct}h(s)\,\mathrm{d}s.$$

It turns out that for the case $k=2,3$ it is also possible to find an explicit solution.

First, I will prove an auxiliary fact, which will help to reduce the number of computations.

Lemma 25.1. *Let v_h denote the solution to the problem.*

$$\begin{aligned} v_{tt} &= c^2\Delta v, \quad \boldsymbol{x}\in\mathbf{R}^k,\\ v(0,\boldsymbol{x}) &= 0,\\ v_t(0,\boldsymbol{x}) &= h(\boldsymbol{x}). \end{aligned} \tag{25.2}$$

Then function

$$w=\frac{\partial}{\partial t}v_g$$

solves

$$\begin{aligned} w_{tt} &= c^2\Delta w, \quad \boldsymbol{x}\in\mathbf{R}^k,\\ w(0,\boldsymbol{x}) &= g(\boldsymbol{x}),\\ w_t(0,\boldsymbol{x}) &= 0. \end{aligned} \tag{25.3}$$

Proof. Indeed, since v_g satisfies the wave equation, taking the derivatives with respect to time on both left and right hand sides and exchanging the order of operators implies that w solves the wave equation. Moreover, $w(0,\boldsymbol{x})=\frac{\partial}{\partial t}v_g(0,\boldsymbol{x})=g(\boldsymbol{x})$ due to the definition of v_g. Finally, I need to show that $w_t(0,\boldsymbol{x})=0$. Consider

$$\frac{\partial}{\partial t}w(0,\boldsymbol{x})=\frac{\partial^2}{\partial t^2}v_g(0,\boldsymbol{x})=c^2\Delta v_g(0,\boldsymbol{x})=0,$$

since, by definition $v_g(0,\boldsymbol{x})=0$. ∎

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Corollary 25.2. *Any solution to* (25.1) *can be written as*

$$u = \frac{\partial}{\partial t} u_g + u_h,$$

where u_h *solves* (25.2)*.*

Proof. The proof follows from the linearity of the original problem. ■

Therefore, all I need to do is to solve problem (25.2). The lemma above holds for any dimension k. From now on, however, I will stick to the case $k = 3$, i.e., I will consider our familiar Euclidian space.

To find a solution to (25.2), I first will find *the fundamental solution* to the three dimensional wave equation. This solution, by definition, solves problem (25.2) with h replaced by $\delta(\boldsymbol{x})$, which is three dimensional delta-function, and which models an unit impulse (disturbance) applied at the point 0. To do this I will use a three dimensional Fourier transform, which is defined for any $f(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbf{R}^3$ as

$$\hat{f}(\boldsymbol{k}) = \frac{1}{(\sqrt{2\pi})^3} \iiint_{\mathbf{R}^3} f(\boldsymbol{x}) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \,\mathrm{d}\boldsymbol{x},$$

where $\boldsymbol{k} \cdot \boldsymbol{x} = k_1 x_1 + k_2 x_2 + k_3 x_3$ in the usual scalar product. The inverse Fourier transform is defined in a similar way with minus replaced by the plus. Denoting $\hat{v}$ the Fourier transform of my unknown function, I get, similarly to one dimensional case

$$\hat{v}_{tt} = -c^2 |\boldsymbol{k}|^2 \hat{v}, \quad \hat{v}(0) = 0, \quad \hat{v}_t(0) = \frac{1}{(\sqrt{2\pi})^3},$$

where

$$|\boldsymbol{k}|^2 = \boldsymbol{k} \cdot \boldsymbol{k} = k_1^2 + k_2^2 + k_3^2$$

is the usual Euclidian norm. Solving this simple ODE I find that

$$\hat{v}(t, \boldsymbol{k}) = \frac{1}{(\sqrt{2\pi})^3 c |\boldsymbol{k}|} \sin(ct|\boldsymbol{k}|).$$

Hence my solution is given by

$$v(t, \boldsymbol{x}) = \frac{1}{8\pi^3 c} \iiint_{\mathbf{R}^3} \sin(ct|\boldsymbol{k}|) e^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{k}}{|\boldsymbol{k}|}.$$

To evaluate this integral I switch to spherical coordinates, chosen such that polar axis coincides with the direction of vector $\boldsymbol{x}$. My spherical coordinates are $k = |\boldsymbol{k}|, \theta, \varphi$. I also denote $|\boldsymbol{x}| = r$. I have, since $\boldsymbol{k} \cdot \boldsymbol{x} = kr\cos\varphi$, due to the choice of the polar axis. Hence my integral is now

$$v(t, \boldsymbol{x}) = \frac{1}{8\pi^3 c} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} \sin(ctk) e^{\mathrm{i}kr\cos\varphi} k^2 \sin\varphi \,\mathrm{d}k \,\mathrm{d}\varphi \,\mathrm{d}\theta.$$

Evaluating the integral for θ and φ, I find (left as an exercise)

$$v(t, \boldsymbol{x}) = \frac{1}{2\pi^2 cr} \int_0^{\infty} \sin kct \sin kr \,\mathrm{d}k.$$

The last integral should be understood in terms of generalizer functions and the inverse Fourier transform. Using the complex exponent to represent sines I end up with

$$v(t,\boldsymbol{x})=\frac{1}{8\pi^2cr}\int_{\mathbf{R}}\left(e^{\mathrm{i}k(ct-r)}-e^{\mathrm{i}k(ct+r)}\right)\mathrm{d}k=\frac{1}{4\pi cr}\left(\delta(ct-r)-\delta(ct+r)\right).$$

I am looking only in the future $t>0$, and hence $\delta(ct+r)=0$. Finally, I get

$$\frac{\delta(ct-r)}{4\pi cr}$$

is *the fundament solution* to the three dimensional heat equation. By a translation argument I get that if my initial velocity would be

$$v_t(0,\boldsymbol{x})=\delta(\boldsymbol{x}-\boldsymbol{\xi}),$$

then my solution is

$$K(t,\boldsymbol{x},\boldsymbol{\xi})=\frac{\delta(ct-|\boldsymbol{x}-\boldsymbol{\xi}|)}{4\pi c|\boldsymbol{x}-\boldsymbol{\xi}|}.$$

Thus the fundamental solution is a traveling wave, initially concentrated at $\boldsymbol{\xi}$ and afterwards on

$$\partial B_{ct}(\xi)=\{\boldsymbol{x}\colon|\boldsymbol{x}-\boldsymbol{\xi}|=ct\},$$

which is the boundary of the ball with the center at $\boldsymbol{\xi}$ and radius ct. This means, among other things, that the wave originates at time $t=0$ at the point $\boldsymbol{\xi}$ will be felt at the point $\boldsymbol{\eta}$ *only* at the time $|\boldsymbol{\eta}-\boldsymbol{\xi}|/c$, which is called the *strong Huygen's principle* and gives a mathematical explanation why sharp signals propagate from a point source. Using the superposition principle I can represent the sought solution to (25.2) as

$$v(t,\boldsymbol{x})=\iiint_{\mathbf{R}^3}K(t,\boldsymbol{x},\boldsymbol{\xi})h(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}.$$

The last integral in spherical coordinates takes the form

$$v(t,\boldsymbol{x})=\int_0^\infty\frac{\delta(ct-r)}{4\pi cr}\int_0^{2\pi}\int_0^\pi h(r,\theta,\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}r.$$

The double integral inside is the integral over the surface of the sphere of radius r with the center at the point x:

$$\int_0^{2\pi}\int_0^\pi h(r,\theta,\varphi)\,\mathrm{d}\varphi\,\mathrm{d}\theta=\iint_{\partial B_r(\boldsymbol{x})}h(\boldsymbol{\sigma})\,\mathrm{d}\boldsymbol{\sigma},$$

therefore, using the main property of the delta function I finally get

Theorem 25.3 (Kirchhoff's formula)**.** *The unique solution to the problem* (25.1) *with* $k=3$ *is given by*

$$u(t,\boldsymbol{x})=t\overline{h}+\frac{\partial}{\partial t}\left(t\overline{g}\right),$$

where $\overline{f}$ *is the average value of function* f *over the sphere of radius* ct *centered at* $\boldsymbol{x}$*,*

$$\overline{f}(t,\boldsymbol{x})=\frac{1}{4\pi c^2t^2}\iint_{\partial B_{ct}(\boldsymbol{x})}f(\boldsymbol{\sigma})\,\mathrm{d}\boldsymbol{\sigma}.$$

Kirchhoff's formula also emphasizes the strong Huygen's principle. To see this (see the figure) assume that the initial disturbance h has a small compact support (that means that it is nonzero only for some small region $U \subseteq \mathbf{R}^3$). I am interested in observing the signal at the point $\boldsymbol{x}$. Initially, for small t_1 the sphere ∂B_{ct_1} will not touch U and hence there will no signal at $\boldsymbol{x}$. At some time t_2 finally the sphere will touch U, and this means that I hear the signal. I continue to experience the signal at $\boldsymbol{x}$ up to the time t_3, when the whole domain U will be inside B_{ct_3}. And since I am integrating over the surface of my ball, after that time, for any $t > t_3$, I will have no indication of the signal at $\boldsymbol{x}$. In other words, the traveling wave in three dimensions has both the leading and the trailing edges sharp.

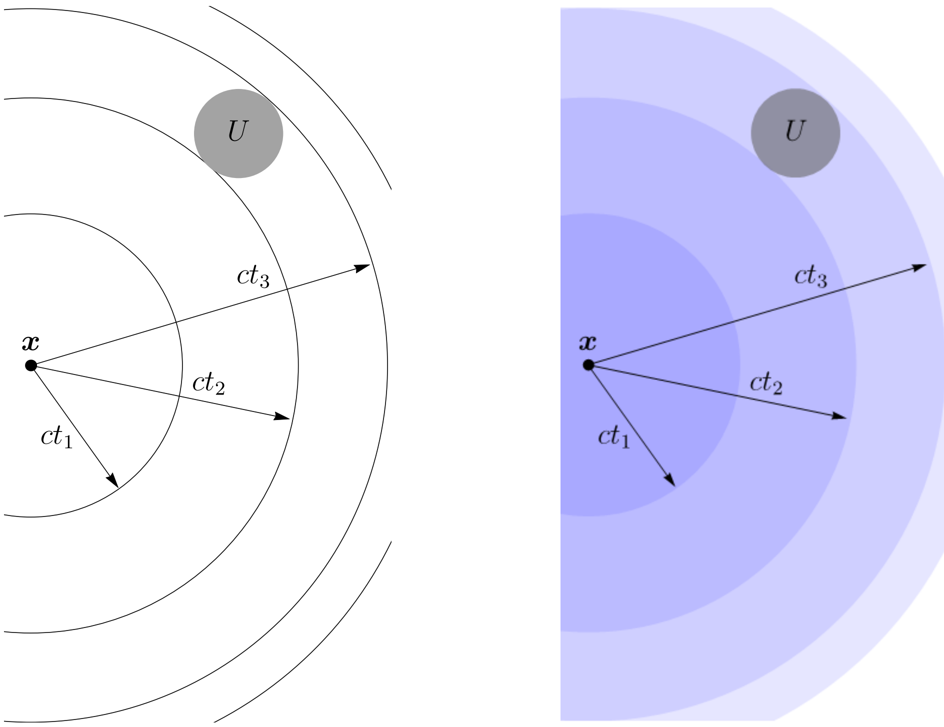

Figure 1: The spread of waves in three (left) and two (right) dimensional spaces. On the left I integrate over the surface of my ball, and on the right I integrate over the whole shaded area

Now I will use the *method of descent* to obtain the explicit solution in the case $\mathbf{R}^2$. The key idea is to consider the problem

$$u_{tt} = c^2 \Delta u, \quad \boldsymbol{x} \in \mathbf{R}^2, \quad u_t(0, \boldsymbol{x}) = h(\boldsymbol{x})$$

as three dimensional and write the points in $\mathbf{R}^3$ as $(\boldsymbol{x}, x_3)$ and set $h(\boldsymbol{x}, x_3) = h(\boldsymbol{x})$. Then by Kirchhoff's formula the solution is given by

$$U(t, \boldsymbol{x}, x_3) = \frac{1}{4\pi c^2 t} \iint_{\partial B_{ct}(\boldsymbol{x}, x_3)} h(\boldsymbol{\sigma}) \, \mathrm{d}\boldsymbol{\sigma}.$$

I claim that this solution is independent of x_3 and hence gives me the solution to the two-dimensional problem for any choice of x_3, for example, $x_3 = 0$. Indeed, I have that my surface consists of two hemispheres defined explicitly as

$$\xi_3 = x_3 \pm \sqrt{c^2t^2 - r^2} = F_{\pm}(\boldsymbol{\xi}), \quad r^2 = (\xi_1 - x_1)^2 + (\xi_2 - x_2)^2.$$

Taking integral by any of these hemispheres and using the fact that h does not depend on x_3 I get

$$\begin{aligned} u(t, \boldsymbol{x}) &= \frac{1}{4\pi c^2 t} \iint_{\partial B_{ct}(\boldsymbol{x}, x_3)} h(\boldsymbol{\sigma})\, \mathrm{d}\boldsymbol{\sigma} \\ &= \frac{1}{2\pi c^2 t} \iint_{B_{ct}(\boldsymbol{x})} h(\boldsymbol{\xi}) \sqrt{1 + |(\partial_{\xi_1} F_{\pm})^2 + (\partial_{\xi_2} F_{\pm})|^2}\, \mathrm{d}\boldsymbol{\xi} \\ &= \frac{1}{2\pi c^2 t} \iint_{B_{ct}(\boldsymbol{x})} \frac{h(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{\xi}|^2}}\,. \end{aligned}$$

This yields

Theorem 25.4 (Poisson's formula)**.** *The unique solution to the problem* (25.1) *with* $k = 2$ *is given by*

$$u(t, \boldsymbol{x}) = \frac{1}{2\pi c^2 t} \left(\frac{\partial}{\partial t} \iint_{B_{ct}(\boldsymbol{x})} \frac{g(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{\xi}|^2}} + \iint_{B_{ct}(\boldsymbol{x})} \frac{h(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}}{\sqrt{c^2t^2 - |\boldsymbol{x} - \boldsymbol{\xi}|^2}} \right).$$

The key fact here is that the integration now not over the surface but over the whole ball itself. That is, the traveling wave has the sharp leading edge but not sharp trailing edge, our integral will always be nonzero for any time $t > t_2$ (see the right panel in the figure) since the initial disturbance will be always inside the ball. The same holds for any space of *even* dimension. You can observe actually this effect experimentally by putting a cork on water surface and dropping a stone nearby. You will see how, after some initial time, the cork will feel the disturbance, but it will not stop and continue to oscillate later.

Returning back to the quotation for this lecture, in even dimensions there is no possibility to talk since the sound ways have no sharp trailing edge. In dimension five, however, the conversations can be made the same way how we talk in our familiar three dimensions.