Two dimensional discrete dynamical systems

Let us now move one dimension up and consider two dimensional discrete dynamical systems of the form \( f : U \to U \) and \( U \subseteq \mathbb{R}^2 \). In coordinates we have

\[
\begin{align*}
  x_1(t + 1) &= f_1(x_1(t), x_2(t)), \\
  x_2(t + 1) &= f_2(x_1(t), x_2(t)),
\end{align*}
\]

where (and below) \( t \in \mathbb{Z} \) or \( t \in \mathbb{Z}_+ \). It is quite clear, given the complexity of one dimensional maps, that two (or more) dimensional maps can produce a wealth of different dynamical regimes. Therefore for the beginning I will restrict myself to linear systems

\[
\begin{align*}
  x_1(t + 1) &= a_{11}x_1(t) + a_{12}x_2(t), \\
  x_2(t + 1) &= a_{21}x_1(t) + a_{22}x_2(t),
\end{align*}
\]

or, in matrix notations

\[
  x(t + 1) = Ax(t),
\]

or

\[
  x \mapsto Ax, \quad x \in U \subseteq \mathbb{R}^2.
\]

Obviously, the fixed points are solutions to

\[
  Ax = x,
\]

or

\[
  (A - I)x = 0,
\]

which, assuming that 1 is not an eigenvalue of \( A \), which I assume in the following, has only trivial solution \( \hat{x} = (0, 0) \).

How to solve (1)? Very easy, by assuming that we are given the initial condition \( x(0) = x_0 \), then

\[
  x(t) = A^t x_0,
\]

and the only problem to find a simple way to compute \( A^t = AA \ldots A \). Recall from our discussion of the \( 2 \times 2 \) dimensional matrices, that any such matrix is similar to one of three possible Jordan normal forms

\[
  J_1 = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \quad J_2 = \begin{bmatrix} \mu & 1 \\ 0 & \mu \end{bmatrix}, \quad J_3 = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.
\]

This means that there exists an invertible matrix \( P \) such that

\[
  J_i = P^{-1} AP, \quad i = 1, \text{ or } 2, \text{ or } 3,
\]

and we also know how to find \( P \). Now we only need

**Proposition 1.** Let \( A \) has Jordan normal form \( J \) with the given transformation matrix \( P \). Then

\[
  A^t = PJ^tP^{-1}.
\]
Proof of this proposition is left as an exercise. Now we just need to find $t$-th powers of Jordan normal forms.

For $J_1$ it is immediate that

$$J_1^t = \begin{bmatrix} \mu_1^t & 0 \\ 0 & \mu_2^t \end{bmatrix},$$

and recall that $\mu_1$ and $\mu_2$ are real eigenvalues of $A$ (recall that they also called the multipliers of $A$).

For $J_2$ a simple induction argument shows that

$$J_2^t = \mu^{t-1} \begin{bmatrix} \mu & t \\ 0 & \mu \end{bmatrix},$$

and here $\mu$ is the only eigenvalue of $A$ multiplicity 2 and $A$ is different from the scalar matrix $\mu I$.

For $J_3$, where $A$ has two complex eigenvalues $\mu_{1,2} = \alpha \pm i\beta$, note that

$$\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \sqrt{\alpha^2 + \beta^2} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix},$$

for some $0 \leq \theta < 2\pi$. From this representation we have that multiplication by $J_3$ rotates a point around the origin by the angle $\theta$ and multiply the distance by $\sqrt{\alpha^2 + \beta^2}$, therefore

$$J_3^t = \rho^t \begin{bmatrix} \cos t\theta & \sin t\theta \\ -\sin t\theta & \cos t\theta \end{bmatrix},$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$.

**Example 2** (Fibonacci numbers). As an example, consider the Fibonacci numbers, defined as

$$n_{t+1} = n_t + n_{t-1},$$

with the initial conditions $n_0 = 0, n_1 = 1$. Reformulate this problem as a two-dimensional discrete system (I use $x_1(t) = n_t$ and $x_2(t) = n_{t-1}$)

$$x_1(t+1) = x_1(t) + x_2(t),$$

$$x_2(t+1) = x_1(t),$$

with the initial condition $x_1(1) = 1, x_2(1) = 0$. In this example we have

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},$$

and the general solution is given by

$$x(t) = A^{t-1} x(1).$$

Eigenvalues of $A$ are $\mu_{1,2} = \frac{1 \pm \sqrt{5}}{2}$ and the corresponding eigenvectors are $((1 + \sqrt{5})/2, 1)^\top$ and $((1 - \sqrt{5})/2, 1)^\top$, therefore, we have

$$P = \begin{bmatrix} \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{\sqrt{5}-1}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}, \quad J = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix},$$

2
which implies
\[ x(t) = P J^{t-1} P^{-1} x(1), \]
and after some algebra yields
\[ x_1(t) = n_t = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^t - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^t \]
for the \( t \)-th Fibonacci number. Since \( |1 - \sqrt{5}|/2 \) is less than 1, therefore, for sufficiently large \( t \)
\[ n_t \approx \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^t, \]
and
\[ \frac{n_{t+1}}{n_t} \approx \frac{1 + \sqrt{5}}{2}, \]
i.e., the ratio of two consequent Fibonacci numbers is approximately equal to the golden ratio (recall that two positive quantities are in golden ratio if the ratio of their sum to the bigger quantity is equal to the ratio of the bigger quantity to the smaller one, deduce from here that \( \varphi = (1 + \sqrt{5})/2 \)).

Now using the explicit representation of \( A^t \), we can state the following important stability result:

**Theorem 3.** Consider (1) and assume that \( \hat{x} = (0, 0) \) is isolated. Then \( \hat{x} \) is asymptotically stable (a sink) if and only if all the eigenvalues of \( A \) satisfy the condition \( |\mu_{1,2}| < 1 \). Fixed point \( \hat{x} \) is unstable if there is at least one eigenvalue of \( A \) such that \( |\mu_i| > 1 \). It is a source if \( |\mu_{1,2}| > 1 \), and it is a saddle if \( |\mu_1| < 1 \) and \( |\mu_2| > 2 \). In the latter case there exist stable and unstable subspaces of \( \mathbb{R}^2 \). Finally \( \hat{x} \) is Lyapunov stable if \( A \) has eigenvalues \( |\mu| = 1 \).

For the following it is natural to call matrix \( A \) hyperbolic if it does not have eigenvalues with the absolute value equal to 1. Hyperbolicity is a generic property in the sense that almost all the matrices are hyperbolic.

Similarly to the case with the differential equations, the linearization of a nonlinear system around a fixed point can be used to infer the stability properties, given that the Jacobi matrix \( Df(\hat{x}) \) is hyperbolic. Formally,

**Theorem 4.** Consider discrete dynamical system \( x \rightarrow f(x), x \in U \subseteq \mathbb{R}^d \), where \( f \in C^1(U; U) \), and let \( \hat{x} \) be a fixed point: \( f(\hat{x}) = \hat{x} \). Then \( \hat{x} \) is asymptotically stable if all the multipliers of the Jacobi matrix \( Df(\hat{x}) \) lie within the unit circle on the complex plane. If at least one multiplier lies outside the unit circle then \( \hat{x} \) is unstable.

**Example 5** (A delayed Ricker’s equation). Consider a difference equation
\[ N_{t+1} = N_t e^{r\left(1 - \frac{N_t}{K}\right)}, \]
which, using new variable \( u_t = N_t/K \) can be reduced to
\[ u_{t+1} = u_t e^{r(1-u_{t-1})}, \quad r > 0. \]
This equation is not a discrete dynamical system since \( u_{t+1} \) depends on two time moments: on the present \( t \) and on the past \( t-1 \), which is quite often reasonable to assume. However, using new notations

\[
x_1(t) = u_t, \quad x_2(t) = u_{t-1},
\]

this equation can be rewritten as

\[
\begin{align*}
x_1(t + 1) &= x_1(t)e^{r(1-x_2(t))}, \\
x_2(t + 1) &= x_1(t).
\end{align*}
\]

The fixed points are found as the solutions to

\[
x_1e^{r(1-x_2)} = x_1, \quad x_1 = x_2,
\]

and therefore we have \( \hat{x}_0 = (0,0) \) and \( \hat{x}_1 = (1,1) \). The Jacobi matrix of the map is

\[
Df = \begin{bmatrix} e^{r(1-x_2)} & -re^{r(1-x_2)} \\
1 & 0 \end{bmatrix}.
\]

Therefore, \( Df(\hat{x}_0) \) has two multiplier \( e^r \) and 0 and hence always unstable and hyperbolic saddle. The multipliers of \( \hat{x}_1 \) are found by solving

\[
\mu^2 - \mu + r = 0,
\]

and therefore

\[
\mu_{1,2} = \frac{1 \pm \sqrt{1 - 4r}}{2} \quad r \leq \frac{1}{4},
\]

and

\[
\mu_{1,2} = \rho e^{\pm i\theta}, \quad \rho = \sqrt{r} \quad \theta = \arctan \frac{1}{\sqrt{4r-1}} \quad r > \frac{1}{4}.
\]

If \( 0 < r \leq \frac{1}{4} \) then the multipliers are real and less then 1, hence \( \hat{x}_1 \) is asymptotically stable. If \( r > \frac{1}{4} \) then the multipliers are complex conjugate and \( \mu_1\mu_2 = |\mu_1|^2 = r \), therefore, for \( r < 1 \) this fixed point is still asymptotically stable (but convergence is oscillatory). At \( r = 1 \)

\[
\mu_{1,2} = e^{\frac{i\pi}{4}},
\]

and both multipliers have absolute values 1 simultaneously, the linearization at \( \hat{x}_1 \) is non-hyperbolic, a bifurcation occurs in full nonlinear system. It can be proved that in this particular case an invariant closed curve is born around the fixed point, which is attracting. The full theory is somewhat similar to the Poincaré–Andronov–Hopf bifurcation and can be found in the cited textbooks. The name of the bifurcation if the Neimark–Saker bifurcation. This situation is generic when there are two complex multipliers, one is conjugate of another, both having absolute value 1.