Introduction to the Theory of Ordinary Differential Equations. A Graduate Course

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Foreword

These lecture notes are written for the introductory graduate course on ordinary differential equation, taught initially in the Fall 2014 at North Dakota State University. They cover the classical theory in both the initial and boundary value problems. The main goal is not to be encyclopedic, but present the core of the theory with (sometimes excessive) all excruciating details. The problems are a significant part of the notes and must be worked out to make sure that the material is digested well.

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Chapter 1

Introduction

1.1 Second Newton's law, ordinary differential equations, and the three-body problem

By the *second Newton's law* the product of the mass[1] of a body and its acceleration is equal to the net force applied to this body:

$$m\boldsymbol{a} = \boldsymbol{F},$$

were m is a scalar, mass of the body, $\boldsymbol{a}(t) \in \mathbf{R}^3$ is a vector with three components at each time moment t, $\boldsymbol{a}\colon \mathbf{R} \longrightarrow \mathbf{R}^3$, and $\boldsymbol{F}(t, \boldsymbol{x}, \boldsymbol{v}) \in \mathbf{R}^3$ is the net force, which may depend in general on the time t, current displacement $\boldsymbol{x}(t) \in \mathbf{R}^3$, or current velocity $\boldsymbol{v}(t) \in \mathbf{R}^3$ of the body. Assume for simplicity that the body under question has only one *degree of freedom*, which means that one coordinate is enough to specify the body's position. In this case the second Newton's law looks simpler:

$$ma = F,$$

where now $a(t)$ and $F(t, x(t), v(t))$ are scalars at the fixed moment t. Since acceleration a is the rate of change of velocity v, and velocity is the rate of change of displacement x, therefore

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}\,, \quad v = \frac{\mathrm{d}x}{\mathrm{d}t}\,,$$

and hence the physical law takes the form

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = F,$$

or

$$m\ddot{x} = F, \tag{1.1}$$

using the notation due to Isaac Newton himself:

$$\ddot{x} := \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\,, \quad \dot{x} := \frac{\mathrm{d}x}{\mathrm{d}t}\,.$$

[1] I assume that the student saw the second Newton's law before and has at least an intuitive understanding of the terms "mass," "body," "force," even if the rigorous definitions cannot be spelled out. If somehow you did not see the second Newton's law before, omit this motivational section and go straight to the next one, with axiomatic-style definitions.

Equation (1.1) is a basic example of an *ordinary differential equation*, in which our goal is to find the function $t \mapsto x(t)$ that has two continuous derivatives and satisfies the equation.

Example 1.1 (No force)**.** Assume that $F = 0$, then $\ddot{x} = 0$, and I can directly integrate twice with respect to t to find the solution

$$x(t) = C_1 t + C_2,$$

where C_1 and C_2 are arbitrary (real, if we stick to the real-valued solutions) constants. To determine these constants I need the initial conditions that can be given, e.g., as $x(0) = x_0$ (initial position) and $\dot{x}(0) = v_0$ (initial velocity). This implies that my solution is the function

$$x(t) = x_0 + v_0 t,$$

which is actually a mathematical manifestation of the *first Newton's law*, that states that "an object at rest stays at rest and an object in motion stays in motion with the same speed and in the same direction unless acted upon by an unbalanced force."

Example 1.2 (Constant force)**.** Now let $F = \text{const}$, then I get (here the minus sign is chosen by tradition)

$$\ddot{x} = -g, \quad g = \text{const} \in \mathbf{R}.$$

I again can simply integrate twice to find, using the same initial conditions as in the previous example, that

$$x(t) = x_0 + v_0 t - \frac{gt^2}{2}$$

is my solution. For example, the faith of a stone thrown vertically down ($v_0 < 0$) or up ($v_0 > 0$) from some height x_0 above the Earth's ground is governed, approximately, by this solution.

Example 1.3 (Linear force)**.** Now let $F \propto x$, where $\propto$ means "proportional." For example, *Hooke's law* states that "the force needed to extend or compress a spring by some distance is proportional to that distance." Hence I have

$$m\ddot{x} = -kx, \quad k > 0,$$

where k is called the spring constant. I can rewrite this differential equation as

$$\ddot{x} + \omega^2 x = 0, \quad \omega^2 := \frac{k}{m}\,.$$

This ordinary differential equation (ODE from now on) cannot be solved by simple consecutive integrations, but one can check that both $t \mapsto \sin \omega t$ and $t \mapsto \cos \omega t$ solve this equation, as well as any their linear combination

$$x(t) = C_1 \sin \omega t + C_2 \cos \omega t.$$

Actually the last solution, as I will prove in this course, is the *general solution* to $\ddot{x} + \omega^2 x = 0$, i.e., *any* solution to this equation can be expressed with this formula for some specific values of C_1 and C_2.

Can any ODE be solved analytically? Not really.

Example 1.4 (Three-body problem)**.** According to *Newton's law of gravity*, two bodies of masses m_1 and m_2 respectively are attracted to each other with the force proportional to the product of their masses and inversely proportional to the square of the distance between them. Hence, if I have three bodies, and the only forces that are applied to them are gravitational then the second Newton's law implies for the first body that

$$m\ddot{\boldsymbol{x}}_1 = G\frac{m_1m_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^2}\cdot\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} + G\frac{m_1m_3}{|\boldsymbol{x}_1-\boldsymbol{x}_3|^2}\cdot\frac{\boldsymbol{x}_3-\boldsymbol{x}_1}{|\boldsymbol{x}_1-\boldsymbol{x}_3|}\,,$$

where G is the constant of proportionality in Newton's law of gravity, $|\boldsymbol{x}| := \sqrt{x_1^2+x_2^2+x_3^2}$ for the vector[2] $\boldsymbol{x}^\top = (x_1, x_2, x_3) \in \mathbf{R}^3$, such that $|\boldsymbol{x}-\boldsymbol{y}|$ is the usual Euclidian distance between points $\boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^3$, and

$$\frac{\boldsymbol{x}_2-\boldsymbol{x}_1}{|\boldsymbol{x}_1-\boldsymbol{x}_2|}$$

is the unit vector in the direction from the first body to the second one. I will also need two more equations for the second and third bodies:

$$m\ddot{\boldsymbol{x}}_2 = G\frac{m_1m_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^2}\cdot\frac{\boldsymbol{x}_1-\boldsymbol{x}_2}{|\boldsymbol{x}_1-\boldsymbol{x}_2|} + G\frac{m_2m_3}{|\boldsymbol{x}_2-\boldsymbol{x}_3|^2}\cdot\frac{\boldsymbol{x}_3-\boldsymbol{x}_2}{|\boldsymbol{x}_2-\boldsymbol{x}_3|}\,,$$
$$m\ddot{\boldsymbol{x}}_3 = G\frac{m_2m_3}{|\boldsymbol{x}_2-\boldsymbol{x}_3|^2}\cdot\frac{\boldsymbol{x}_2-\boldsymbol{x}_3}{|\boldsymbol{x}_2-\boldsymbol{x}_3|} + G\frac{m_1m_3}{|\boldsymbol{x}_1-\boldsymbol{x}_3|^2}\cdot\frac{\boldsymbol{x}_1-\boldsymbol{x}_3}{|\boldsymbol{x}_1-\boldsymbol{x}_3|}\,.$$

Since I deal with the three dimensional Euclidean space, I have total of 9 equations, and I need 18 initial conditions. It turns out that we are still lacking a full understanding of the behavior of solutions to this system of ODE. Here is a short digest of the history of this problem.

1687 Isaac Newton in his *Principia* (Philosophiæ Naturalis Principia Mathematica) was the first one to formulate the problem of determining the positions of three massive bodies (the Sun, the Earth, and the Moon).

1747 Jean d'Alembert and Alexis Clairaut published first analytical analysis of the three-body problem.

1760 Leonhard Euler considered a special case of the general three-body problem, which can be exactly solved, and found some periodic solutions. Later his problem was also analyzed by many others, including Joseph-Louis Lagrange.

1887 Ernst Bruns proved that there existed no general analytical solution given by algebraic expressions and integrals.

1890 Henri Poincaré won a contest on the best solution to the three-body problem. He did not solve, of course, this problem, but formulated the beginning of the theory that eventually led to what is often called the *chaos theory* nowadays.

1912 A power series solution was constructed, which means that, in a sense, we do have an analytical solution to the three-body problem in the form of an infinite series. The usefulness of this solution, however, is doubtful at best.

[2] Note the distinction between vector $\boldsymbol{x}_1$ with three coordinate, say (x_1^1, x_2^1, x_3^1), and the coordinate x_1 of the vector $\boldsymbol{x}^\top = (x_1, x_2, x_3)$. Also note that all the vectors, if not stated otherwise, are *column-vectors*, and hence I use the transposition notation $\boldsymbol{x}^\top$ to write this vector as a *row-vector*.

1970 A new family of exact periodic solutions was found.

2013 By means of elaborate numerical calculations 18 new families of specific solutions were found[3].

To conclude, the history of attempts to analyze the three body problem is definitely not over yet (try to google the latest achievements).

Exercise 1.1. The exact dynamics of a stone thrown vertically on the Earth, if I disregard the air resistance, is described by *Newton's gravitational law*

$$m\ddot{x} = -G\frac{mM}{|x+R|^2}\,,$$

where m is the mass of the stone, M is the mass of the Earth, and R is the radius of the Earth. If I assume that $x + R \approx R$, then I get an approximate equation

$$\ddot{y} = -g, \quad g := G\frac{M}{R^2}\,.$$

Assuming that $x(0) = y(0) = h > 0$, $x'(0) = y'(0) = 0$, then for which equation the time to reach the surface of the Earth is smaller? Please note that you do not need to actually solve the equations.

1.2 Basic definitions and geometric interpretation of solutions

1.2.1 Definitions

Definition 1.5. *An ordinary differential equation is an equation of the form*

$$F(t, x, x', \ldots, x^{(k)}) = 0, \tag{1.2}$$

for a given $F\colon U \subseteq \mathbf{R}^{k+2} \longrightarrow \mathbf{R}$. *A solution to* (1.2) *on the interval* $I = (a, b)$ *is a function*[4] $t \mapsto \phi(t) \in \mathcal{C}^{(k)}(I;\mathbf{R})$ *such that* (1.2) *turns into the identity*

$$F\bigl(t, \phi(t), \phi'(t), \ldots, \phi^{(k)}(t)\bigr) = 0 \quad \text{for all } t \in I.$$

The order of (1.2) *is the order of the highest derivative in it.*

In this course I will always assume that (1.2) can be rewritten as

$$x^{(k)} = f(t, x, x', \ldots, x^{(k-1)}), \tag{1.3}$$

which frequently can be done, at least locally, thanks to the *implicit function theorem.*

Any equation of the form (1.3) can be written as an equivalent system of k first order ODE. To wit, let

$$y_1 = x, \quad y_2 = x', \quad \ldots, \quad y_k = x^{(k-1)},$$

[3]See Milovan Šuvakov, V. Dmitrašinović, Three Classes of Newtonian Three-Body Planar Periodic Orbits, http://arxiv.org/abs/1303.0181.

[4]I use the standard notation $\mathcal{C}^{(p)}(X;Y)$ for the set of p times continuously differentiable functions mapping X into Y, and also several abbreviations: $\mathcal{C}(X;Y) := \mathcal{C}^{(0)}(X;Y)$ for the set of continuous functions, $\mathcal{C}^{\infty}(X;Y) := \bigcap_{p=1}^{\infty}\mathcal{C}^{(p)}(X;Y)$, $\mathcal{C}^{(p)}(X) := \mathcal{C}^{(p)}(X;\mathbf{R})$, $\mathcal{C}^{(p)}(a,b) := \mathcal{C}^{(p)}((a,b);\mathbf{R})$, and $\mathcal{C}^{(p)}[a,b] := \mathcal{C}^{(p)}([a,b];\mathbf{R})$.

then I have

$$\begin{aligned}\dot y_1 &= y_2,\\ \dot y_2 &= y_3,\\ &\dots\\ \dot y_k &= f(t, y_1, \dots, y_k).\end{aligned}$$

Therefore, it is convenient (for theoretical, as well as for numerical purposes) to concentrate attention on the systems of ODE of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f}\colon J \times X \longrightarrow \mathbf{R}^k, \tag{1.4}$$

where $J \subseteq \mathbf{R}$ in an interval of the real line, on which the right hand side of (1.4) is defined with respect to t. If the right hand side of (1.4) does not depend on t explicitly, then the system is called *autonomous*. Any non-autonomous system can be written as an autonomous one by adding one more equation and one more variable. For example, a scalar non-autonomous equation $\dot x = f(t, x)$ can be turned into a system of two first order autonomous equations

$$\begin{aligned}\dot x_1 &= f(x_2, x_1),\\ \dot x_2 &= 1.\end{aligned}$$

Note that this procedure can yield *nonlinear* problems although the original one was *linear*[5].

Example 1.6. Let[6]

$$\dot x = f(t), \quad f \in \mathcal{C}(J).$$

By the fundamental theorem of calculus, I have

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau)\,\mathrm{d}\tau,$$

if the solution satisfies the initial condition $x(t_0) = x_0$. Sometimes it is more convenient to use the indefinite integral, and the general solution will be

$$x(t) = \int f(t)\,\mathrm{d}t + C,$$

where C is an arbitrary constant.

Example 1.7. Consider the following ODE

$$\dot x = ax, \quad a \in \mathbf{R}.$$

I claim that any solution to this equation has the form $t \mapsto C\exp(at)$. Here $\exp\colon t \mapsto e^t$.

Lemma 1.8. *Any solution to $\dot x = ax$ is given by $t \mapsto Ce^{at}$.*

[5]The exact definition of what is called linear will be given later in the course.

[6]When writing ODE, I will follow the historical trend and abuse the notation by typing $f(t)$ to mean function $f\colon J \longrightarrow \mathbf{R}$. The correct way to write this would be f or $t \mapsto f(t)$, but I will stick to the tradition.

Proof. Let u be a solution to $\dot{x} = ax$. Consider

$$(u(t)e^{-at})' = u'(t)e^{-at} - au(t)e^{-at} = au(t)e^{-at} - au(t)e^{-at} = 0,$$

hence

$$u(t)e^{-at} = C \iff u(t) = Ce^{at}.$$

■

In case I have the initial condition $x(t_0) = x_0$, the solution is $t \mapsto x_0 e^{a(t-t_0)}$.

Exercise 1.2. Let

$$x^{(k)} = f(x, x', \ldots, x^{(k-1)})$$

be an autonomous equation. Show that if $t \mapsto \phi(t)$ is a solution, then so is $t \mapsto \phi(t - t_0)$, where t_0 is an arbitrary constant.

Exercise 1.2 shows that for the autonomous equations (and hence for the autonomous systems) I can always take the initial conditions at $t = 0$, other solutions can be obtained by translations.

Exercise 1.3. What is the general solution to the ODE

$$\dot{x} = 0?$$

Is is necessarily a constant function? What else one needs to require for the solution to be a constant function in addition to the ODE?

Exercise 1.4. Does there exist a solution to the problem

$$\dot{x} = \begin{cases} 1, & x < 0, \\ -1, & x \geq 0, \end{cases}$$

with the initial condition $x(0) = 0$? Justify your answer. (If this problem is a little confusing, reading the next section may help.)

Exercise 1.5. In all the examples I gave so far the general solution to an ODE depended on arbitrary constants whose number coincided with the order of the equation. This is true for most examples but not for all. Can you provide an example of an ODE that has *no* solution at all?

1.2.2 Geometric interpretation of the first order scalar ODE

It is convenient to have at hands a geometric interpretation of the solutions to the first order scalar ODE of the form

$$\dot{x} = f(t, x), \quad x(t) \in X \subseteq \mathbf{R}, \quad f \colon J \times X \longrightarrow \mathbf{R}. \tag{1.5}$$

Consider in $J \times X$ the *direction field* defined by $(t, x) \mapsto f(t, x)$; this literally means that at each point $(t, x) \in J \times X$ the slope (direction) is given by $f(t, x)$. A curve $t \mapsto \phi(t)$ is said to belong to the given direction field, if its slope at the point $(t, \phi(t))$ is given exactly by $f(t, \phi(t))$. A curve belonging to a direction field is called an *integral curve* (see Fig. 1.1).

Lemma 1.9. *Consider a differentiable function $t \mapsto \phi(t)$ defined on some interval $I \subseteq J$. Its graph is an integral curve of the direction field defined by $(t, x) \mapsto f(t, x)$ in $J \times X$ if and only if ϕ solves* (1.5).

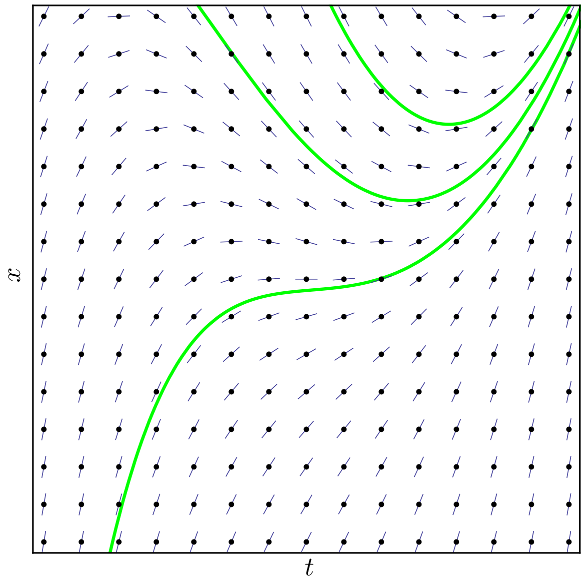

Figure 1.1: The direction field defined by $f(t,x) = t^2 - x^2$ and three integral curves.

Proof. Let $t \mapsto \phi(t)$ be an integral curve. Since the slope at the point t is given by $\phi'(t)$ and since it is an integral curve, the same slope is given by $f(t,\phi(t))$, which means that $\phi'(t) = f(t,\phi(t))$ for all $t \in I$, which implies that $t \mapsto \phi(t)$ is a solution to (1.5).

In the opposite direction. Let $t \mapsto \phi(t)$ solve (1.5) on $t \in I$, therefore $\phi'(t) = f(t,\phi(t))$, i.e., the slope at the point $t \in I$ is equal to $f(t,\phi(t))$, which yields that the graph of $t \mapsto \phi(t)$ belongs to the direction field defined by $f(t,x)$ in $J \times X$, i.e., its graph is an integral curve. ■

Using Lemma 1.9 I can formulate the geometric meaning of the first order ODE: To geometrically solve (1.5) amounts to finding integral curves belonging to the direction field defined by f. There are infinitely many such curves. To geometrically solve the initial value problem (IVP from now on) with $x(t_0) = x_0$ amounts to finding an integral curve belonging to the direction field and passing through the point (t_0, x_0). In the following I will use the terms "solutions to ODE" and "integral curves" interchangeably.

Exercise 1.6. Sketch several integral curves of the differential equations

$$(a) \quad \dot{x} = \frac{tx}{|tx|}\,, \quad (b) \quad \dot{x} = \frac{|t+x|}{t+x}\,. \quad (c) \quad \dot{x} = -\frac{t+|t|}{x+|x|}\,, \quad (d) \quad \dot{x} = \begin{cases} 0, & t \neq x, \\ 1, & t = x. \end{cases}$$

Exercise 1.7. Sketch several integral curves of the differential equation

$$\dot{x} = \frac{x-t}{x+t}\,.$$

Can two integral curves intersect? Obviously, no (for reasonable, say, continuous, f), because this would mean that at the same point I have two different slopes. However, two integral curves can be tangent to each other at some point (t_0, x_0). In general, from applied point of view, this situation

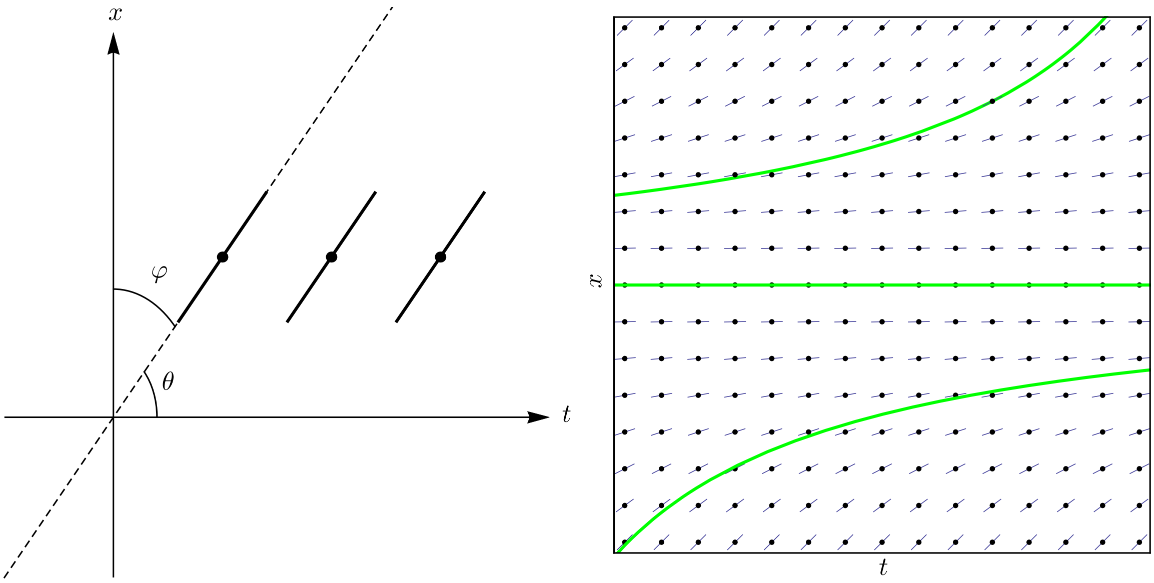

Figure 1.2: The direction field invariant with respect to translations along the t-axis. Left: Relation between θ and φ for an autonomous direction field. Right: An example of the direction field defined by $(t, x) \mapsto x^2$ with three integral curves.

is not pleasant, since it means that more than one solution is defined by the initial conditions and the mathematical model. Therefore, it is important to identify the conditions when only one integral curve passes through each point.

Note that the direction field defined by f in Example 1.6 does not depend on x and therefore invariant with respect to translations along the x-axis. This fact is exactly what has allowed me to integrate my differential equation in Example 1.6: at each point t I have the slope $f(t)$ and my task is to find a function with this slope at t.

What if the direction field is invariant with respect to translations along the t-axis? This means that this direction field does not depend on t explicitly (it is *autonomous*). Now, since the slope at the point (t, x) is equal to the tangent of the angle θ between the t-axis and the tangent line at this point ($\tan\theta = f(x)$), to find the tangent of the angle between the same tangent line and the x-axis amounts to finding $\tan\varphi = \tan(\pi/2 - \theta)$ (see Fig. 1.2, left panel). I have

$$\tan(\pi/2 - \theta) = \frac{1}{\cot(\pi/2 - \theta)} = \frac{1}{\tan\theta} = \frac{1}{f(x)}\,.$$

Therefore, to integrate an autonomous direction field (i.e., invariant with respect to translations along the t-axis), I need to evaluate one integral (compare it with Example 1.6)

$$t = t_0 + \int_{x_0}^{x} \frac{\mathrm{d}s}{f(s)}\,, \tag{1.6}$$

which gives the solution to $\dot{x} = f(x)$ in an implicit form.

The reasonings above can be summarized as

Lemma 1.10. *Let $\dot{x} = f(x)$, $x(t) \in X \subseteq \mathbf{R}$, $f \in \mathcal{C}(X)$, $f(x) \neq 0$ for $x \in X$. Then the solution to $\dot{x} = f(x)$, $x(t_0) = x_0$ exists and unique for any $x_0 \in X$ and given by* (1.6).

Remark 1.11. The usual way to derive (1.6) in introductory ODE courses is to write

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x),$$

manipulate

$$\frac{\mathrm{d}x}{f(x)} = \mathrm{d}t,$$

and integrate

$$\int_{x_0}^{x} \frac{\mathrm{d}s}{f(s)} = \int_{t_0}^{t} \mathrm{d}\tau,$$

to recover (1.6). This procedure treating the derivative as a fraction does not lead to any errors, but may be confusing for those who try to understand the logic behind every step in the reasonings, because definitely the usual derivative is *not* a fraction.

A better, and completely rigorous, approach to integrating the equation $\dot{x} = f(x)$ mechanically is to rearrange it as $\frac{\dot{x}}{f(x)} = 1$, assuming $f(x) \neq 0$, and to integrate both sides of this last equality with respect to variable t:

$$\int_{t_0}^{t} \frac{\dot{x}(\tau)}{f(x(\tau))} \mathrm{d}\tau = \int_{t_0}^{t} \mathrm{d}\tau,$$

and now make a substitution $s = x(\tau)$, which would end up in (1.6)

$$\int_{x_0}^{x} \frac{\mathrm{d}s}{f(s)} = t - t_0.$$

Remark 1.12. Treating expressions like $\frac{\mathrm{d}x}{\mathrm{d}t}$ as fractions quite often yields correct results. Think, for instance, about the chain rule for the composite function $t \mapsto f(u(t))$:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}t}.$$

It should be stressed out, however, that unless you clearly have an understanding what $\mathrm{d}x$ and $\mathrm{d}t$ mean, and what it means to divide one by the other one, you should avoid "intuitively plausible" arithmetic with these quantities. Here is, for instance, one example when simple canceling of identical terms leads to a senseless result: Consider an implicitly defined function $F(x, y) = 0$. Then, as it is shown in many calculus books, assuming that $\frac{\partial F}{\partial y} \neq 0$ at a given point,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}.$$

If you still insist on treating $\mathrm{d}x$ and $\mathrm{d}t$ separately, then probably the most frequent interpretation for them that allows such treatment is to assume that $\mathrm{d}x$ and $\mathrm{d}t$ are *differential forms*, but this would take us too far away from the main line of our course.

Using similar to the above reasoning I find that

Lemma 1.13. *Consider the ODE*

$$\dot{x} = f_1(t)f_2(x), \quad t \in J,\, x(t) \in X, f_1 \in \mathcal{C}(J),\, f_2 \in \mathcal{C}(X). \tag{1.7}$$

Assume that $f_2(x) \neq 0$ for any $x \in X$. Then for any point $(t_0, x_0) \in J \times X$ there exists a unique integral curve passing through this point. This curve can be found with two integrals

$$\int_{x_0}^{x} \frac{\mathrm{d}s}{f_2(s)} = \int_{t_0}^{t} f_1(\tau)\,\mathrm{d}\tau.$$

Exercise 1.8. Prove Lemma 1.13.

Exercise 1.9. Show that all the solutions to $\dot{x} = f(x)$ are monotonous functions.

What if there is an $\hat{x} \in X$ such that $f(\hat{x}) = 0$ in $\dot{x} = f(x)$ or $f_2(\hat{x}) = 0$ in (1.7)? Note that in this case the function $t \mapsto \hat{x}$ is a solution (it satisfies the equation). Then I am tempted to generalize, e.g., Lemma 1.10 in the following (incorrect!) way

Lemma 1.14 (This statement is incorrect!)**.** *Let $\dot{x} = f(x)$, $x(t) \in X \subseteq \mathbf{R}$, $f \in \mathcal{C}(X)$. Then for any point $(t_0, x_0) \in \mathbf{R} \times X$ the solution to $\dot{x} = f(x)$ passing through this point exists and unique. It is given by* (1.6) *if $f(x_0) \neq 0$ or by $t \mapsto x_0$ if $f(x_0) = 0$.*

Example 1.15 (Non-uniqueness of solutions to an IVP)**.** Consider the following IVP

$$\dot{x} = x^{1/3}, \quad x(0) = 0.$$

First I notice that $t \mapsto 0$ is a solution. However, using (1.6) I find that

$$t \mapsto \left(\frac{2t}{3}\right)^{3/2}$$

is also a solution passing through the point $(0, 0)$! Moreover, the function defined by

$$t \mapsto \begin{cases} 0, & t \leq a, \\ \dfrac{2(t-a)^{3/2}}{3}, & t > a, \end{cases}$$

is also a solution passing through $(0, 0)$ for any constant $a > 0$ (check this statement carefully).

By actually carefully analyzing the reason why the solution in the last example is not unique (do the integration step by step!), I conclude that a correct statement should be

Lemma 1.16. *Let $\dot{x} = f(x)$, $x(t) \in X \subseteq \mathbf{R}$, $f \in \mathcal{C}(X)$. Then for any point $(t_0, x_0) \in \mathbf{R} \times X$ the solution, given by* (1.6)*, passing through this point, exists and unique if $f(x_0) \neq 0$. If $f(x_0) = 0$ and the improper integral*

$$\int_{x_0}^{x} \frac{\mathrm{d}s}{f(s)}$$

diverges, then the unique solution for $x(0) = x_0$ is given by $t \mapsto x_0$.

Exercise 1.10. Prove Lemma 1.16.

Note that a sufficient condition for this integral to diverge is to have a continuous derivative at $x = x_0$. This can be stated as

Lemma 1.17. *Consider* (1.7) *and assume that* $f_1 \in \mathcal{C}(J)$ *and* $f_2 \in \mathcal{C}^{(1)}(X)$. *Then for any point* $(t_0, x_0) \in J \times X$ *there exists a unique integral curve passing through this point.*

It turns out that a very similar statement can be formulated in much more general situations (see the next chapter).

Before finishing this section, I would like to remark that I never discussed the interval of existence of my solutions. This is an important point, as the following example shows.

Example 1.18. Consider

$$\dot{x} = x^2.$$

This equation satisfies the conditions of the last lemma everywhere in $\mathbf{R}$ with respect to both t and x, and therefore for any initial conditions (t_0, x_0) there must be a unique integral curve through this point. It is $t \mapsto 0$ if $x_0 = 0$ and

$$t \mapsto \frac{1}{C - t},$$

for an arbitrary constant C. If $t \to C$ then my solution approaches infinity (it *blows up*). Therefore, the solution is defined on some smaller interval $(-\infty, C) \subset \mathbf{R}$ (se Fig. 1.2, the right panel, for the direction field defined by $(t, x) \mapsto x^2$).

1.3 Analytical methods to solve ODE

1.3.1 Separable equations

Definition 1.19. *A first order ordinary differential equation of the form*

$$\dot{x} = f_1(t) f_2(x), \quad t \in J \subseteq \mathbf{R}, \quad x(t) \in X \subseteq \mathbf{R}$$

is called separable.

From the previous section the solution to the separable equation in given by

$$\int \frac{\mathrm{d}x}{f_2(x)} = \int f_1(t)\,\mathrm{d}t,$$

if $f_2(x) \neq 0$. If $\hat{x}$ is such that $f_2(\hat{x}) = 0$ then there is also solution $t \mapsto \hat{x}$. The existence and uniqueness of solutions is determined by the properties of f_1 and f_2 (Lemma 1.16).

Exercise 1.11. Solve the following differential equations:

$$(a) \quad \dot{x} = x^3, \quad (b) \quad \dot{x} = x(1 - x).$$

Exercise 1.12. Investigate uniqueness of the solutions to the differential equation

$$\dot{x} = \begin{cases} -t\sqrt{|x|}, & x \geq 0, \\ t\sqrt{|x|}, & x \leq 0. \end{cases}$$

Show that the initial value problem $x(0) = x_0$ has a unique *global solution* (the solution is called *global* if it is defined for all $t \in \mathbf{R}$) for any $x_0 \in \mathbf{R}$. However, show that the global solutions still intersect! (*Hint:* Note that if $t \mapsto x(t)$ is a solution so are $t \mapsto -x(t)$ and $t \mapsto x(-t)$, so it suffices to consider $x_0 \geq 0$ and $t \geq 0$).

Exercise 1.13. Consider the free fall with air resistance modeled by

$$\ddot{x} = \eta \dot{x}^2 - g, \quad \eta > 0.$$

Solve this equation. (*Hint:* introduce the velocity $v = \dot{x}$ as a new independent variable.) Is there a limit to the speed the object can attain? If yes, find it.

Exercise 1.14. (Before solving this problem, you should solve the previous one.) A parachutist jumps from 1.5 km and opens his parachute at 0.5 km. How long was he falling before opening his parachute? Take into account that the maximal speed of a falling human body in the air of the normal density is 50 m/sec (meters per second).

Exercise 1.15. A body cooled down from 100° to 60° for 10 minutes. The temperature of the surrounding air is constant and equal to 20°. How long will it take for the body to cool down to 25°? (Assume that the rate of cooling is proportional to the difference of the body temperature and the temperature of the environment, which is known as *Newton's cooling law.*)

Exercise 1.16. Find the equations of the curves for which the area of the triangle generated by the t-axis, the tangent line, and the t-coordinate of the tangent line, is constant and equal to a^2. (Consider two cases $x' > 0$ and $x' < 0$.)

Exercise 1.17. It took 30 days for some radioactive substance to reduce 50% from the initial condition. How long will it take for this radioactive substance to reduce to 1% of the initial condition? (Use the *law of the radioactive decay* that says that the rate of decay is proportional to the available amount.)

Exercise 1.18. Show that every integral curve of the equation

$$\dot{x} = \sqrt[3]{\frac{x^2 + 1}{t^4 + 1}}$$

has two horizontal asymptotes.

Exercise 1.19. For which a each solution to

$$\dot{x} = |x|^a,$$

is defined globally (i.e., for all $t \in \mathbf{R}$)?

Exercise 1.20. Solve the differential equation describing the shape y of a hanging chain suspended at two points (this curve is called a *catenary*):

$$y'' = a\sqrt{1 + (y')^2}, \quad a > 0.$$

For simplicity assume that the coordinate system is chosen such that $y'(0) = 0$, i.e., the slope of the catenary at the point $x = 0$ is zero.

1.3.2 Linear equations

Definition 1.20. *A first order ordinary differential equation of the form*

$$\dot{x} + p(t)x = q(t), \quad p, q \in \mathcal{C}(J)$$

is called linear. If $q \equiv 0$ *then it is called linear homogeneous, whereas in the opposite case it is called nonhomogeneous.*

Remark 1.21. The notation $\dot{x} + p(t)x = q(t)$ is slightly abusive since both $p(t)$ and $q(t)$ are not functions but the values of the functions p, q at the point t. I will use this kind of notation in the following.

Lemma 1.22. *Assume that* $p, q \in \mathcal{C}(J)$ *in the linear ODE. Then there exists a unique solution to the IVP with* $x(t_0) = x_0$, $t_0 \in J$ *defined on all* J.

I will give a constructive proof of this lemma by actually presenting the solution by the method called the *variation of the parameter* or *variation of the constant.*

Proof. First, consider the homogeneous linear equation

$$\dot{x} = -p(t)x,$$

which is separable and can be integrated to get

$$x(t) = x_0 e^{-\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau}.$$

This is a unique solution because the integral $\int_0^x \frac{\mathrm{d}s}{s}$ diverges.

Now let us look for the solution to the full non-homogeneous linear ODE in the form

$$x(t) = C(t) e^{-\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau}, \tag{1.8}$$

where $t \mapsto C(t)$ is some unknown function of t (compare with the above solution to the homogeneous equation). It will be a solution if and only if

$$C'(t) = q(t) e^{\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau}.$$

Indeed, assume (1.8) is a solution. After plugging this expression into the equation I get

$$C'(t) e^{-\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau} - p(t)C(t) e^{-\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau} + p(t)C(t) e^{-\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau} = q(t) \iff C'(t) = q(t) e^{\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau}.$$

In the opposite direction, assume that $C'(t) = q(t) e^{\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau}$, then (1.8) is a solution by direct substitution.

Clearly the condition $x(t_0) = x_0$ and assumption (1.8) are equivalent to $C(t_0) = x_0$. Therefore

$$C(t) = x_0 + \int_{t_0}^{t} q(\tau) e^{\int_{t_0}^{\tau} p(\xi)\,\mathrm{d}\xi}\,\mathrm{d}\tau.$$

Putting everything together I conclude that

$$x(t) = x_0 e^{-\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau} + e^{-\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau} \int_{t_0}^{t} q(\tau) e^{\int_{t_0}^{\tau} p(\xi)\,\mathrm{d}\xi}\,\mathrm{d}\tau,$$

is the unique solution to my IVP defined on the whole interval J. ■

Remark 1.23. Often the solution to the linear ODE that I found above is written with indefinite integrals as

$$x(t) = Ce^{-\int p(t)\,\mathrm{d}t} + e^{-\int p(t)\,\mathrm{d}t} \int q(t) e^{\int p(t)\,\mathrm{d}t}\,\mathrm{d}t.$$

Example 1.24. Solve

$$(t + x^2)\dot{x} = x.$$

At a first glance this problem is *not* linear. However, I can again play with the derivative as a fraction:

$$(t + x^2)\frac{\mathrm{d}x}{\mathrm{d}t} = x \iff \frac{\mathrm{d}t}{\mathrm{d}x} - \frac{t}{x} = x,$$

i.e., this equation is of the form

$$\dot{t} + p(x)t = q(x).$$

(A rigorous justification of this manipulation can be made with the help of the *inverse function theorem.*) Now

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{x} \iff \log|t| = \log|Cx| \iff t = Cx,$$

and I point that the arbitrary constant C can be different in different parts of the equality (as it happened in the previous line). Now I look for the solution in the form $t = C(x)x$ and get

$$C'(x) = 1 \iff C(x) = x + A,$$

which implies that $t = x^3 + Ax$ is the general solution to the original equation, where A is an arbitrary constant.

Exercise 1.21. Consider the IVP

$$\dot{x} + \frac{2}{3}x = 1 - \frac{t}{2}, \quad x(0) = x_0.$$

Find the value(s) of x_0 such that the solutions touches, but does not cross the t-axis.

Exercise 1.22. Find a bounded solution to

$$\dot{x} - x = \cos t - \sin t.$$

Exercise 1.23. Consider the family of the integral curves of a linear equation $\dot{x} + p(t)x = q(t)$. Show that the tangent lines to the integral curves at the points with the same t-coordinate cross at the same point.

Exercise 1.24. Show that the equation

$$\dot{x} + x = f(t),$$

where f is a continuous and bounded function (i.e., $|f(t)| \leq M$ for any $t \in \mathbf{R}$ for some constant M), has only one bounded solution for any $t \in \mathbf{R}$. Find this solution. Show that if f is periodic then this solution is also periodic. (*Hint:* Express the solution through an integral with an infinite limit.)

Exercise 1.25. Consider the equation

$$\dot{x} + a(t)x = f(t),$$

where $a(t) \geq C > 0$ for all t, $f(t) \to 0$ as $t \to \infty$, and functions a, f are continuous for any $t > t_0$. Show that any solution of this equation approaches 0 as $t \to \infty$. (*Hint:* Use the explicit formula.)

Exercise 1.26. Let x_1 and x_2 be two particular (different) solutions to the linear equation $x'+p(t)x = q(t)$. Express the general solution to this equation through x_1 and x_2.

Exercise 1.27. Find a periodic solution to the differential equation

$$y' = 2y\cos^2 t - \sin t.$$

(*Hint:* It may be useful, but this is definitely not the only way to solve this problem, to show first that the equation $x' = f(t, x)$ with T-periodic f (i.e., $f(t, x) = f(t + T, x)$ for any t) has a T-periodic solution ϕ if and only if $\phi(0) = \phi(T)$ (this actually means that the problem to find a periodic solution boils down to a *boundary value problem*). Another approach to this problem is to express the general solution through an integral with the infinite (upper) limit.)

Exercise 1.28. Show that only one solution to

$$t\dot{x} - (2t^2 + 1)x = t^2$$

tends to a finite limit for $t \to \infty$ and find this limit.

Exercise 1.29. Consider the linear homogeneous equation

$$\dot{x} = a(t)x.$$

Find the conditions on function a for the solutions to this equation to be periodic.

Exercise 1.30. Consider the linear non-homogeneous equation

$$\dot{x} = a(t)x + b(t),$$

where both a, b are T-periodic functions. Assume that $\int_0^T a(\tau)\,\mathrm{d}\tau \neq 0$. Show that in this case there exists only one T-periodic solution to the original equation.

1.3.3 Exact equations

Going back to the geometric interpretation of the first order ODE, I remark that if I consider

$$\dot{x} = f(t, x),$$

then I actually ignore the directions parallel to the x-axis (the slope is infinite). If, as in Example 1.24, I take the point of view that

$$\dot{t} = g(t, x),$$

then I ignore the direction parallel to the t-axis.

There is a general way to take care of both of these directions by writing the first order ODE in the symmetric form

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0,$$

where M and N are some given functions. Now the direction is not defined only at those points (t, x) at which *both* M and N vanish.

Definition 1.25. *A first order ODE of the form*

$$M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x = 0$$

is called exact if there is a function $F\colon U \subseteq \mathbf{R}^2 \longrightarrow \mathbf{R}$ *such that*

$$\mathrm{d}F = M(t,x)\,\mathrm{d}t + N(t,x)\,\mathrm{d}x.$$

Here $\mathrm{d}F$ is the full differential of the function of two variables, which is defined to be

$$\mathrm{d}F = \frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial x}\,\mathrm{d}x.$$

From the definition I immediately have that the general solution to the exact equation is given by

$$F(t,x) = \text{const}.$$

From analysis I know that if U is "nice" (an open simply connected subset of the plane, i.e., without "holes") then a necessary and sufficient condition for the equation to be exact is

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial t}$$

for all points $(t,x) \in U$. In this case function F can be found by integration.

Exercise 1.31. Check that the following differential equation exact and find its solution:

$$3x^2(1+\ln y)\,\mathrm{d}x = \left(2y - \frac{x^3}{y}\right)\mathrm{d}y.$$

Remark 1.26. In some sense *exactness* is what allows one integrating an ODE in a closed form. Clearly, a separable equation $\dot{x} = f_1(t)f_2(x)$ written as

$$f_1(t)\,\mathrm{d}t - \frac{1}{f_2(x)}\,\mathrm{d}x = 0$$

is exact.

Linear equation $\dot{x} + p(t)x = q(t)$ is not exact, but it becomes exact after multiplication by an *integrating factor* $\mu(t) = e^{\int_{t_0}^{t} p(\tau)\,\mathrm{d}\tau}$. Indeed, the ODE

$$\mu(t)\,\mathrm{d}x + (\mu(t)p(t)x - q(t)\mu(t))\,\mathrm{d}t$$

is exact (check it).

1.3.4 Substitutions

Most of the ODE are neither separable, nor linear, nor exact. However it is often possible to come up with a substitution to turn the given first order ODE into one of these three types. Here are just two examples.

Definition 1.27. *A first order ODE of the form*

$$\dot{x} = f\left(\frac{x}{t}\right)$$

is called homogeneous.

The substitution $x(t) = u(t)t$ turns it into a separable equation for the new unknown function u.

Definition 1.28. *A first order ODE of the form*

$$\dot{x} + p(t)x = q(t)x^n, \quad n \in \mathbf{R}\backslash\{0, 1\}$$

is called Bernoulli's equation.

The substitution $u(t) = x^{1-n}(t)$ turns it into a linear equation for the new unknown function u.

Modern computer algebra systems, like Maple® or Mathematica®, can solve a great deal of different ODE in explicit form.

Exercise 1.32. Solve the following homogeneous differential equation

$$\dot{x} = \frac{t + x}{t - x}\,.$$

Sketch the integral curves. (*Hint:* For the last point it is better to switch to polar coordinates $t = \rho\cos\theta$, $x = \rho\sin\theta$.)

Exercise 1.33. Recall that function ψ is called homogeneous of degree $m \in \mathbf{R}$ if for any $\tau \in (a, b)$ $\psi(\tau t, \tau x) = \tau^m\psi(t, x)$. Prove that the equation

$$M(t, x)\,\mathrm{d}t + N(t, x)\,\mathrm{d}x = 0$$

is homogeneous if and only if M and N are homogeneous functions of the same degree.

Exercise 1.34. The solutions to $\dot{x} = f(t)$ are invariant with respect to translations along x-axis, the solutions to $\dot{x} = f(x)$ are invariant with respect to translations along t-axis. Which geometric transformation leaves the integral curves of the homogeneous equation invariant?

1.3.5 Liouville's theorem

Can I always find an analytical solution (a formula) to a first order ODE? The answer is negative, as it was proved by Joseph Liouville in 1839[7]. To be precise, he showed that there are some specific second order linear differential equation with non-constant coefficients that cannot be solved in elementary functions. To wit, the following theorem[8] holds

Theorem 1.29 (Liouville)**.** *The solutions of the equation $y'' + xy = 0$ cannot be obtained from the field of rational functions of x by any sequence of finite algebraic extensions, adjunctions of integrals, and adjunctions of exponentials of integrals.*

[7] Mémoire sur l'intégration d'une classe d'équations différentielles du second ordre en quantités finies explicites, Journal de Mathématiques pures et appliquées, 1839.

[8] I am borrowing this theorem from Irving Kaplansky, An introduction to differential algebra, 1957, where all the glorious details can be found.

It turns out that, after a specific substitution, *Riccati's equation*

$$\dot{x} = q_0(t) + q_1(t)x + q_2(t)x^2$$

can be rewritten as a second order linear ODE. First it can be rewritten, using the new variable $v = q_2 x$, as

$$\dot{v} = v^2 + p_1(t)v + p_2(t),$$

and after this, using the substitution $v = -u'/u$, as

$$u'' - p_1(t)u' + p_2(t)u = 0,$$

which is a second order linear ODE. If I take $\dot{x} = x^2 + t$ then the corresponding second order linear equation is

$$u'' + tu = 0,$$

for which Liouville's theorem holds.

Exercise 1.35. Show that if one knows a particular solution x_p to Riccati's equation then the general solution can be found by quadratures (integration).

1.4 Appendix: Additional exercises

In this appendix to the first chapter I collect a number of exercises in elementary ODE theory.

Exercise 1.36. Sketch the integral curves of the equation

$$\dot{x} = \frac{t^2 + x^2}{2} - 1.$$

Exercise 1.37. Consider the equation

$$\dot{x} = \frac{x}{t} + \varphi\left(\frac{t}{x}\right).$$

Find such φ that the general solution to this equation is given by

$$x(t) = \frac{t}{\log |Ct|}.$$

Exercise 1.38. Sketch the graphs of solutions to

$$\sin \dot{x} = 0.$$

Exercise 1.39. Prove that the differential equation for the algebraic curves of the second order has the form

$$\left(\left(x''\right)^{-2/3}\right)''' = 0.$$

Exercise 1.40. Prove that the equation

$$\dot{x} = 2\sqrt[3]{tx}$$

has more than one solutions passing through the origin.

Exercise 1.41. Prove that all the solutions to

$$\dot{x} = \frac{1}{1 + t^2 + x^2}$$

are bounded for all $t \in \mathbf{R}$.

Exercise 1.42. Solve

$$x'' - tx' - x = 0.$$

Exercise 1.43. Prove that the equation

$$\dot{x} = x^2 + t$$

with the initial condition $x(0) = 0$ has no solution on the interval $(0, 3)$.

Exercise 1.44. Let $x \in \mathcal{C}^{(2)}(a, b) \cap \mathcal{C}[a, b]$ satisfy the equation

$$x'' = e^t x$$

and the boundary conditions $x(a) = x(b) = 0$. Find x. How many solutions exist?