Chapter 2

Fundamental theorems

2.1 Dynamical systems, phase flows, and differential equations

A *dynamical system* is a mathematical formalization of an *evolutionary deterministic* process. An evolutionary (i.e., changing with time) process is called *deterministic* if, given the initial state, the past and future of the process can be inferred. Sometimes *semi-deterministic* processes are considered, when the initial condition determines only the future, but not the past. The evolutionary processes I will consider in these notes will always be deterministic.

For example, consider a pendulum. It is known from physics that the knowledge of the initial angle θ and the initial angular velocity $\dot{\theta}$ uniquely define both the position and velocity of the pendulum in the past and in the future. The pair of variables $(\theta, \dot{\theta})$ is called a *phase point* belonging to the *phase* or *state* space X, which in this particular example can be taken as the cylinder $\mathbf{S}^1 \times \mathbf{R}$ because I need to know the angle only modulo 2π ($\mathbf{S}^1$ denotes the unit circle).

If I take, e.g., an ecological system then sometimes it is reasonable to assume that the phase point is just one number $n \in \mathbf{R}_+ := \{n \in \mathbf{R} \colon n \geq 0\}$, if the future and the past population sizes can be determined by its current population size n_0.

The mentioned two examples (as well as many others) lead to the conclusion that the mathematical formalization of an evolutionary deterministic process should include three ingredients: the ordered time set T, the state space X, and the law that actually determines the evolution, or, more precisely, the one-parameter family of operators $\{\varphi^t\}$ acting on the state space X, $\varphi^t \colon X \longrightarrow X$ for each fixed $t \in T$. To make sure that the one-parameter family of evolutionary operators corresponds to our intuitive understanding of the deterministic process, it is necessary to demand that[1]

$$\varphi^0 x = x, \quad x \in X, \tag{2.1}$$

i.e., that φ^0 is the identity operator on X. This means that the dynamical system does not change its state "spontaneously." Secondly, it is naturally needed to require that

$$\varphi^{\tau+t} x = \varphi^{\tau} \varphi^t x = \varphi^t \varphi^{\tau} x, \quad x \in X, \quad \tau, t, \tau + t \in T, \tag{2.2}$$

i.e., that it is the same to evolve the system for $t + \tau$ time units, or first evolve it for t time units, and then for τ time units. Properties (2.1), (2.2) mean that the one-parameter family of evolutionary

[1]Since my evolutionary operator is acting on an abstract space X, I abuse the notation by writing $\varphi^t x$ to denote $\varphi^t(x)$.

operators $\{\varphi^t\}$ forms a *transformation group* (actually, an abelian group) acting on the state space X. Indeed, from (2.2) it follows that for each $t \in T$ the unique inverse for φ^t element is φ^{-t} since

$$\varphi^t \circ \varphi^{-t} = \varphi^0,$$

assuming that $-t \in T$.

Definition 2.1. *A dynamical system is a triple $\{T, X, \varphi^t\}$, where T is an ordered time set, X is the state space and $\varphi^t\colon X \longrightarrow X$, $t \in T$ is a one-parameter family of evolutionary operators that satisfy* (2.1) *and* (2.2).

I emphasize that this definition gives a rigorous mathematical description of what we intuitively call a deterministic system, however in order to apply it to real situations it is necessary to have at hands the information on T, X, and, most importantly, on $\{\varphi^t\}$. Unfortunately, the cases when we know the family $\{\varphi^t\}$ explicitly are quite rare.

Set T can be usually taken as $\mathbf{Z}$ or $\mathbf{R}$ (or $\mathbf{Z}_+$ and $\mathbf{R}_+$ for semi-deterministic systems, which are also called *non-invertible* dynamical systems). In the former case one speaks of a *discrete* dynamical system, in the latter case — of a *continuous* dynamical system. For continuous dynamical systems the family $\{\varphi^t\}$ is often called the *flow* or *phase flow* of the dynamical system.

The underlying state space usually has a natural structure of a *vector space* and can be classified as *finite* or *infinite* dimensional. Consequently, I deal with *finite* or *infinite* dimensional dynamical systems respectively. In this course I will be talking only about finite dimensional dynamical systems, and X is always an open subset of the Euclidian space $\mathbf{R}^k$, if not stated otherwise.

Before turning to continuous dynamical systems, consider a prototypical example of a non-invertible discrete dynamical system. In this case the family of evolutionary operators $\{\varphi^t\}$ forms a *semigroup* (elements of the set do not have inverses).

Example 2.2. Consider $T = \mathbf{Z}_+ = \{0, 1, 2, \ldots\}$, $X = [0, 1]$, and

$$\varphi^1\colon x \mapsto \lambda x(1 - x), \quad 0 \leq \lambda \leq 4.$$

This is a discrete dynamical system, which is defined in terms of 1-operator φ^1. Knowledge of this operator allows me to get φ^n for any $n \in \mathbf{Z}_+$ as n times composition of φ^1:

$$\varphi^n = \varphi^1 \circ \ldots \circ \varphi^1.$$

Obviously, (2.1) and (2.2) are satisfied. Note that the map φ^1 is non-invertible, and therefore I can only predict future, not the past. I will not discuss the discrete dynamical systems in this course, just mention that they turn out to be very important for understanding complicated behavior of systems of ODE studied in more advanced courses.

Now consider a simplest example of a continuous dynamical system.

Example 2.3. Let $T = \mathbf{R}$, $X = \mathbf{R}$, $\varphi^t\colon x \mapsto e^{at}x$ for some fixed $a \in \mathbf{R}$. Check that properties (2.1) and (2.2) are satisfied. Note that the evolutionary operator here solves the scalar ODE $\dot{x} = ax$.

While considering continuous dynamical systems I usually require that $\varphi\colon T \times X \longrightarrow X$ be continuously differentiable with respect to both t and x. Since the inverse for φ^t is given by φ^{-t} and therefore also differentiable, $\{\varphi^t\}$ is a one-parameter group of *diffeomorphisms* (recall that a map

$g\colon U \longrightarrow V$ is a diffeomorphism if $g \in \mathcal{C}^{(1)}(U;V)$, has an inverse, and the inverse is a $\mathcal{C}^{(1)}(V;U)$ map). If $\varphi \in \mathcal{C}^{(1)}(T \times X; X)$, then I speak of a *smooth* or *differentiable* dynamical system or *smooth* or *differentiable flow* (in the latter case "smooth" and "differentiable" are usually omitted).

For a smooth finite dimensional dynamical system the vector of the phase velocity is defined at each point $\boldsymbol{x} \in X$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t \boldsymbol{x}\Big|_{t=0} = \boldsymbol{f}(\boldsymbol{x}), \tag{2.3}$$

and the vector $\boldsymbol{f}$ of the phase velocity defines a *smooth vector field* everywhere in X, except the points $\hat{\boldsymbol{x}}$ where $\boldsymbol{f}(\hat{\boldsymbol{x}}) = 0$.

Now assume that $X \subseteq \mathbf{R}^k$, $\boldsymbol{x}_0 \in X$ and consider evolution of $\boldsymbol{x}_0$ under the action of the flow $\{\varphi^t\}$ as a function of t. In other words, I consider the mapping

$$\boldsymbol{\phi}\colon \mathbf{R} \longrightarrow X, \quad \boldsymbol{\phi}(t) = \boldsymbol{\phi}(t; \boldsymbol{x}_0) := \varphi^t \boldsymbol{x}_0.$$

Theorem 2.4. *Mapping $\boldsymbol{\phi}$ solves the IVP*

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

where the vector field on the right hand side is defined by (2.3).

Proof. This follows from (2.2):

$$\frac{\mathrm{d}\boldsymbol{\phi}}{\mathrm{d}\tau}(\tau) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi^t \boldsymbol{x}_0\Big|_{t=\tau} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\varphi^{\tau+\epsilon}\boldsymbol{x}_0\Big|_{\epsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\epsilon}\varphi^{\epsilon}\varphi^{\tau}\boldsymbol{x}_0\Big|_{\epsilon=0} = \boldsymbol{f}(\varphi^\tau \boldsymbol{x}_0) = \boldsymbol{f}(\boldsymbol{\phi}(\tau)).$$

■

In words, for each smooth dynamical system defined on a finite dimensional state space $X \subseteq \mathbf{R}^k$ I found a system of autonomous ordinary differentiable equations defined by the vector field $\boldsymbol{f}$ of the phase velocity. Solutions to this system are the images of the movement of the phase points under the action of the phase flow. Therefore, my attempt to give a mathematical formalization for evolutionary deterministic processes showed that actually the deterministic laws of nature are not necessary to formulate in terms of the transformation group $\{\varphi^t\}$ acting on the state space X, but it is also possible to do in terms of the corresponding vector field $\boldsymbol{f}$ or, equivalently, in terms of an autonomous system of ODE. The latter approach reduces any problem of the process evolution to a geometric problem about the behavior of the flow of the given vector field.

Example 2.5. Returning to the example of the pendulum, the second Newton's law leads to the second order differential equation (if I disregard the friction)

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

This equation is equivalent to the following autonomous system of two first order ODE:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1, \end{aligned}$$

where $(x_1, x_2) = (\theta, \dot{\theta}) \in X = \mathbf{S}^1 \times \mathbf{R}$. The vector field of this system defines the corresponding flow $\{\varphi^t\}$. I will not be able to write φ down explicitly, but this specific system of ODE will allow me to study many of its properties!

The main question is whether the converse is true? That is, does an arbitrary smooth vector field (an autonomous system of ODE) generate a smooth dynamical system? A complete answer to this question requires a careful analysis of the properties of the solutions to the general ODE system, and this will be my task for the next several lectures.

2.2 A motivation for the existence and uniqueness theorem proof

My first goal is to establish the fundamental theorem that states that if the right hand side of the system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f}\colon J \times X \longrightarrow \mathbf{R}^k, \tag{2.4}$$

is "good enough" (here, $J \subseteq \mathbf{R}$), then the initial value problem (2.4) and (2.5):

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in X, \quad t_0 \in J, \tag{2.5}$$

has a unique solution.

A crucial step in establishing this fact is to note

Lemma 2.6. *Assuming that* $\boldsymbol{f} \in \mathcal{C}(J \times X; \mathbf{R}^k)$*, problem* (2.4), (2.5) *is equivalent to the system of integral equations*

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) \,\mathrm{d}\tau. \tag{2.6}$$

More precisely, every solution to (2.4), (2.5) *solves* (2.6). *Any continuous on* $I := (t_0 - \delta, t_0 + \delta)$, $I \subseteq J$, $\delta > 0$ *solution to* (2.6) *solves* (2.4), (2.5) *on* I.

Proof. Indeed, let $\boldsymbol{x}$ solve (2.4), (2.5) on I and let $t_0, t \in I$. Then, by integrating (2.4) from t_0 to t and taking into account (2.5) I get (2.6).

Now, let $\boldsymbol{x}$ be a continuous on I solution to (2.6). Then I have $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ by direct substitution. The right hand side of (2.6) is continuously differentiable as an integral of a continuous function, and therefore $\boldsymbol{x}$ on the right hand side is also continuously differentiable. By differentiating both sides of (2.6) I obtain (2.4). ∎

Now I can guess that at least sufficiently close to t_0 the constant $\boldsymbol{x}_0$ closely approximates unknown $\boldsymbol{x}$, therefore,

$$\boldsymbol{x}_1(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{x}_0) \,\mathrm{d}\tau.$$

It is my hope that $\boldsymbol{x}_1$ approximates $\boldsymbol{x}$ "better" than $\boldsymbol{x}_0$. Or, in general, that the n-th iteration

$$\boldsymbol{x}_n(t) = \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{x}_{n-1}(\tau)) \,\mathrm{d}\tau$$

gives "almost" the solution to the original problem.

The sequence[2] $(\boldsymbol{x}_n)_{n=0}^{\infty}$ is called *Picard's iterates*, and there is hope that this sequence will converge to the unknown solution $\boldsymbol{x}$ as the following example shows.

[2]The correct notation $(\boldsymbol{x}_n)_{n=0}^{\infty}$ for the sequence $\boldsymbol{x}_0, \boldsymbol{x}_1, \dots$ will be often abbreviated as $(\boldsymbol{x}_n)$ in the rest of these notes.

Example 2.7. Consider the IVP

$$\dot{x} = x, \quad x(0) = x_0.$$

Picard's iterates take the form

$$\begin{aligned}
x_1(t) &= x_0 + x_0 t = (1+t)x_0, \\
x_2(t) &= \left(1 + t + \frac{t^2}{2}\right) x_0, \\
x_3(t) &= \left(1 + t + \frac{t^2}{2} + \frac{t^3}{3!}\right) x_0, \\
&\dots \\
x_n(t) &= \left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!}\right) x_0.
\end{aligned}$$

These are the partial sums of *Taylor's series* for the exponential function

$$e^t = \sum_{i=0}^{\infty} \frac{t^i}{i!},$$

which converges absolutely for any $t \in \mathbf{R}$. And indeed, as I already know (Lemma 1.8), $t \mapsto e^t$ solves my initial value problem.

It is actually possible to proceed in a similar way with the IVP (2.4), (2.5) by building Picard's iterates and proving that they, under some additional assumptions on $\boldsymbol{f}$, converge uniformly (see the next section for the exact definitions) to some function $\boldsymbol{x}$ that, therefore, solves the integral equation (2.6). Since each Picard's iterate is a continuous function by construction, the uniform convergence implies that $\boldsymbol{x}$ is continuous, and hence, according to Lemma 2.6 above, solves (2.4), (2.5). However, I will take a more abstract rout and present a conceptual proof that, firstly, shows what is at work exactly here, and, secondly, is widely applicable out of the area of ODE. For this proof I will need some basic notions of analysis.

Exercise 2.1. Find three first Picard iterates for

1. $$\dot{x} = t - x^2, \quad x(0) = 0.$$

2. $$\dot{x}_1 = 2t + x_2, \quad \dot{x}_2 = x_1, \quad x_1(1) = 1, \quad x_2(1) = 0.$$

Exercise 2.2. Derive the Taylor series for $x(t) = \sin t$ by applying the Picard method to the first order system corresponding to

$$\ddot{x} = -x, \quad x(0) = 0, \quad \dot{x}(0) = 1.$$

2.3 Auxiliary facts from analysis

A basic mathematical structure that underlines most of the modern mathematics[3] is a *set*. To make a progress in describing various models, I need to add additional structures to the sets I deal with. For example, a *topological structure* implies that I can talk about *open* and *closed* sets, and can introduce the notions of *convergence* and *continuity*. Arguably, a simplest way to introduce a topological structure (but not the most general one!) is to consider a *metric*, or a *distance* function, on the given set.

To be precise, a *metric space* is a pair (X, d), where X is the underlying set, and d is a function

$$d\colon X \times X \longrightarrow \mathbf{R}_+ := \{x \in \mathbf{R}\colon x \geq 0\},$$

that gives a "distance" between any two points x, y of X and satisfies the following properties:

$$\begin{aligned}
&d(x,y) \geq 0, \quad d(x,y) = 0 \Longleftrightarrow x = y, \quad (\textit{positivity})\\
&d(x,y) = d(y,x), \quad (\textit{symmetry})\\
&d(x,y) \leq d(x,z) + d(y,z). \quad (\textit{triangle inequality})
\end{aligned}$$

These are the natural properties of a distance that we got used to in our Euclidian space $\mathbf{R}^3$ with the standard metric

$$d(\boldsymbol{x},\boldsymbol{y}) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}, \quad \boldsymbol{x},\boldsymbol{y} \in \mathbf{R}^3.$$

Using the distance function I immediately get the notions of a convergent sequence and a continuous mapping. In particular, a sequence $(x_n)_{n=0}^{\infty}$ of points of the metric space (X, d) is said to *converge* to $x \in X$, written as usual

$$\lim_{n\to\infty} x_n = x, \quad \text{or} \quad x_n \to x,$$

if for any $\epsilon > 0$ there exists $N = N(\epsilon)$ such that for all $n \geq N$

$$d(x_n, x) < \epsilon.$$

Consider now a mapping $F\colon X \longrightarrow Y$ between two metric spaces (X, d_1) and (Y, d_2). Mapping F is *continuous* at the point x_0 if for any $\epsilon > 0$ there exists a $\delta = \delta(x_0, \epsilon) > 0$ such that for any $x \in X$ for which $d_1(x, x_0) < \delta$ I have

$$d_2(F(x), F(x_0)) < \epsilon.$$

F is *continuous* if it is continuous at each point of its domain (this is an important point since functions continuous on some open sets are of the most interest).

The connection between continuity and convergent sequences can be stated as follows: Let F be a mapping from (X, d_1) to (Y, d_2). Then F is continuous if and only if

$$F(\lim x_n) = \lim F(x_n)$$

for any convergent sequence $(x_n)_{n=0}^{\infty}$ in (X, d_1). In other words, a mapping F is continuous if and only if it preserves convergent sequences.

[3]This is a very light refresher. If a lot of the definitions are new in this section, the student needs to pick up an introductory analysis book. My favorites are T. W. Körner, A Companion to Analysis: A Second First and First Second Course in Analysis, AMS, 2003, and V.A. Zorich, Mathematical Analysis, Vols. I and II, Springer, 2004.

Using the distance function I can define *open* and *closed* sets. To wit, let $B_\delta(a) = \{x \in X : d(a,x) < \delta\}$ be a *ball* with center a of radius $\delta > 0$, or a *δ-neighborhood* of the point a. A set $U \subseteq X$ is *open* in the metric space (X, d) if for each point $x \in U$ there exists a ball $B_\delta(x)$ such that $B_\delta(x) \subseteq U$ (can you prove that a ball is an open set?). A set $V \subseteq X$ is *closed* if its complement $X \backslash V$ is open. A point $a \in X$ is a *limit point* of the set $W \subseteq X$ if for any neighborhood $O(a)$ (that is, for any open set containing a) of a the intersection $O(a) \cap W$ is an infinite set. The union of the set W and all its limit points is called the *closure* of the set W in X and denoted $\overline{W}$. It is true (and left as an exercise) that set X is closed if and only if it contains all its limit points (X is closed if and only if $\overline{X} = X$). Further information on the notions of convergence and continuity and examples will be given below.

Another basic structure that can be added to a set X is an algebraic structure of *linear* or *vector* space X. A non-empty set X is a vector (linear) space over field $\mathbf{F}$ (you can think of $\mathbf{F} := \mathbf{R}$, the set of real numbers, or $\mathbf{F} := \mathbf{C}$, the set of complex numbers) if for any two elements $x, y \in X$ and any scalar $\alpha \in \mathbf{F}$ the expression

$$\alpha x + y$$

is defined and belongs to X (i.e., I can add the elements of X, which are called vectors, and multiply them by scalars). Moreover, the following axioms hold for any $x, y, z \in X$ and $\alpha, \beta \in \mathbf{F}$:

$$\begin{aligned}
x + y &= y + x, \\
(x + y) + z &= x + (y + z), \\
\text{there exists } 0 \in X,\ 0 + x &= x, \\
\text{there exists } -x \in X,\ x + (-x) &= 0, \\
\alpha(\beta x) &= (\alpha\beta)x, \\
1x &= x, \\
\alpha(x + y) &= \alpha x + \alpha y, \\
(\alpha + \beta)x &= \alpha x + \beta x.
\end{aligned}$$

The basic example of a real vector space is $\mathbf{R}^k$, where the vectors[4] are $\boldsymbol{x} = (x_1, \ldots, x_k)^\top$. Linear spaces are studied in the course of *linear algebra*.

One of the key notions of the vector spaces is a *basis*. I say that the list of vectors $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ is *linearly independent* if any linear combination of these vectors

$$\alpha_1 \boldsymbol{x}_1 + \ldots + \alpha_k \boldsymbol{x}_k$$

is equal to zero only if $\alpha_1 = \ldots = \alpha_k = 0$. Otherwise, this list is called *linearly dependent*. A *basis* of a vector space X is a linearly independent list of vectors that *spans* X, that is, for any $\boldsymbol{x} \in X$ there exist a unique list of scalars $\{\alpha_1, \ldots, \alpha_k\}$ such that

$$\boldsymbol{x} = \alpha_1 \boldsymbol{x}_1 + \ldots + \alpha_k \boldsymbol{x}_k.$$

It is possible to have different bases for the vector space X, however, the number of vectors in any basis is the same and called the *dimension* of X. Vector spaces that possess bases in the sense of the given definition are called *finite dimensional*. All other vector spaces are called *infinite dimensional*,

[4] I use the bold fond exclusively to denote the vectors from $\mathbf{R}^k$. Similarly, the bold font will be used to denote the mappings of $\mathbf{R}^l$ to $\mathbf{R}^k$, $k > 1$.

and the notion of the basis for those requires more careful consideration (I will talk about it in the last part of the course).

The following exercises are suggested for the student who feels that their knowledge of the topics of the metric spaces and vector spaces is somewhat shaky.

Exercise 2.3. Let (X, d) be a metric space. Is X open? What about $\emptyset$?

Exercise 2.4. Prove that both ball $B_\delta(a)$ and its exterior $\{x \colon d(a, x) \geq \delta\}$ are open sets.

Exercise 2.5. Let (X, d) be a metric space. Is X closed? What about $\emptyset$?

Exercise 2.6. Prove that the union of any number of open sets (finite or infinite) is open; the intersection of a finite number of open sets is open; the intersection of any number of closed sets is closed; the union of finite number of closed sets is closed.

Exercise 2.7. Prove that the set is closed if and only if it contains all its limit points.

Exercise 2.8. Prove that the set W in a metric space X is closed if and only if every convergent sequence (x_n) with $x_n \in W$ for any n has its limit in W.

Exercise 2.9. Prove that a mapping is continuous if and only if it preserves the convergent sequences.

Exercise 2.10. Let X be a vector space and let $\{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\}$ be a basis of X. Let $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m$ be elements of X and assume that $m > k$. Prove that the list $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_m\}$ is linearly dependent.

Exercise 2.11. Show that any basis of X has the same number of elements.

Now finally I can start presenting the main mathematical structure, which will be of most use in this course. For this structure, which is called a *normed vector space*, I unite both the topological (metric) and algebraic properties of linear spaces.

Definition 2.8. *Let X be a vector space over real or complex numbers. A function $\|\cdot\| \colon X \longrightarrow \mathbf{R}_+$ is called a norm in X if for any $x, y \in X$ and $\alpha \in \mathbf{F}$ it satisfies*

$$\|x\| \geq 0,\ \|x\| = 0 \Longleftrightarrow x = 0, \quad (\textit{positivity})$$
$$\|\alpha x\| = |\alpha|\|x\|, \quad (\textit{homogeneity})$$
$$\|x + y\| \leq \|x\| + \|y\|. \quad (\textit{triangle inequality})$$

Definition 2.9. *A vector space X with a norm defined on it is called a normed vector space and denoted $(X, \|\cdot\|)$ or simply X if no confusion arises.*

Every normed space has the natural metric

$$d(x, y) = \|x - y\|,$$

and therefore I immediately have the notions of convergence, continuity, and open and closed sets in a normed vector space. For example, $(x_n)_{n=1}^{\infty}$ converges to $x \in X$ in $(X, \|\cdot\|)$ if the sequence of real numbers $\|x_n - x\|$ converges to zero. A mapping $F \colon X \longrightarrow Y$ is continuous at the point $x_0 \in X$ if for any $\epsilon > 0$ there exists a $\delta > 0$ such that for any $x \in X$ satisfying $\|x - x_0\|_X < \delta$ I have that $\|F(x_n) - F(x)\|_Y < \epsilon$ holds, note two different norms used.

Here are several examples of normed vector spaces.

Example 2.10. Consider $X = \mathbf{R}$ with the norm

$$\|x\| := |x|.$$

The axioms are obvious.

Example 2.11. Let $X = \mathbf{R}^k$ be k-dimensional vector space with the norm

$$\|\boldsymbol{x}\|_2 := \sqrt{\sum_{i=1}^{k} x_i^2}.$$

To prove that $\|\boldsymbol{x}\|_2$ is actually a norm, it is useful first to establish the fact that, if $\boldsymbol{x} \cdot \boldsymbol{y}$ the usual dot product in $\mathbf{R}^k$ (i.e., $\boldsymbol{x} \cdot \boldsymbol{y} := \sum_{i=1}^{k} x_i y_i$) then $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\|_2 \|\boldsymbol{y}\|_2$ and use the fact that $\|\boldsymbol{x}\|_2^2 = \boldsymbol{x} \cdot \boldsymbol{x}$ and properties of the dot product.

Example 2.12. Let again $X = \mathbf{R}^k$ and consider now

$$\|\boldsymbol{x}\|_p := \left(\sum_{i=1}^{k} |x_i|^p\right)^{\frac{1}{p}}, \quad p \geq 1.$$

If the case $p = 2$ I obtain the norm from the previous example. To prove that $\|\boldsymbol{x}\|_p$ is a norm, one actually needs first to establish the so-called *Minkowski's inequality* (pick up an analysis book).

Example 2.13. I can formally take the limit $p \to \infty$ above and obtain

$$\|\boldsymbol{x}\|_\infty := \max_i |x_i|,$$

which is a legitimate norm as can be checked.

The last examples show that it is possible to have different normed linear spaces on the same underlying vector space X. However, for a finite dimensional vector space X one can prove that all the norms are equivalent (two norms $\|\cdot\|_a$ and $\|\cdot\|_b$, defined on the same vector space X, are equivalent if there exist constants $c > 0$ and C such that $c\|x\|_a \leq \|x\|_b \leq C\|x\|_a$ for any $x \in X$), i.e., one can speak of convergence and continuity in $\mathbf{R}^k$ with respect simply to a norm, without specifying which norm is talked about. The convergence in the normed vector spaces mentioned so far is the usual element-wise convergence.

Exercise 2.12. Show that

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\|.$$

Exercise 2.13. Let $(X, \|\cdot\|)$ be a normed vector space. Show that the norm, vector addition, and multiplication by scalars are continuous.

Exercise 2.14. Prove that for $X = \mathbf{R}^k$ the expression

$$\|\boldsymbol{x}\| = \left(x_1^2 + \ldots + x_k^2\right)^{1/2}$$

is a norm.

Exercise 2.15. Prove that any two norms on a finite dimensional vector space are equivalent.

Example 2.14. Consider the set $X := \mathcal{BC}(U;\mathbf{R})$, which I will also denote simply $\mathcal{BC}(U)$, of bounded continuous functions $x\colon U \longrightarrow \mathbf{R}$. This set is a real vector space if the operations of addition and multiplication by scalars are taken pointwise. A norm on X can be defined as (check the norm axioms)

$$\|x\|_\infty := \sup_{t\in U} |x(t)|.$$

If U is *compact* (assuming that U is a subset of $\mathbf{R}^k$ it means that that U is bounded and closed), then this norm can be defined as

$$\|x\|_\infty := \max_{t\in U} |x(t)|,$$

on the set $\mathcal{C}(U)$ of continuous functions, since as it is known from analysis a continuous mapping reaches its minimal and maximal values on a compact set. Hence, if, for simplicity, $U = [a, b]$ is a closed interval of the real line $\mathbf{R}$, then set $X = \mathcal{C}[a, b]$ becomes a normed vector space with the norm

$$\|x\|_\infty := \max_{t\in[a,b]} |x(t)|.$$

A different norm on $\mathcal{C}[a, b]$ is given by (check that it is a norm)

$$\|x\|_1 := \int_a^b |x(t)|\,\mathrm{d}t.$$

Vector space $\mathcal{C}[a, b]$ is infinite dimensional (i.e., for any $n \in \mathbf{N}$ it is possible to find n linearly independent vectors), and below I will show that the normed vector spaces $(\mathcal{C}[a, b], \|\cdot\|_\infty)$ and $(\mathcal{C}[a, b], \|\cdot\|_1)$ are fundamentally different.

Exercise 2.16. Show that for the vector space $\mathcal{C}[a, b]$

$$\|x\|_\infty = \max_{t\in[a,b]} |x(t)|,$$

and

$$\|x\|_1 = \int_a^b |x(t)|\,\mathrm{d}t$$

are norms.

To establish this let me first recall the notions of *pointwise* and *uniform convergence.*

The sequence of functions $(x_n)_{n=0}^\infty$ *converges pointwise* on $[a, b]$ to a function x, if, given any point $t \in [a, b]$ and any $\epsilon > 0$ there exists $N(\epsilon, t)$ such that for any $n \geq N$ I have $|x_n(t) - x(t)| < \epsilon$. Note that different t may require different N. A sequence of functions $(x_n)_{n=0}^\infty$ *converges uniformly* to x if for any $\epsilon > 0$ there exists $N(\epsilon)$ such that for any $n \geq N$ and any $t \in [a, b]$ I have $|x_n(t) - x(t)| < \epsilon$. Note that now N works for any t. That is, since

$$|x_n(t) - x(t)| \leq \max_{t\in[a,b]} |x_n(t) - x(t)| = \|x_n - x\|_\infty,$$

this means that the notion of the uniform convergence is exactly the convergence in the normed vector space $(\mathcal{C}[a, b], \|\cdot\|_\infty)$.

One of the most important properties of the uniform convergence can be stated as follows.

Lemma 2.15. *If the sequence $(x_n)_{n=0}^{\infty}$ converges uniformly to x and each $x_n \in \mathcal{C}[a,b]$ then $x \in \mathcal{C}[a,b]$.*

Proof. I need to prove that x is continuous for any $t \in [a,b]$, i.e., I need to find $\delta(\epsilon) > 0$ such that $|x(t) - x(s)| < \epsilon$ for $|t - s| < \delta$ for any $\epsilon > 0$.

Pick any $\epsilon > 0$. Since $(x_n)_{n=0}^{\infty}$ converges uniformly, I can always find n such that for any $t \in [a,b]$

$$|x_n(t) - x(t)| < \frac{\epsilon}{3}.$$

Since each x_n is continuous, it means that for given $\epsilon/3$ I can always pick $\delta > 0$ such that

$$|x_n(t) - x_n(s)| < \frac{\epsilon}{3},$$

for $|t - s| < \delta$. Now

$$|x(t) - x(s)| \leq |x(t) - x_n(t)| + |x_n(t) - x_n(s)| + |x_n(s) - x(s)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

■

Definition 2.16. *A sequence $(x_n)_{n=0}^{\infty}$ is called fundamental or Cauchy if for any $\epsilon > 0$ there exists $N(\epsilon)$ such that for any $n, m \geq N$*

$$\|x_n - x_m\| < \epsilon. \quad (d(x_n, x_m) < \epsilon)$$

Definition 2.17. *A normed vector space $(X, \|\cdot\|)$ (a metric space (X, d)) is called complete if every fundamental sequence converges.*

Definition 2.18. *A complete normed vector space is called a Banach space.*

Lemma 2.19. *Normed vector space $(\mathcal{C}[a,b], \|\cdot\|_\infty)$ is a Banach space.*

Proof. Let $(x_n)_{n=0}^{\infty}$ be a fundamental sequence. Since norm is a continuous mapping, I can pass to the limit under the norm:

$$\lim_{n\to\infty} \|x_m - x_n\|_\infty = \|x_m - \lim_{n\to\infty} x_n\|_\infty.$$

Since $(x_n(t))_{n=0}^{\infty}$ for each fixed t is a fundamental sequence in $\mathbf{R}$, and $\mathbf{R}$ is complete, then there exists at least a pointwise limit

$$\lim_{n\to\infty} x_n = x.$$

Since $\|x_m - x\|_\infty < \epsilon$ for any $\epsilon > 0$ and all $m \geq N$ for some N then it means that the sequence $(x_n)_{n=0}^{\infty}$ converges to x uniformly, therefore x is continuous and hence belongs to $\mathcal{C}[a,b]$. ■

Finally, I will show that $(\mathcal{C}[a,b], \|\cdot\|_1)$ is not a Banach space.

Example 2.20. Consider the sequence $(x_n)_{n=0}^{\infty} \in \mathcal{C}[-1,1]$ defined by

$$x_n(t) = \begin{cases} -1, & -1 \leq t \leq -\frac{1}{n}, \\ nt, & -\frac{1}{n} < t < \frac{1}{n}, \\ 1, & \frac{1}{n} \leq t \leq 1. \end{cases}$$

I have that all $x_n \in \mathcal{C}[a,b]$ and the sequence $(x_n)_{n=0}^{\infty}$ is fundamental in $(\mathcal{C}[a,b], \|\cdot\|_1)$ (check it!). However, a direct observation implies that the pointwise limit is given simply by

$$x(t) = \begin{cases} -1, & 1 \leq t \leq 0, \\ 1, & 0 < t \leq 1, \end{cases}$$

which is a discontinuous function, and hence $(x_n)_{n=0}^{\infty}$ does not converge in $(\mathcal{C}[a,b], \|\cdot\|_1)$.

Exercise 2.17. Fill in the missed details in the last example.

2.4 Contraction mapping principle

Definition 2.21. *A mapping $K\colon X \longrightarrow X$ of a normed vector space $(X, \|\cdot\|)$ (of a metric space (X, d)) is called a contraction, if there exists a number q, $0 < q < 1$, such that the inequality*[5]

$$\|Kx_1 - Kx_2\| \leq q\|x_1 - x_2\| \qquad (d(Kx_1, Kx_2) \leq q d(x_1, x_2))$$

holds for any points $x_1, x_2 \in X$.

Definition 2.22. *Point $\hat{x} \in X$ is called a fixed point of a mapping $K\colon X \longrightarrow X$ if*

$$\hat{x} = K\hat{x}.$$

Theorem 2.23. *Let Y be a Banach space (a complete metric space), and $X \subseteq Y$ be closed. Let $K\colon X \longrightarrow X$ be a contraction on X. Then K has a unique fixed point $\hat{x}$.*

Proof.
(i) Uniqueness. Let $\hat{x}$ and $\hat{y}$ be such that $K\hat{x} = \hat{x}$ and $K\hat{y} = \hat{y}$. Then, by

$$\|\hat{x} - \hat{y}\| = \|K\hat{x} - K\hat{y}\| \leq q\|\hat{x} - \hat{y}\|$$

we have that $\|\hat{x} - \hat{y}\| = 0$, i.e., $\hat{x} = \hat{y}$.

(ii) Existence. Consider a sequence $(x_n)_{n=0}^{\infty}$ defined as

$$x_{n+1} = Kx_n, \quad x_0 \in X.$$

I estimate

$$\|x_{n+1} - x_n\| = \|Kx_n - Kx_{n-1}\| \leq q\|x_n - x_{n-1}\| \leq \ldots \leq q^n\|x_1 - x_0\|.$$

Now, for $m \geq n$,

$$\|x_m - x_n\| \leq \sum_{j=n}^{m-1} \|x_{j+1} - x_j\| \leq \sum_{j=n}^{m-1} q^j \|x_1 - x_0\| \leq \frac{q^n}{1-q}\|x_1 - x_0\|.$$

Therefore, $(x_n)_{n=0}^{\infty}$ is fundamental, and since Y is complete, there is the limit $x_n \to \hat{x} \in Y$. But since all $x_n \in X$ and X is closed then $\hat{x} \in X$.

Since contraction is continuous (prove it), I can exchange the operations of limit and mapping. I.e.,

$$x_{n+1} = Kx_n \implies \lim_{n\to\infty} x_{n+1} = K \lim_{n\to\infty} x_n \implies \hat{x} = K\hat{x},$$

which proves that $\hat{x}$ is indeed a unique fixed point of K. ∎

[5] Here, as before with the evolutionary operator φ^t, I abuse the notation and write Kx to denote $K(x)$.

Remark 2.24. Now it should be clear what is the exact strategy for my ultimate goal. I have the integral equation (2.6) and need to make its right hand side a contraction. More precisely, I need a Banach space (which is quite obviously should be $\mathcal{C}([a,b];\mathbf{R}^k)$ with the uniform norm $\|\cdot\|_\infty$), I need a mapping $K\colon \boldsymbol{x} \mapsto \boldsymbol{x}_0 + \int_{t_0}^t \boldsymbol{f}(\tau, \boldsymbol{x}(\tau))\,\mathrm{d}\tau$, and I need a closed subset $X \subseteq Y$, on which K is a contraction. This will guarantee both existence and uniqueness of the solution to the initial value problem (2.4), (2.5).

2.5 Existence and uniqueness theorem

2.5.1 Proof of the theorem of existence and uniqueness for a system of the first order ODE

I aim to prove that the problem

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f}\colon J \times X \longrightarrow \mathbf{R}^k, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in X, \tag{2.7}$$

where $J \subseteq \mathbf{R}$, has a unique solution.

In the following I will use two norms. First, for any element $\boldsymbol{x}$ of $X \subseteq \mathbf{R}^k$ I will use

$$\|\boldsymbol{x}\| = |\boldsymbol{x}| := \max_i |x_i|,$$

note the usual absolute value notation for this norm. Second, for a continuous vector function $\boldsymbol{x}\colon J \longrightarrow X$ I will use the uniform norm

$$\|\boldsymbol{x}\|_\infty := \max_{t\in J} |\boldsymbol{x}(t)|,$$

where the absolute value is exactly the previous norm.

Recall from the last section that the normed vector space $(\mathcal{C}([a,b];X), \|\cdot\|_\infty)$ is Banach for any $[a,b] \subseteq J$.

I will also need

Lemma 2.25. *If $\boldsymbol{x} \in \mathcal{C}([a,b];X)$ then*

$$\left|\int_a^b \boldsymbol{x}(t)\,\mathrm{d}t\right| \leq \left|\int_a^b |\boldsymbol{x}(t)|\,\mathrm{d}t\right|.$$

Proof. I have

$$\left|\int_a^b x_j(t)\,\mathrm{d}t\right| \leq \left|\int_a^b |x_j(t)|\,\mathrm{d}t\right| \leq \left|\int_a^b |\boldsymbol{x}(t)|\,\mathrm{d}t\right|$$

since

$$|x_j(t)| \leq \max_i |x_i(t)| = |\boldsymbol{x}(t)|.$$

The right hand side does not depend on j. Taking maximum through j on the left hand side I get the needed inequality. ∎

Definition 2.26. *Let $U \subseteq \mathbf{R} \times \mathbf{R}^k$. Function $\boldsymbol{f}\colon U \longrightarrow \mathbf{R}^k$ is called Lipschitz continuous in the second argument in U if*

$$|\boldsymbol{f}(t, \boldsymbol{x}_1) - \boldsymbol{f}(t, \boldsymbol{x}_2)| \leq L|\boldsymbol{x}_1 - \boldsymbol{x}_2|$$

for any points $(t, \boldsymbol{x}_1)$ and $(t, \boldsymbol{x}_2)$ in U. Constant L is independent of the points $(t, \boldsymbol{x}_1)$ and $(t, \boldsymbol{x}_2)$ and called the Lipschitz constant.

Exercise 2.18. For each of the following functions, find a Lipschitz constant on the region indicated, or prove there is none:

1.
$$f(x) = |x|, \quad x \in \mathbf{R},$$

2.
$$f(x) = x^{1/3}, \quad -1 \leq x \leq 1.$$

3.
$$f(x) = \frac{1}{x}, \quad 1 \leq x \leq \infty.$$

Exercise 2.19. More generally, function f is called *locally Lipschitz continuous* in U if for any compact subset V of U it is true that
$$|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| \leq L|\boldsymbol{x}_1 - \boldsymbol{x}_2|$$
for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in V$, and now constant L can depend on V.

Prove that if $f \in \mathcal{C}^{(1)}(I; \mathbf{R})$, where $I \subseteq \mathbf{R}$, then f is locally Lipschitz continuous.

Now I am ready to prove

Theorem 2.27. *Consider problem* (2.4) *and assume that* $\boldsymbol{f}$ *is continuous and Lipschitz continuous in* $\boldsymbol{x}$ *in domain* $U \subseteq J \times X$ *such that* $(t_0, \boldsymbol{x}_0) \in U$. *Then problem* (2.4), (2.5) *has a unique solution* $t \mapsto \boldsymbol{\phi}(t)$ *defined on some interval* $(t_0 - \delta, t_0 + \delta)$, $\delta > 0$.

Proof. Let P be the $k+1$ dimensional cube $|\boldsymbol{x} - \boldsymbol{x}_0| \leq b$, $|t - t_0| \leq a$ such that $P \subseteq U$. Denote $M = \max_P |\boldsymbol{f}(t, \boldsymbol{x})|$. As the Banach space Y I take the set of continuous vector functions $\mathcal{C}([t_0 - \delta, t_0 + \delta]; X)$, where $0 < \delta \leq a$ will be defined later. Consider also the set W, which is given by those $\boldsymbol{x} \in Y$ that satisfy $\|\boldsymbol{x} - \boldsymbol{x}_0\|_\infty \leq b$. The set W is closed by construction. Recall that problem (2.4) is equivalent to the integral equation $\boldsymbol{x} = K\boldsymbol{x}$, where operator K is given by
$$K \colon \boldsymbol{x} \mapsto \boldsymbol{x}_0 + \int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{x})\, \mathrm{d}\tau.$$

(i) I will show that if δ is small enough then $K \colon W \longrightarrow W$. Consider for $t \in [t_0, t_0 + \delta)$ (the other side is treated similarly)
$$|(K\boldsymbol{x})(t) - \boldsymbol{x}_0| = \left|\int_{t_0}^{t} \boldsymbol{f}(\tau, \boldsymbol{x}(\tau))\, \mathrm{d}\tau\right| \leq \left|\int_{t_0}^{t} |\boldsymbol{f}(\tau, \boldsymbol{x}(\tau))|\, \mathrm{d}\tau\right| \leq \left|\int_{t_0}^{t} M\, \mathrm{d}\tau\right| \leq M\delta,$$
therefore
$$\|K\boldsymbol{x} - \boldsymbol{x}_0\|_\infty \leq M\delta.$$
If I choose δ to satisfy
$$\delta \leq \frac{b}{M},$$
then $K \colon W \longrightarrow W$.

(*ii*) I will show that if δ is small enough then $K\colon W \longrightarrow W$ is a contraction. Consider

$$
\begin{aligned}
|(K\boldsymbol{x})(t) - (K\boldsymbol{y})(t)| &\leq \left|\int_{t_0}^{t} |\boldsymbol{f}(\tau, \boldsymbol{x}(\tau)) - \boldsymbol{f}(\tau, \boldsymbol{y}(\tau))| \,\mathrm{d}\tau\right| \leq L \left|\int_{t_0}^{t} |\boldsymbol{x}(\tau) - \boldsymbol{y}(\tau)| \,\mathrm{d}\tau\right| \\
&\leq L \left|\int_{t_0}^{t} \|\boldsymbol{x} - \boldsymbol{y}\|_\infty \,\mathrm{d}\tau\right| \leq L\delta\|\boldsymbol{x} - \boldsymbol{y}\|_\infty,
\end{aligned}
$$

which proves that if

$$\delta < \frac{1}{L}\,,$$

then K is a contraction. If I choose $0 < \delta < \min\{a, b/M, 1/L\}$ then K satisfies the conditions of the contraction mapping principle in W and therefore has a unique fixed point, which is a unique continuous solution to the integral equation (2.6), and therefore the unique solution to the IVP (2.4), (2.5). ■

2.5.2 Some remarks and extensions

There are a lot of points which can be added to the theorem of existence and uniqueness from the previous subsection.

1. Since any k-th order ODE of the form

   $$x^{(k)} = f(t, x, x', \dots, x^{(k-1)})$$

   is equivalent to a system of k first order equations, then I also proved that if f is continuous in t and Lipschitz continuous in all other variables, then the solution to the corresponding IVP exists and unique.

   Exercise 2.20. Formulate theorems of existence and uniqueness of solutions for a k-th order ODE and for a system of m ODE of the k-th order.

2. Not surprisingly, to be Lipschitz continuous is more demanding than to be simply continuous. Actually the following is true for a compact U:

   $$
   \begin{aligned}
   &\text{continuously differentiable in } U \subset \text{Lipschitz continuous in } U \\
   &\qquad\qquad \subset \text{uniformly continuous in } U \subset \text{continuous in } U
   \end{aligned}
   $$

   In particular, if $\boldsymbol{f}$ is continuously differentiable with respect to $\boldsymbol{x}$ in $U \subset \mathbf{R}^k$, where U is compact, then $\boldsymbol{f}$ is Lipschitz continuous with

   $$L = k \max_{ij} \left(\max_{U} \left|\frac{\partial f_i(\boldsymbol{x})}{\partial x_j}\right|\right).$$

3. A slightly more involved argument shows (see the textbook) that the condition $\delta < 1/L$ is superfluous. Actually one can always take $\delta = \min\{a, b/M\}$.

Exercise 2.21. There is another way to show that the condition $\delta < 1/L$ in unnecessary. Let X be the set of continuous functions from $[t_0 - a, t_0 + a]$ to the ball $B_\delta(x_0)$ with the norm

$$\|\boldsymbol{g}\|_w := \sup_{|t-t_0|\leq a} \left\{ e^{-2L|t-t_0|} |\boldsymbol{g}(t)| \right\}.$$

Show that this norm is equivalent to $\|\cdot\|_\infty$ and hence this vector space is complete.

Use this norm to establish that K is contraction in X if $\delta = \min\{a, b/M\}$.

4. For the existence of solutions it is enough to ask that $\boldsymbol{f}$ is continuous. This fact is called *Peano's theorem.*

5. The theorem is local, which is important since even the simplest examples show that solutions can blow-up (tend to infinity for finite t).

   Consider, e.g., $\dot{x} = x^2$, $x(0) = 1$. The theorem guarantees the existence and uniqueness of solution on some small interval. If I put an upper bound b on my solution, then $M = \max_{x:|x-1|\leq b} x^2 = (b+1)^2$, therefore, the solution is defined on $|t| \leq \delta$, where

$$\delta = \min\left\{\infty, \frac{b}{(b+1)^2}\right\} = \frac{b}{(b+1)^2},$$

   which is equal, e.g., to $2/9$ for $b = 2$, i.e., for the solution with the upper bound 3. The exact solution is

$$x(t) = \frac{1}{1-t},$$

   and reaches the same upper bound at $t = 2/3$, and the *maximal solution* is defined on $(-\infty, 1)$.

6. It is possible to find weaker conditions on $\boldsymbol{f}$ to guarantee uniqueness. In particular, let $u \mapsto \phi(u)$ be continuous for $u \geq 0$, $\phi(0) = 0$, $\phi(u) > 0$ for $u > 0$ and

$$\int_0^a \frac{\mathrm{d}u}{\phi(u)} = \infty$$

   for any $a > 0$. Assume that for any points $(t, \boldsymbol{x}_1)$ and $(t, \boldsymbol{x}_2)$ function $\boldsymbol{f}$ is such that

$$|\boldsymbol{f}(t, \boldsymbol{x}_1) - \boldsymbol{f}(t, \boldsymbol{x}_2)| \leq \phi(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)$$

   for $(t, \boldsymbol{x}_1)$, $(t, \boldsymbol{x}_2)$ from U. Then problem (2.1) has a unique solution for any $(t_0, \boldsymbol{x}_0) \in U$. This is *Osgood's theorem*[6]. It is possible to find infinitely many such functions, e.g.,

$$\phi(u) = Lu, \quad \phi(u) = Lu|\ln u|^\alpha,\ \alpha \geq 1, \quad \phi(u) = Lu|\ln u||\ln|\ln u||, \ \ldots.$$

7. The usual title for the existence and uniqueness theorem is the *Picard–Lindelöf theorem.* Historically Augustin-Louis Cauchy was probably the first to provide a rigorous proof of the existence and uniqueness theorem. He dealt with *analytic* right hand side. Recall that the function

[6] If the reader wants to see the proof of this and many other facts, which are difficult to find anywhere else, I recommend Philip Hartman, Ordinary differential equations.

$z \mapsto w(z)$ of complex variable z is called analytic at the point z_0, if it can be represented by the power series

$$w(z) = \sum_{n=0}^{\infty} w_n (z - z_0)^n,$$

convergent in some neighborhood of z_0. Analytic at z_0 function is analytic is some circle $|z-z_0| < r$.

Consider a first order differential equation and the initial condition

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f(z, w), \quad w(z_0) = w_0,$$

and assume that f is analytic at (z_0, w_0). Then, as was proved by Cauchy, this problem has a unique solution w, which is analytic at z_0.

8. From the point of view of applications, the right hand side f is often not continuous, let alone analytic. For example, in *control theory* it is natural to consider an equation of the form

$$\dot{x} = f(t, x) + u(t) =: g(t, x),$$

where u is a control function, which can be only in two states, say, $+1$ and -1. In this case the right hand side of our differential equation is not even continuous, and it is not immediately clear what to call a solution. A usual approach is to replace the ODE with the corresponding integral equation

$$x(t) = x_0 + \int_{t_0}^{t} g(\tau, x(\tau))\,\mathrm{d}\tau,$$

and call the continuous solution to the integral equation to be the "solution" to the original ODE (it does not have to be differentiable anymore). It can be proved that if g is Lipschitz continuous with respect to the second argument, and for each point of continuity x, g is *piecewise* continuous as a function of t, then the solution to the IVP (understood as a continuous solution to the integral equation) exists and unique. It is possible to make further generalizations, but this would require the notion of *Lebesgue's integral.*

9. As a simple corollary to the main theorem, it can be proved that

Corollary 2.28. *Consider the IVP (2.7), and assume that* $\boldsymbol{f} \in \mathcal{C}^{(p)}(J \times X; X)$*, i.e., function* $\boldsymbol{f}$ *is* p *times continuously differentiable with respect to all its arguments. Then any solution to this problem is* $p+1$ *times continuously differentiable.*

Exercise 2.22. Prove this corollary.

Exercise 2.23. For which initial conditions there exists a unique solution to

$$(a) \quad \ddot{x} = \tan x + \sqrt[3]{t}, \quad (b) \quad (t+1)\ddot{x} = x + \sqrt{x}, \quad (c) \quad \ddot{x} - x\dddot{x} = \sqrt[5]{\dot{x} - t}?$$

Exercise 2.24. Can the graphs of two solutions of a given ODE cross on the plane (t, x)?

$$(a) \quad \dot{x} = t + x^2, \quad (b) \quad \ddot{x} = t + x^2.$$

Exercise 2.25. For which n the equation

$$x^{(n)} = f(t, x),$$

where $f \in \mathcal{C}^{(1)}$, can have solutions $x_1(t) = t$, $x_2(t) = t + t^4$?

Exercise 2.26. How many derivatives do the solutions to the following equations have in a neighborhood of the origin?

$$(a) \quad \dot{x} = t + x^{7/3}, \quad (b) \quad \dddot{x} = x - t\sqrt[3]{t}.$$

Exercise 2.27. For which non-negative a the uniqueness of solutions to $\dot{x} = |x|^a$ is broken and at which points?

Exercise 2.28. Can the graphs of two solutions to $x'' = t + x^2$ intersect at some point (t_0, x_0) in the plane tOx? Can the graphs of two solutions to $x'' = t + x^2$ be tangent to each other at some point (t_0, x_0)?

2.6 Dependence on the parameters and initial conditions

Returning to the interpretation of the system of ordinary differential equations as a mathematical formalization of a finite dimensional deterministic evolutionary process, it should be clear that existence and uniqueness of solutions are not the only desirable properties of solutions. It is also reasonable to expect that solutions will depend continuously on the initial conditions and parameters of the system.

Definition 2.29. *A problem is called well posed is it has a solution, the solution is unique, and depends continuously on the initial data and parameters of the problem.*

The goal of this section is to prove that

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f} \colon J \times X \longrightarrow \mathbf{R}^k, \quad J \subseteq \mathbf{R}, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in X \tag{2.8}$$

is well posed.

The problem (2.8) does not include any parameters. This can be done because any system that depends explicitly on parameters, such as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu}), \quad \boldsymbol{\mu} \in \mathbf{R}^m, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0,$$

can be reformulated as

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{y}), \quad \dot{\boldsymbol{y}} = 0, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad \boldsymbol{y}(t_0) = \boldsymbol{\mu},$$

i.e., in the form (2.8). Conversely, if one is given a system in the form (2.8), then by introducing $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$ I find a new system

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(t, \boldsymbol{y} + \boldsymbol{x}_0) =: \boldsymbol{g}(t, \boldsymbol{y}, \boldsymbol{x}_0), \quad \boldsymbol{y}(t_0) = 0,$$

i.e., I can talk about the initial data and parameters interchangeably.

First I will need *Gronwall's inequality.*

Lemma 2.30. *Let* $u\colon [0,T] \longrightarrow \mathbf{R}$ *be continuous and nonnegative. Suppose that* $\alpha \geq 0$, $\beta \geq 0$ *are such that*

$$u(t) \leq \alpha + \int_0^t \beta u(\tau)\,\mathrm{d}\tau, \quad t \in [0,T].$$

Then

$$u(t) \leq \alpha e^{\beta t}, \quad t \in [0,T].$$

Proof. Let $\alpha > 0$ and hence

$$U(t) := \alpha + \int_0^t \beta u(\tau)\,\mathrm{d}\tau > 0.$$

By the assumption $u(t) \leq U(t)$. By differentiating $U(t)$

$$U'(t) = \beta u(t),$$

or

$$\frac{U'(t)}{U(t)} = \frac{\beta u(t)}{U(t)} \leq \beta \Longrightarrow \frac{\mathrm{d}}{\mathrm{d}t} \log U(t) \leq \beta \Longrightarrow \log U(t) \leq \log U(0) + \beta t.$$

Since $U(0) = \alpha$ then by exponentiation

$$U(t) \leq \alpha e^{\beta t} \Longrightarrow u(t) \leq \alpha e^{\beta t}.$$

If $\alpha = 0$ then I apply the above argument for a sequence of positive α_i that tend to zero. ■

Exercise 2.29. Prove that if

$$\phi(t) \leq \delta_1 \int_{t_0}^t \psi(s)\phi(s)\,\mathrm{d}s + \delta_3,$$

for continuous functions ϕ and ψ, $\phi(t) \geq 0$ and $\psi(t) \geq 0$, $\delta_1 > 0$, $\delta_3 > 0$, then

$$\phi(t) \leq \delta_3 e^{\delta_1 \int_{t_0}^t \psi(s)\,\mathrm{d}s}.$$

Additionally prove that if

$$\phi(t) \leq \delta_2(t - t_0) + \delta_1 \int_{t_0}^t \phi(s)\,\mathrm{d}s + \delta_3,$$

for a continuous nonnegative ϕ, and $\delta_1 > 0, \delta_2 \geq 0, \delta_3 \geq 0$, then

$$\phi(t) \leq \left(\frac{\delta_2}{\delta_1} + \delta_3\right) e^{\delta_1(t-t_0)} - \frac{\delta_2}{\delta_1}.$$

Theorem 2.31. *Assume that* $\boldsymbol{f}$ *is continuous and Lipschitz continuous in* $\boldsymbol{x}$ *with constant* L. *Let* $t \mapsto \boldsymbol{\phi}(t;\boldsymbol{x}_1)$ *and* $t \mapsto \boldsymbol{\phi}(t;\boldsymbol{x}_2)$ *be solutions to* $\dot{\boldsymbol{x}} = \boldsymbol{f}(t,\boldsymbol{x})$ *with the initial conditions* $\boldsymbol{x}(t_0) = \boldsymbol{x}_1$ *and* $\boldsymbol{x}(t_0) = \boldsymbol{x}_2$ *respectively. Then*

$$|\boldsymbol{\phi}(t;\boldsymbol{x}_1) - \boldsymbol{\phi}(t;\boldsymbol{x}_2)| \leq |\boldsymbol{x}_1 - \boldsymbol{x}_2| e^{L(t-t_0)}.$$

Proof. Define

$$v(t) := |\boldsymbol{\phi}(t; \boldsymbol{x}_1) - \boldsymbol{\phi}(t; \boldsymbol{x}_2)|.$$

Since

$$\boldsymbol{\phi}(t; \boldsymbol{x}_1) - \boldsymbol{\phi}(t; \boldsymbol{x}_2) = \boldsymbol{x}_1 - \boldsymbol{x}_2 + \int_{t_0}^{t} \Big(\boldsymbol{f}(\tau, \boldsymbol{\phi}(\tau; \boldsymbol{x}_1)) - \boldsymbol{f}(\tau, \boldsymbol{\phi}(\tau; \boldsymbol{x}_2))\Big)\,\mathrm{d}\tau,$$

then

$$v(t) \le v(t_0) + \int_{t_0}^{t} Lv(\tau)\,\mathrm{d}\tau.$$

Applying Lemma 2.30 to the function $u(t) = v(t + t_0)$ yields

$$v(t) \le v(t_0)e^{L(t-t_0)},$$

which concludes the proof. ■

It is also useful to know that if $\boldsymbol{f} \in \mathcal{C}^{(p)}$, then the solution to the problem (2.5) depends on the initial conditions differentiably. Since the initial conditions and parameters of the problem can be interchanged then a similar statement applies to a parameter dependent problem. In particular, it is true that

Theorem 2.32. *Let $\boldsymbol{f} \in \mathcal{C}^{(p)}(U; \mathbf{R}^k)$, $U = J \times X \times W$. Then the problem $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{\mu})$, $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ has p times continuously differentiable solution $\boldsymbol{\phi}(t; t_0, \boldsymbol{x}_0, \boldsymbol{\mu})$ with respect to the variables $(t, t_0, \boldsymbol{x}_0, \boldsymbol{\mu})$.*

Proof. See the textbook. ■

This theorem is important because it justifies some of the basic *perturbation methods*. Assume that a problem of the form

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \epsilon), \quad \boldsymbol{x}(0) = \boldsymbol{\eta}$$

is given, depending on the scalar (for simplicity) parameter ϵ. Assume that $\boldsymbol{f} \in \mathcal{C}^{(p)}$. Then it means that there are p continuously differentiable derivatives of the solution with respect to this parameter, in particular this means that the solution can be represented by the Taylor formula

$$\boldsymbol{x}(t) = \boldsymbol{x}_0(t) + \boldsymbol{x}_1(t)\epsilon + \boldsymbol{x}_2(t)\epsilon^2 + \ldots + \boldsymbol{x}_p(t)\epsilon^p + \mathcal{O}(|\epsilon|^{p+1}),$$

where $\boldsymbol{x}_i$ is related to the i-th derivative of the solution with respect to ϵ and evaluated at $\epsilon = 0$. Here $g := \mathcal{O}(f)$ when $u \to a$ is a function such that $|g(u)| \le C|f(u)|$ for some constant C and $|u - a| < \delta > 0$. Hence $\boldsymbol{x}_0$ is the solution of the unperturbed problem

$$\dot{\boldsymbol{x}}_0 = \boldsymbol{f}(t, \boldsymbol{x}_0, 0) = \boldsymbol{f}_0(t, \boldsymbol{x}_0), \quad \boldsymbol{x}_0(0) = \boldsymbol{\eta}.$$

To find the equations to determine other $\boldsymbol{x}_i$ I can either formally differentiate the original equation with respect to ϵ, or, simpler, plug $\boldsymbol{x}$ into the original equation, expand the corresponding series and equal the coefficients at the same powers of ϵ.

Example 2.33. Consider the IVP

$$\ddot{x} + \omega^2 x = \epsilon x^3, \quad x(0) = 1, \quad \dot{x}(0) = 0,$$

where $\omega > 0$ is a constant. Let me find $\frac{\partial x}{\partial \epsilon}(t,\epsilon)\big|_{\epsilon=0} =: x_1(t)$. First, I note that if $\epsilon = 0$ then I have

$$\ddot{x}_0 + \omega^2 x_0 = 0, \quad x_0(0) = 1, \quad \dot{x}_0(0) = 0,$$

with the solution

$$x_0(t) = \cos \omega t.$$

Differentiating the original equation with respect to ϵ I find

$$\ddot{x}_1 + \omega^2 x_1 = \cos^3 \omega t = \frac{1}{4}(\cos 3\omega t + 3\cos \omega t),$$

with the initial conditions

$$x_1(0) = 0, \quad \dot{x}_1(0) = 0.$$

The last problem can be solved to find that

$$x_1(t) = \frac{\partial x}{\partial \epsilon}(t)\bigg|_{\epsilon=0} = \frac{3}{8\omega} t \sin \omega t + \frac{1}{32\omega^2}(\cos \omega t - \cos 3\omega t).$$

Finally, I can use the found expression to approximate the unknown solution to the original problem as

$$x(t) = x_0(t) + x_1(t)\epsilon + \mathcal{O}(|\epsilon|^2).$$

Exercise 2.30. Find the derivative of the solution to $\dot{x} = x^2 + x \sin t$ with respect to the initial condition $x(0) = a$ at $a = 0$.

Answer: $e^{1-\cos t}$.

Exercise 2.31. Find the derivative of the solution to the pendulum equation $\ddot{\theta} + \sin\theta = 0$ with the initial condition $\theta(0) = a$, $\dot{\theta}(0) = 0$ with respect to a at $a = 0$.

Answer: $\cos t$.

Exercise 2.32. Let

$$\dot{x} = f(t, x), \quad x(\tau) = a,$$

and assume that the solution $x = x(t, \tau, a) = x(t, \tau)$. Assume also that the solution at a specific time moment $\tau = t_0$ is known and given by $t \mapsto x_0(t, t_0)$. Find the initial value problem for

$$x_1(t, t_0) := \frac{\partial x}{\partial \tau}(t, \tau)|_{\tau = t_0}.$$

Answer:

$$\dot{x}_1(t, t_0) = \frac{\partial f}{\partial x}(t, x_0(t, t_0)) x_1(t, t_0), \quad x_1(t_0, t_0) = -f(t_0, a).$$

2.7 On extending solutions

Consider

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \quad \boldsymbol{x}(t) \in X \subseteq \mathbf{R}^k, \quad \boldsymbol{f}\colon J \times X \longrightarrow X, \quad J \subseteq \mathbf{R}, \tag{2.9}$$

where $J \times X$ is an open subset of $\mathbf{R}^{k+1}$, together with the initial conditions

$$\boldsymbol{x}(t_0) = \boldsymbol{x}_0, \quad (t_0, \boldsymbol{x}_0) \in J \times X. \tag{2.10}$$

Assume that in $J \times X$ the conditions of the existence and uniqueness theorem are satisfied (i.e., $\boldsymbol{f}$ is continuous in t and locally Lipschitz continuous in $\boldsymbol{x}$ in X). I showed in the first chapter by the way of an example that solutions to a perfectly fine differential equation can be defined only on a finite interval (e.g., $\dot{x} = 1 + x^2$), and the existence and uniqueness theorem is essentially local. Therefore, an important question is what can be said about extending solutions to a bigger interval than it is promised by the existence and uniqueness theorem, which usually gives quite a conservative estimate.

I start with an almost obvious lemma.

Lemma 2.34. *Suppose two solutions $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ to* (2.9), (2.10) *are defined on the same open interval I containing t_0. Then $\boldsymbol{\phi}_1(t) = \boldsymbol{\phi}_2(t)$ for all $t \in I$.*

Proof. By the existence and uniqueness theorem $\boldsymbol{\phi}_1(t) = \boldsymbol{\phi}_2(t)$ in some open interval containing t_0; the union of all such open intervals is the largest interval I^* around t_0 on which $\boldsymbol{\phi}_1(t) = \boldsymbol{\phi}_2(t)$. Let $I^* \neq I$. Then I^* has an end point $t_1 \in I$, and by continuity $\boldsymbol{\phi}_1(t_1) = \boldsymbol{\phi}_2(t_1)$, therefore, by the existence and uniqueness theorem $\boldsymbol{\phi}_1(t) = \boldsymbol{\phi}_2(t)$ in some I', an open interval containing t_1, then $\boldsymbol{\phi}_1(t) = \boldsymbol{\phi}_2(t)$ in $I^* \cup I'$ which is larger than I^*. A contradiction. ■

This lemma implies that for each $\boldsymbol{x}_0 \in X$ there exists a maximal open interval (α, β), containing t_0, on which there is unique solution $\boldsymbol{\phi}$ with $\boldsymbol{\phi}(t_0) = \boldsymbol{x}_0$. This $\boldsymbol{\phi}$ is called the *maximal* solution through $(t_0, \boldsymbol{x}_0)$. If $(\alpha, \beta) = (-\infty, \infty)$ then the maximal solution is called the *global* solution. Each global solution is maximal but not vise versa.

Next, I would like to prove that the solution to (2.9), (2.10) leaves any compact subset of $J \times X$ when $t \to \beta$ or $t \to \alpha$. To make things a little simpler, I will prove this for an autonomous system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in X, \quad \boldsymbol{f}\colon X \longrightarrow X, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0, \tag{2.11}$$

(but do remember that any non-autonomous system can be turned into an autonomous system with the same maximal interval of existence).

Theorem 2.35. *Let $t \mapsto \boldsymbol{\phi}(t)$ be the maximal solution to* (2.11) *on $(\alpha, \beta) \subset \mathbf{R}$ with $\alpha > -\infty$ or $\beta < +\infty$ or both. Then given any compact set $U \subset X$, solution $\boldsymbol{\phi}$ leaves U as $t \to \alpha$ or $t \to \beta$.*

The proof is by contradiction. I assume that $\boldsymbol{\phi}(t) \in U$ for all $t \in (\alpha, \beta)$ and show that $I = (\alpha, \beta)$ cannot be the maximal interval.

Proof. Let $\boldsymbol{\phi}(t) \in U$ for all t, and since $\boldsymbol{f}$ is continuous then there exists $M > 0$ such that $|\boldsymbol{f}(\boldsymbol{x})| \leq M$ for any $\boldsymbol{x} \in U$. The equality

$$|\boldsymbol{\phi}(t_0) - \boldsymbol{\phi}(t_1)| = \left| \int_{t_0}^{t_1} \boldsymbol{\phi}'(\tau)\, \mathrm{d}\tau \right| \leq \int_{t_0}^{t_1} |\boldsymbol{f}(\boldsymbol{\phi}(\tau))|\, \mathrm{d}\tau \leq |t_1 - t_0| M$$

implies that $\boldsymbol{\phi}$ is uniformly continuous on I, and hence extends to a continuous map $[\alpha, \beta] \longrightarrow X$ (see Exercise 2.33).

Now I will look only at the right part of my interval, the left one can be treated similarly. Let $\gamma \in [\alpha, \beta]$, and I state that

$$\boldsymbol{\phi}(t) = \boldsymbol{\phi}(\gamma) + \int_{\gamma}^{t} \boldsymbol{f}(\boldsymbol{\phi}(\tau))\, \mathrm{d}\tau$$

for all $t \in [\gamma, \beta]$. The only problem can be at $t = \beta$, but since

$$\begin{aligned}\boldsymbol{\phi}(\beta) &= \boldsymbol{\phi}(\gamma) + \lim_{t \to \beta} \int_{\gamma}^{t} \boldsymbol{\phi}'(\tau)\, \mathrm{d}\tau = \\ &= \boldsymbol{\phi}(\gamma) + \lim_{t \to \beta} \int_{\gamma}^{t} \boldsymbol{f}(\phi(\tau))\, \mathrm{d}\tau = \\ &= \boldsymbol{\phi}(\gamma) + \int_{\gamma}^{\beta} \boldsymbol{f}(\phi(\tau))\, \mathrm{d}\tau,\end{aligned}$$

therefore $\boldsymbol{\phi}(t)$ is differentiable at $t = \beta$ and satisfies (2.11), which implies, due to the existence and uniqueness theorem, that there is a solution on the interval $[\beta, \delta)$ with $\delta > 0$, which contradicts that I is the maximal interval. ■

Exercise 2.33. There is still a gap in the given proof. Prove that if $f\colon (\alpha, \beta) \longrightarrow \mathbf{R}$ is a uniformly continuous function then f can be extended to a continuous function on the domain $[\alpha, \beta]$. What if f is just continuous?

Now I state several important corollaries to the proved theorem.

Corollary 2.36. *If for the autonomous problem* (2.11) *the solutions are contained in a compact set they must be global.*

Corollary 2.37. *If for the maximal interval of existence its left or right (or both) boundaries are finite, then the solution blows up (its norm approaches infinity).*

Finally we can discuss what happens in non-autonomous cases. Note that in this case we also must consider a possibility that $\lim_{t \to \beta} \boldsymbol{x}(t)$ does not exists. For example, it can be directly checked that the function $x(t) = \sin(1/t)$ solves $\dot{x} = -t^2 \cos(1/t)$ in the domain $t < 0, -\infty < x < \infty$. $\lim_{t \to -0} x(t)$ however does not exist. In words

Corollary 2.38. *Consider problem* (2.9), (2.10), *with* $J = (c, d)$ (*such that* J *can also have* $c = -\infty$ *and* $d = +\infty$), *and assume that* X *is a bounded set with the boundary* ∂X. *Then either* $\beta = d$ *or* $\boldsymbol{x}(t) \to \partial X$ *as* $t \to \beta$, *and either* $c = \alpha$ *or* $\boldsymbol{x}(t) \to \partial X$ *as* $t \to \alpha$. *If* $X = \mathbf{R}^k$ *then either* $\beta = d$ *or* $|\boldsymbol{x}(t)| \to \infty$ *as* $t \to \beta$, *and either* $c = \alpha$ *or* $|\boldsymbol{x}(t)| \to \infty$ *as* $t \to \alpha$.

Here is a very useful corollary

Corollary 2.39. *Assume that for problem* (2.9), (2.10) $J = \mathbf{R}$, $X = \mathbf{R}^k$ *and for any constant* $T > 0$ *there are constants* $M(T)$, $L(T)$ *such that*

$$|\boldsymbol{f}(t, \boldsymbol{x})| \leq M(T) + L(T)|\boldsymbol{x}|, \quad (t, \boldsymbol{x}) \in [-T, T] \times \mathbf{R}^k.$$

Then all the solutions to (2.9), (2.10) *are global.*

Proof. Using the estimate above, I find that

$$|\boldsymbol{\phi}(t)| \leq |\boldsymbol{x}_0| + \int_0^t (M + L|\boldsymbol{\phi}(\tau)|)\,\mathrm{d}\tau, \quad t \in [0, T].$$

Gronwall-type inequality inequality implies (see Exercise 2.29)

$$|\boldsymbol{\phi}(t)| \leq |\boldsymbol{x}_0| e^{LT} + \frac{M}{L}(e^{LT} - 1).$$

Therefore, for each T the solutions lie in a bounded set, and therefore defined for all $t \in [0, T]$. Since T is arbitrary, the solutions can be extended for $T \to +\infty$. Similar considerations for the left half interval finish the proof. ∎

Exercise 2.34. For which a any solution to

$$\dot{x} = (x^2 + e^t)^a$$

can be extended to the real line $-\infty < t < +\infty$?

Exercise 2.35. Let $\dot{x} = f(t, x)$ and $f\colon \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$. Suppose that $xf(t, x) < 0$ for $x > R$. Show that all the solutions exist for all $t > 0$.

Finally, consider again the autonomous system (2.11) defined on all $\mathbf{R}^k$. I know that it is possible to have solutions not defined globally. Together with (2.11) consider

$$\dot{\boldsymbol{x}} = \frac{\boldsymbol{f}(\boldsymbol{x})}{1 + |\boldsymbol{f}(\boldsymbol{x})|}.$$

For the last system I have that

$$|\dot{\boldsymbol{x}}| \leq 1,$$

and therefore the conditions from the last corollary are satisfied, and the solutions to this systems are defined globally. It is possible to show that if I have two vector fields $\boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} \mapsto h(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x})$, where $h(\boldsymbol{x}) > 0$, then the images of the corresponding solutions in the phase space coincide, which in a sense shows that I can always to reparametrize the time in autonomous system to guarantee that all the solutions are global.

2.8 Autonomous systems of ODE and dynamical systems

Recall that a differentiable (finite dimensional) dynamical system is a triple $\{\mathbf{R}, X, \varphi^t\}$, where φ^t is a $\mathcal{C}^{(1)}$ map, $\{\varphi^t\}\colon \mathbf{R} \times X \longrightarrow X$, X is an open subset of $\mathbf{R}^k$, called the phase space, and the one-parameter family of evolutionary operators $\{\varphi^t\}$, which is also called a flow, satisfies

$$\begin{aligned} &\varphi^0 \boldsymbol{x} = \boldsymbol{x}, \\ &\varphi^{\tau + t}\boldsymbol{x} = \varphi^\tau \varphi^t \boldsymbol{x} = \varphi^t \varphi^\tau \boldsymbol{x}, \quad \tau, t \in \mathbf{R}. \end{aligned}$$

Also recall that every differentiable dynamical system defines the vector field

$$\boldsymbol{f}(\boldsymbol{x}) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\varphi^t \boldsymbol{x}\right|_{t=0},$$

and therefore an autonomous system of ODE

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}), \quad \boldsymbol{x}(t) \in X. \tag{2.12}$$

Now I am prepared to state a converse implication. Consider system (2.11) and assume that $\boldsymbol{f} \in \mathcal{C}^{(1)}(X;X)$. It means that for any point $\boldsymbol{x}_0 \in X$ there exists a unique solution, which I will denote as

$$(t, \boldsymbol{x}_0) \mapsto \boldsymbol{x}(t; \boldsymbol{x}_0)$$

to emphasize the dependence on the initial condition. I do not have to include the initial time moment t_0 in this notation, since for the autonomous system (2.11) if $t \mapsto \boldsymbol{\phi}(t)$ is a solution then $t \mapsto \boldsymbol{\phi}(t-t_0)$ is also a solution (recall one of the earlier exercises), and therefore I can always shift the initial condition to zero. This solution can be extended on the maximal interval of existence $I(\boldsymbol{x}_0)$, which depends on $\boldsymbol{x}_0$. The map

$$\boldsymbol{x} \colon \Omega \longrightarrow X,$$

where

$$\Omega = \{(t, \boldsymbol{x}_0) \in \mathbf{R} \times X \mid t \in I(\boldsymbol{x}_0)\}$$

defines the flow of the system (2.12). Indeed, I know that if $\boldsymbol{f} \in \mathcal{C}^{(1)}$ then $\boldsymbol{x} \colon \Omega \longrightarrow X$ is also $\mathcal{C}^{(1)}$ (recall the differentiable dependence on the initial conditions).

By the definition I have that

$$\boldsymbol{x}(0; \boldsymbol{x}_0) = \boldsymbol{x}_0,$$

i.e., the first property of the flow is fulfilled.

Finally, I will show that

$$\boldsymbol{x}(t_2 + t_1; \boldsymbol{x}_0) = \boldsymbol{x}(t_2; \boldsymbol{x}(t_1; \boldsymbol{x}_0)) = \boldsymbol{x}(t_1; \boldsymbol{x}(t_2; \boldsymbol{x}_0)), \quad t_1,\, t_2,\, t_2 + t_1 \in I(\boldsymbol{x}_0).$$

To prove it, consider two solutions

$$\boldsymbol{\phi}_1(t) := \boldsymbol{x}(t; \boldsymbol{x}(t_1; \boldsymbol{x}_0)), \quad \boldsymbol{\phi}_2(t) := \boldsymbol{x}(t + t_1; \boldsymbol{x}_0).$$

Since

$$\boldsymbol{\phi}_1(0) = \boldsymbol{\phi}_2(0)$$

then, by the existence and uniqueness theorem,

$$\boldsymbol{\phi}_1(t) = \boldsymbol{\phi}_2(t),$$

where they defined, and in particular at $t = t_2$, which proves the first of the equalities. The second one is proved in a similar way.

Summarizing, the autonomous system (2.12) defines a *local* differentiable dynamical system, whose flow is defined as

$$\varphi^t \boldsymbol{x}_0 = \boldsymbol{x}(t; \boldsymbol{x}_0), \quad t \in I(\boldsymbol{x}_0),$$

and satisfies the group property

$$\boldsymbol{x}(t + \tau; \boldsymbol{x}_0) = \boldsymbol{x}(t; \boldsymbol{x}(\tau; \boldsymbol{x}_0)), \quad \tau, t, \tau + t \in I(\boldsymbol{x}_0)$$

in the sense that if one side of this equality is defined, so is the other, and they are equal.

If all the solutions to (2.12) are global, then system (2.12) defines a global differentiable dynamical system (i.e., the group property is valid for all $t, \tau \in \mathbf{R}$).

2.9 Appendix

2.9.1 Inverse and implicit function theorems

Several times in this course I refer to either *inverse* or *implicit function* theorems. In this appendix I state these theorems and also show how they are connected to the theory of ODE.

Consider the following system of n nonlinear equations with n unknowns:

$$\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{y}), \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbf{R}^n,$$

and assume that $\boldsymbol{f} \in \mathcal{C}^{(1)}$ in some neighborhood of the point $\boldsymbol{a} \in \mathbf{R}^n$, $\boldsymbol{b} = \boldsymbol{f}(\boldsymbol{a})$. The inverse function theorem deals with the situation when unknown $\boldsymbol{y}$ can be expressed locally as the function of $\boldsymbol{x}$:

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}), \quad \boldsymbol{a} = \boldsymbol{g}(\boldsymbol{b}).$$

The function $\boldsymbol{g}$ is called the inverse function to $\boldsymbol{f}$, and the existence of such function is equivalent to saying that $\boldsymbol{f}$ is a bijection (one-to-one and onto) from some neighborhood U of $\boldsymbol{a}$ to some neighborhood V of $\boldsymbol{b}$.

Heuristically, using Taylor's formula for $\boldsymbol{f}$:

$$\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{a}) + \boldsymbol{f}'(\boldsymbol{a})(\boldsymbol{y} - \boldsymbol{a}) + o(|\boldsymbol{y} - \boldsymbol{a}|),$$

where

$$\boldsymbol{f}'(\boldsymbol{a}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\boldsymbol{a}) & \cdots & \frac{\partial f_n}{\partial x_n}(\boldsymbol{a}) \end{bmatrix}$$

is the *Jacobi matrix* of $\boldsymbol{f}$ evaluated at $\boldsymbol{a}$. Dropping the terms $o(|\boldsymbol{y} - \boldsymbol{a}|)$ I get

$$\boldsymbol{x} - \boldsymbol{b} = \boldsymbol{f}'(\boldsymbol{a})(\boldsymbol{y} - \boldsymbol{a}),$$

which is a linear system and therefore has a unique solution if and only if $\det \boldsymbol{f}'(\boldsymbol{a}) \neq 0$. It turns out that exactly the same condition is sufficient for $\boldsymbol{f}$ to be a bijection in U.

Theorem 2.40. *Let $\boldsymbol{a}, \boldsymbol{b} \subseteq \mathbf{R}^n$ and $\boldsymbol{f} \in \mathcal{C}^{(1)}$ in some neighborhood of $\boldsymbol{a}$, $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{b}$. Assume that* $\det \boldsymbol{f}'(\boldsymbol{a}) \neq 0$. *Then there exist open $U, V \subseteq \mathbf{R}^n$, $\boldsymbol{a} \in U$, $\boldsymbol{b} \in V$ such that*
(a) *$\boldsymbol{f}$ is a bijection from U to V;*
(b) *The inverse function $\boldsymbol{g} \in \mathcal{C}^{(1)}(V; U)$.*

Exercise 2.36. Prove part (a) of Theorem 2.40. *Hint*: Assume without loss of generality that $\boldsymbol{a} = \boldsymbol{b} = 0$ and consider the equation $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{y}$, where the operator $\boldsymbol{A}$ is given by $\boldsymbol{A}\colon \boldsymbol{y} \mapsto \boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{h}(\boldsymbol{y}))$, where $\boldsymbol{B} = \boldsymbol{f}'(\boldsymbol{a})$, and $\boldsymbol{h}(\boldsymbol{y}) = o(|\boldsymbol{y} - \boldsymbol{a}|)$. Apply the contraction mapping principle to this equation.

Now consider the system

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = 0, \quad \boldsymbol{F}\colon \mathbf{R}^{n+m} \longrightarrow \mathbf{R}^m,\ \boldsymbol{x} \in \mathbf{R}^n,\ \boldsymbol{y} \in \mathbf{R}^m.$$

I need to identify the conditions when I can express uniquely variables $\boldsymbol{y}$ through $\boldsymbol{x}$:

$$\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}),$$

or

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) = 0.$$

Theorem 2.41. *Assume that* $\boldsymbol{a} \in \mathbf{R}^n$, $\boldsymbol{b} \in \mathbf{R}^m$ *and* $\boldsymbol{F}(\boldsymbol{a}, \boldsymbol{b}) = 0$. *Assume that* $\boldsymbol{F} \in \mathcal{C}^{(1)}$ *in some neighborhood of the point* $(\boldsymbol{a}, \boldsymbol{b})$ *and* $\det \boldsymbol{F'}_{\boldsymbol{y}}(\boldsymbol{a}, \boldsymbol{b}) \neq 0$, *where* $\boldsymbol{F'}_{\boldsymbol{y}}(\boldsymbol{a}, \boldsymbol{b})$ *is the Jacobi matrix of* $\boldsymbol{F}$ *with respect to* $\boldsymbol{y}$. *Then there exist open* $U, V \subseteq \mathbf{R}^n$, $\boldsymbol{a} \in U$, $\boldsymbol{b} \in V$ *such that*

(*a*) *There exists a unique* $\boldsymbol{g} \colon U \longrightarrow V$, *which satisfies*

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) \equiv 0, \quad \boldsymbol{x} \in U;$$

(*b*) $\boldsymbol{g} \in \mathcal{C}^{(1)}(U; V)$.

Exercise 2.37. Prove Theorem 2.41. *Hint*: Consider an auxiliary problem

$$\boldsymbol{F}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{z}, \quad \boldsymbol{x} = \boldsymbol{x},$$

and apply to it Theorem 2.40.

Exercise 2.38. Prove Theorem 2.40 from Theorem 2.41.

The stated theorems and exercises show that implicit and inverse function theorems are equivalent, and their proof can be performed by the same device that I used to prove the existence and uniqueness theorem for ODE. These facts imply that there is a strong connection between these theorems and the theory of differential equations[7]. In particular, I can even prove the (version of) implicit function theorem using my main result on ODE theory.

Theorem 2.42. *Let* $U \subseteq \mathbf{R}^{k+1}$ *and* $\boldsymbol{F} \in \mathcal{C}^{(2)}(U; \mathbf{R}^k)$. *Assume that* $\boldsymbol{F}(t_0, \boldsymbol{x}_0) = 0$, $(t_0, \boldsymbol{x}_0) \in \mathbf{R}^{k+1}$ *and the Jacobi matrix* $\boldsymbol{F'}_{\boldsymbol{x}}(t_0, \boldsymbol{x}_0)$ *is nonsingular. Then there exists an open interval* $(t - \delta, t + \delta) =: I$ *and* $\boldsymbol{\phi} \in \mathcal{C}^{(1)}(I; \mathbf{R}^k)$ *such that* $\boldsymbol{\phi}(t_0) = \boldsymbol{x}_0$ *and*

$$\boldsymbol{F}(t_0, \boldsymbol{\phi}(t)) \equiv 0, \quad t \in I.$$

Proof. I will consider only the case $k = 1$, more can be found in the given reference.

Pick $a, b > 0$ such that $(t - a, t + a) \times (x - b, x + b) \in U$ and $\partial F / \partial x(t, x) \neq 0$ in this rectangular. Hence I can define

$$f(t, x) = -\frac{F'_t(t, x)}{F'_x(t, x)}.$$

Consider the ODE

$$\dot{x} = f(t, x), \quad x(t_0) = x_0.$$

I know that there exists a unique on some interval I solution ϕ to this problem. Note that $F(t_0, \phi(t_0)) = F(t_0, x_0) = 0$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} F(t, \phi(t)) = \frac{\partial F}{\partial t}(t, \phi(t)) + \phi'(t) \frac{\partial F}{\partial x}(t, \phi(t)) = 0,$$

therefore

$$F(t, \phi(t)) \equiv 0, \quad t \in I.$$

■

Remark 2.43. Actually, it can be shown that $\boldsymbol{F} \in \mathcal{C}^{(1)}(U; \mathbf{R}^k)$ is enough to prove the theorem.

Remark 2.44. I showed that the implicit function theorem is a consequence of the existence and uniqueness theorem for ODE. The opposite is also true, but I will postpone the discussion of this fact till the end of the next chapter.

[7]Much more can be found in Krantz, S. G., & Parks, H. R. (2012). *The implicit function theorem: history, theory, and applications.* Springer Science & Business Media.