3.7 Non-autonomous linear systems of ODE. General theory

Now I will study the ODE in the form
\[ \dot{x} = A(t)x + g(t), \quad x(t) \in \mathbb{R}^k, \quad A, g \in C(I), \]  
where now the matrix \( A \) is time dependent and continuous on some \( I \subseteq \mathbb{R} \).

The initial condition is now
\[ x(t_0) = x_0, \quad (t_0, x_0) \in I \times \mathbb{R}^k. \]

**Theorem 3.20.** Let the matrix-function \( A \) and the vector-function \( g \) be continuous on some interval \( I \subseteq \mathbb{R} \). Then the solution to (3.10), (3.11) exists, unique and extends to the whole interval \( I \).

**Proof.** Problem (3.10), (3.11) satisfies the conditions of the existence and uniqueness theorem. Moreover, since
\[ |A(t)x + g(t)| \leq \|A(t)\| |x| + |g(t)| \leq L|x| + M, \]
for some \( L > 0, M > 0 \), therefore, by Corollary 2.37, this solution can be extended to the whole interval \( I \). \( \blacksquare \)

Note the global character of the theorem.

Together with (3.10) consider the corresponding homogeneous system
\[ \dot{x} = A(t)x, \quad x(t) \in \mathbb{R}^k, \quad A \in C(I), \]

**Exercise 3.42.** For the first order linear homogeneous ODE
\[ \dot{x} = a(t)x \]
the solution is given by
\[ x(t) = x_0e^{\int_{t_0}^t a(\tau) \, d\tau}. \]

A naive approach would be to solve problem (3.12) by writing
\[ x(t) = e^{\int_{t_0}^t A(\tau) \, d\tau} x_0. \]

Consider the matrix
\[ A(t) = \begin{bmatrix} 0 & 0 \\ 1 & t \end{bmatrix} \]
and find its solution directly. Also find \( e^{\int_{t_0}^t A(\tau) \, d\tau} \) and show that at least in this particular case this formula does not give a solution to the problem.

Explain, what when wrong in this example and give a condition on matrix \( A(t), t \in I \) such that the matrix exponent formula would work.

**Theorem 3.21** (Principle of superposition).

(a) If \( x_1, x_2 \) solve (3.12) then their linear combination \( \alpha_1 x_1 + \alpha_2 x_2 \) also solves (3.12).

(b) If \( x_1, x_2 \) solve (3.10) then their difference \( x_1 - x_2 \) solves (3.12).

(c) Any solution to (3.12) can be represented as a sum of a particular (fixed) solution to (3.10) and some solution to (3.12).
Proof. (a) and (b) follow from the linearity of the operator \( \frac{d}{dt} - A(t) \) acting on the space of continuously differentiable on \( I \) vector functions \( x: I \rightarrow \mathbb{R}^k \). To show (c) fix some solution \( x_0 \) to (3.10). Assume that arbitrary solution to (3.10) is given by \( x = x_0 + x_h \) for some function \( x_h \). From this, \( x_h = x - x_0 \) and therefore, due to (b), solves (3.12).

Actually the first point in the last theorem, together with the fact that \( x = 0 \) solves (3.12), can be restated as: The set of solutions to the homogeneous linear system (3.12) is a vector space. Therefore, it would be nice to figure out what is the dimension of this vector space (in this case any solution can be represented as a linear combination of basis vectors).

Let me first recall the notion of linear dependence and independence specifically applied to functions and vector functions.

**Definition 3.22.** The list of functions \( x_1, \ldots, x_k \) defined on \( I = (a, b) \) is called linearly dependent on \( I \) if there exist scalars \( \alpha_1, \ldots, \alpha_k \), not equal to zero simultaneously, such that

\[
\alpha_1 x_1(t) + \ldots + \alpha_k x_k(t) \equiv 0, \quad t \in I.
\]

If this list of functions is not linear independent then it is called linearly dependent on \( I \).

**Example 3.23.** Consider, e.g., the functions \( 1, t, t^2, \ldots, t^k \). These functions are linearly independent on any \( I \).

Another example of linearly independent functions on any \( I \) is given by \( e^{\lambda_1 t}, \ldots, e^{\lambda_k t} \), where all \( \lambda_j \) are distinct.

**Exercise 3.43.** Prove the statements from the example above.

**Exercise 3.44.** Decide whether these functions are linearly independent or not:

1. \( t + 2, \quad t - 2 \).
2. \( x_1(t) = t^2 - t + 3, \quad x_2(t) = 2t^2 + t, \quad x_3(t) = 2t - 4 \).
3. \( \log t^2, \quad \log 3t, \quad 7, \quad t \geq 0 \).
4. \( \sin t, \quad \cos t, \quad \sin 2t \).

The definition of linear independency verbatim can be used for the vector functions \( x_1, \ldots, x_k \) on \( I \) (write it down).

Let \( (x_j)_{j=1}^k, \quad x_j: I \rightarrow \mathbb{R}^k \) be a list of vector functions. The determinant

\[
W := \det(x_1|\ldots|x_k): I \rightarrow \mathbb{R},
\]

is called the Wronskian. I have the following important lemma.
Lemma 3.24.

(a) If the Wronskian of \((x_j)_{j=1}^k\) is different from zero at least at one point \(t_0 \in I\) then these functions are linearly independent.

(b) If \((x_j)_{j=1}^k\) are linearly dependent then their Wronskian is identically zero on \(I\).

(c) Let \((x_j)_{j=1}^k\) be solutions to linear system (3.12). If their Wronskian is equal to zero at least at one point \(t_0 \in I\) then these vector functions are linearly dependent.

Proof. (a) and (b) are the consequences of the standard facts from linear algebra and left as exercises. To show (c), assume that \(t_0\) is such that \(W(t_0) = 0\). It means that the linear combination

\[ x = \alpha_1 x_1 + \ldots + \alpha_k x_k \]

is such that \(x(t_0) = 0\) with not all \(\alpha_j\) equal to zero simultaneously. Due to the superposition principle, \(x\) solves (3.12) with \(x(t_0) = 0\). On the other hand, a vector function \(\tilde{x} \equiv 0\) also solves the same problem. Due to the uniqueness theorem \(x \equiv \tilde{x}\) and therefore \(\{x_1, \ldots, x_k\}\) are linearly dependent. ■

Exercise 3.45. Fill in the missed details in the proof above.

Remark 3.25. For arbitrary vector functions statement (c) from the lemma is not true. Consider, e.g.,

\[ x_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0 \\ t \end{bmatrix}, \]

which are linearly independent; their Wronskian, however, is identically zero.

Lemma 3.26. Let matrix \(X \in C^{(1)}(I; \mathbb{R}^{k^2})\) be invertible at \(t = t_0\). Then at \(t = t_0\)

\[ \frac{\det X'}{\det X} = \text{tr} \left( X'X^{-1} \right), \]

where the prime denotes the derivative with respect to \(t\).

Proof. Taylor’s formula tells me

\[ X(t_0 + h) = X(t_0) + hX'(t_0) + o(h), \quad h \to 0. \]

Now calculate the determinant

\[ \det X(t_0 + h) = \det X(t_0) \det (I + hB + o(h)), \]

where

\[ B := X'(t_0)X^{-1}(t_0). \]

Since, due to Lemma 3.5, \(\det(I + hB + o(h)) = 1 + h \text{tr} B + o(h)\), I have

\[ \frac{\det X(t_0 + h) - \det X(t_0)}{h} = \det X(t_0)(\text{tr} B + o(1)), \]

which proves the lemma. ■
Theorem 3.27 (Liouville’s formula or Abel’s identity). Let \( \mathbf{x}_1, \ldots, \mathbf{x}_k \) solve (3.12) and \( W \) be their Wronskian. Then

\[
W(t) = W(t_0) \exp \left( \int_{t_0}^{t} \text{tr} \, A(\tau) \, d\tau \right). \tag{3.13}
\]

Proof. If \( \mathbf{x}_1, \ldots, \mathbf{x}_k \) are linearly dependent, then \( W(t) \equiv 0 \) and the formula is true. Assume that \( \mathbf{x}_1, \ldots, \mathbf{x}_k \) are linearly independent and \( \mathbf{X} = (\mathbf{x}_1| \ldots | \mathbf{x}_k) \) be the matrix, whose \( j \)-th column is \( \mathbf{x}_j \). This matrix by construction solves the matrix differential equation

\[
\dot{\mathbf{X}} = A(t) \mathbf{X}.
\]

From the previous lemma I have

\[
\frac{W'(t)}{W(t)} = \text{tr} \left( \mathbf{X}' \mathbf{X}^{-1} \right) = \text{tr} \left( A(t) \mathbf{X} \mathbf{X}^{-1} \right) = \text{tr} \left( A(t) \right),
\]

which, after integration, implies (3.13).

Finally I am ready to prove the main theorem of the theory of linear homogeneous systems of ODE.

Definition 3.28. A fundamental system of solutions to (3.12) is the set of \( k \) linearly independent solutions. A fundamental matrix solution is the matrix composed of the fundamental set of solutions:

\[
\mathbf{X} = (\mathbf{x}_1| \ldots | \mathbf{x}_k).
\]

Theorem 3.29. The set of all solutions to (3.12) is a vector space of dimension \( k \).

This theorem basically states that to solve system (3.12) one needs to come up with a fundamental system of solutions, which form the basis of the space of solutions. To find any solution I need to find \( k \) (linearly independent) solutions. This is not true for nonlinear systems, and if I know a hundred (or more) of solutions to \( \dot{\mathbf{x}} = f(t, \mathbf{x}) \) it will not help me finding one more solution from those that I have.

Proof. First, I will show that the fundamental system of solutions exists. For this consider \( k \) IVPs for (3.12) with

\[
\mathbf{x}_j(t_0) = \mathbf{e}_j, \quad j = 1, \ldots, k,
\]

where \( \mathbf{e}_j \in \mathbb{R}^k \) are the standard unit vectors with 1 at the \( k \)-th position and 0 everywhere else. By construction, \( W(t_0) \neq 0 \) and hence \( \{ \mathbf{x}_j \} \) forms a fundamental system of solutions.

Now consider a solution \( \mathbf{x} \) to (3.12) with \( \mathbf{x}(t_0) = \mathbf{x}_0 \). Since \( \mathbf{e}_j \) are linearly independent, I have

\[
\mathbf{x}(t_0) = \alpha_1 \mathbf{x}_1(t_0) + \ldots + \alpha_k \mathbf{x}_k(t_0).
\]

Consider now the function

\[
\tilde{\mathbf{x}}(t) = \alpha_1 \mathbf{x}_1(t) + \ldots + \alpha_k \mathbf{x}_k(t),
\]

which by the superposition principle solves (3.12) and also satisfies \( \tilde{\mathbf{x}}(t_0) = \mathbf{x}(t_0) \), which, by the uniqueness theorem, implies that \( \mathbf{x}(t) \equiv \tilde{\mathbf{x}}(t) \), which means that any solution can be represented as a linear combination of the solutions in the fundamental system.
**Corollary 3.30.** If $X$ is a fundamental matrix solution, then any solution to (3.12) can be represented as

$$x(t) = X(t)\xi, \quad \xi \in \mathbb{R}^k,$$

where $\xi$ is an arbitrary constant vector.

Any two fundamental matrix solutions are related as

$$X(t) = \tilde{X}(t)C,$$

where $C$ is a constant matrix.

A fundamental matrix solution $X$ satisfying the condition $X(t_0) = I$ is called the **principal matrix solution** (at $t_0$) and can be found as

$$\Phi(t, t_0) = X(t)X^{-1}(t_0).$$

Using the variation of the constant method, it can be shown that if $\Phi(t, t_0)$ is the principal matrix solution to (3.12) then the general solution to (3.10) with the initial condition (3.11) can be written as

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)g(\tau)\,d\tau.$$

**Exercise 3.46.** Prove the last formula.

### 3.8 Linear $k$-th order equations with non-constant coefficients

#### 3.8.1 The general theory

Consider a linear $k$-th order differential equation

$$x^{(k)} + a_{k-1}(t)x^{(k-1)} + \ldots + a_1(t)x' + a_0(t)x = g(t), \quad (3.14)$$

where $a_j, g$ are assumed to be continuous on $I = (a, b)$. Together with (3.14) consider a linear homogeneous equation

$$x^{(k)} + a_{k-1}(t)x^{(k-1)} + \ldots + a_1(t)x' + a_0(t)x = 0, \quad (3.15)$$

and initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = x_1, \ldots, x^{(k-1)}(t_0) = x_{k-1}. \quad (3.16)$$

I know that problem (3.14), (3.16) (or (3.15), (3.16)) can be rewritten in the form of a system of $k$ first order equations, and therefore all the previous consideration can be applied. Let me spell them out.

Consider a system of $k-1$ times continuously differentiable functions $x_1, \ldots, x_k$. Their Wronskian is defined as

$$W(t) = \det \begin{bmatrix} x_1(t) & x_2(t) & \ldots & x_k(t) \\ x'_1(t) & x'_2(t) & \ldots & x'_k(t) \\ \vdots & \vdots & \ddots & \vdots \\ x_{k-1}'(t) & x_{k-1}'(t) & \ldots & x_{k-1}'(t) \end{bmatrix}.$$
• If \( W(t_0) \neq 0 \) then \((x_j)_{j=1}^k\) are linearly independent.

• Let \( x_1, \ldots, x_k \) be solutions to (3.15). If \( W = 0 \) at least at one point then these solutions are linearly dependent.

• Consider vector functions \( x_1, \ldots, x_k \) with components \((x_j, x'_j, \ldots, x_{(k-1)}^j)\), \(1 \leq j \leq k\). Then \((x_j)_{j=1}^k\) and \((x_j)_{j=1}^k\) are linearly dependent or independent simultaneously.

• The set of solutions to (3.15) is a vector space of dimension \( k \). The set of \( k \) linearly independent solutions to (3.15) is called the fundamental system of solutions.

• If \( W \) is the Wronskian of the solutions \( x_1, \ldots, x_k \) then I have Liouville's formula

\[
W(t) = W(t_0) \exp \left( -\int_{t_0}^t a_{k-1}(\tau) \, d\tau \right).
\]

• Using the formula for a particular solution to the nonhomogeneous system, I can write an explicit solution to (3.15), details are left as an exercise.

Exercise 3.47. Provide proofs for all the statements above.

3.8.2 Examples

Here I will discuss a few approaches of analysis of linear ODE, which can be used for specific equations.

Example 3.31 (Second order equation). Consider

\[
x'' + a(t)x' + b(t)x = 0.
\]

If \( x_1, x_2 \) solve this equation then

\[
W(t) = \begin{vmatrix} x_1(t) & x_2(t) \\
x_1'(t) & x_2'(t) \end{vmatrix}
\]

and Liouville’s formula takes the form

\[
\begin{vmatrix} x_1(t) & x_2(t) \\
x_1'(t) & x_2'(t) \end{vmatrix} = C \exp \left( -\int_{t_0}^t a(\tau) \, d\tau \right).
\]

Sometimes, if one particular solution is known, the second one can be found through the formula above.

For the special case

\[
x'' + q(t)x = 0
\]

I have

\[
\begin{vmatrix} x_1(t) & x_2(t) \\
x_1'(t) & x_2'(t) \end{vmatrix} = C.
\]

Or, after simplification,

\[
x_2'(t) - \frac{x_2(t)}{x_1(t)} x_1'(t) = \frac{C}{x_1(t)},
\]

which gives for \( x_2 \) a linear first order ODE, provided I know \( x_1 \).
Exercise 3.48. Two particular solutions

\[ y_1(t) = t - 1, \quad y_2(t) = \frac{t^2 - t + 1}{t} \]

are known for the differential equation

\[ (t^2 - 2t)y'' + 4(t - 1)y' + 2y = 6t - 6. \]

Find the general solution.

Example 3.32 (Solving nonhomogeneous equation). Assume that I need to solve

\[ x'' + a(t)x' + b(t)x = f(t), \]

and let \( x_1, x_2 \) be a fundamental system of solutions to the homogeneous equation. Let me look for a solution to the non-homogeneous equation in the form

\[ x(t) = c_1(t)x_1(t) + c_2(t)x_2(t), \]

where \( c_1, c_2 \) are unknown functions to be determined.

I have

\[ x' = c_1 x_1' + c_2 x_2' + [c_1' x_1 + c_2' x_2]. \]

I choose functions \( c_1, c_2 \) such that the expression in the square brackets is equal to zero. Then, plugging \( x \) into the original equation, I find

\[ c_1' x_1 + c_2' x_2 = 0, \]

\[ c_1' x_1' + c_2' x_2' = f. \]

Finally, after solving the last system for \( c_1, c_2 \), I find a particular solution.

Exercise 3.49. Show that the equation

\[ t^2 x'' + tx' - x = f(t), \quad t > 0 \]

has the general solution

\[ x(t) = C_1 t + \frac{C_2}{t} + \frac{t}{2} \int_{t_0}^{t} \frac{f(\tau)}{\tau^2} \, d\tau - \frac{1}{2t} \int_{t_0}^{t} f(\tau) \, d\tau. \]

Hint: to solve the homogeneous equation use the ansatz \( x(t) = t^\lambda \) and find \( \lambda \).

Exercise 3.50. Show that the equation

\[ t^2 x'' + tx' + x = f(t), \quad t > 0 \]

has the general solution

\[ x(t) = C_1 \cos \log t + C_2 \sin \log t + \int_{t_0}^{t} \frac{f(\tau)}{\tau} \sin \log \frac{t}{\tau} \, d\tau. \]
Example 3.33 (Reduction of order). If one non-trivial solution to the homogeneous linear ODE is known then the order of this equation can be reduced by one.

Consider
\[ x^{(k)} + a_{k-1}(t)x^{(k-1)} + \ldots + a_1(t)x' + a_0(t)x = 0, \]
and let \( x_1 \neq 0 \) solves it. Use the substitution \( x(t) = x_1(t)v(t) \), where \( v \) is a new unknown function. The equation for \( v \) takes the form (fill in the details)
\[ b_k(t)v^{(k)} + \ldots + b_1(t)v' = 0, \]
and hence another substitution \( w = v' \) reduces its order by one.

Exercise 3.51. Solve the equation
\[(1 + t^2)x'' - 2tx' + 2x = 0,\]
if one solution is given by \( x_1(t) = t \).

Exercise 3.52. Solve the equation
\[(2t + 1)x'' + 4tx' - 4x = 0.\]
*Hint:* Look for a solution in the form \( x(t) = e^{pt} \).

Exercise 3.53. Similarly, the same trick (reduction of order) can be used to solve systems of linear equations. Solve the system
\[
\begin{bmatrix}
  & \\
\end{bmatrix} = \mathbf{A}(t)\mathbf{x}
\]
with
\[
\mathbf{A}(t) = \begin{bmatrix} t^2 & -1 \\ 2t & 0 \end{bmatrix},
\]
if one of the solutions is \( \phi_1(t) = (1, t^2)^T \). *Hint:* make a substitution \( x(t) = Q(t)y(t) \), where \( Q(t) = (\phi_1(t) \mid e_2) \), and \( e_2 = (0, 1)^T \).

Exercise 3.54. Functions
\[ x_1 = t, \quad x_2 = t^5, \quad x_3 = |t|^5 \]
solve the differential equation
\[ t^2x'' + 5tx' + 5x = 0. \]
Are they linearly independent on \((-1, 1)\)?

Exercise 3.55. Let \( y \) and \( z \) be the solutions to
\[ y'' + q(t)y = 0, \quad z'' + Q(t)z = 0 \]
with the same initial conditions \( y(t_0) = z(t_0), \ y'(t_0) = z'(t_0) \). Assume that \( Q(t) > q(t), \ y(t) > 0 \) and \( z(t) > 0 \) for all \( t \in [t_0, t_1] \). Prove that the function
\[ \frac{z(t)}{y(t)} \]
is decreasing in \([t_0, t_1]\).

Exercise 3.56. Prove that two solutions to \( x'' + p(t)x' + q(t)x = 0 \), where \( p, q \in \mathcal{C}(I) \), that achieve maximum at the same value \( t_0 \in I \) are linearly dependent on \( I \).

Exercise 3.57. Let \( x_1(t) = 1 \) and \( x_2(t) = \cos t \). Come up with a linear ODE, which has these two functions as particular solutions. Try to find an ODE of the least possible order.

Exercise 3.58. Generalize the previous exercise.
3.9 Linear systems with periodic coefficients

In this section I will consider the systems of the form

\[ \dot{x} = A(t)x, \quad x(t) \in \mathbb{R}^k, \quad (3.17) \]

where \( A \) is a continuous periodic matrix function, i.e., there exists \( T > 0 \) such that \( A(t) = A(t + T) \) for all \( t \). The fundamental result about such systems belongs to Floquet and can be formulated in the following form.

**Theorem 3.34 (Floquet).** If \( X \) is a fundamental matrix solution for \( (3.17) \) then so is \( \Xi \), where

\[ \Xi(t) := X(t + T). \]

Corresponding to each such \( X \) there exists a periodic nonsingular matrix \( P \) with period \( T \), and a constant matrix \( B \) such that

\[ X(t) = Pe^{tB}. \quad (3.18) \]

**Proof.** I have

\[ \dot{\Xi}(t) = \dot{X}(t + T) = A(t + T)X(t + T) = A(t)\Xi(t), \]

which proves that \( \Xi \) is a fundamental matrix solution since \( \det \Xi(t) = \det X(t + T) \neq 0 \). Therefore, there exists a nonsingular matrix \( C \) such that

\[ X(t + T) = X(t)C, \]

and moreover there exists a constant matrix \( B \) such that \( C = e^{TB} \) (this matrix is called the logarithm of \( B \) and does not have to be real).

Now define

\[ P(t) := X(t)e^{-tB}. \]

Then

\[ P(t + T) = X(t + T)e^{-t+T}B = X(t)e^{TB}e^{-(t+T)B} = X(t)e^{-tB} = P(t). \]

Since \( X(t) \) and \( e^{-tB} \) are nonsingular, then \( P(t) \) is nonsingular, which completes the proof. \( \blacksquare \)

**Exercise 3.59.** Show that if matrix \( C \) is nonsingular then there exists matrix \( B \), possibly complex, such that \( e^{B} = C \).

**Remark 3.35.** Actually, if \( A(t) \) is real and the system \( \dot{x} = A(t)x \) is considered as \( 2T \)-periodic, then it is possible to find \( P_1(t) \) and \( B_1 \) such that \( P_1(t + 2T) = P_1(t), \ X(t) = P_1(t)\exp(B_1t) \) and \( B_1 \) is real. I will leave a proof of this fact to the reader.

The matrix \( C \), which was introduced in the proof, is called the *monodromy* matrix of equation (3.17), the eigenvalues \( \rho_j \) of \( C \) are called the *characteristic multipliers*, and the quantities \( \lambda_j \) such that

\[ \rho_j = e^{\lambda_jT} \]

are called the *characteristic exponents* (or *Floquet exponents*). The imaginary part of the characteristic exponents is not determined uniquely (recall that the exponent has period \( 2\pi i \)). I can always choose the characteristic exponents such that they coincide with the eigenvalues of \( B \).
Exercise 3.60. Carefully note that for different $X$ one will get different $C$. Explain why this does not influence the conclusions of the theorem and the last paragraph.

Exercise 3.61. Show that the change of variables $x = P(t)y$ for the matrix $P(t) = X(t)e^{-tB}$, where $X(t)$ is the principal matrix solution, turns $x = A(t)x$ in a linear system with constant coefficients.

The notion of stability verbatim translates to the linear systems with non-constant coefficients. In particular, it should be clear that the existence of periodic solutions to (3.17) or the stability of this system are both determined by the eigenvalues of $B$, because the Floquet theorem implies that the solutions are composed of products of polynomials in $t$, $e^{\lambda_j t}$ and $T$-periodic functions. I can formulate, leaving the details of the proof to the reader, the following

Theorem 3.36. Consider system

$$\dot{x} = A(t)x, \quad x(t) \in \mathbb{R}^k, \quad A(t) = A(t + T), \quad T > 0, \quad A \in \mathbb{C}(\mathbb{R}_+; \mathbb{R}^k \times \mathbb{R}^k), \quad t > 0.$$

(a) This system is asymptotically stable if and only if all the characteristic multipliers are in modulus less than one.

(b) This system is Lyapunov stable if and only if all the characteristic multipliers are in modulus less than or equal to one, and those with one have equal algebraic and geometric multiplicities.

(c) This system is unstable if and only if it has a characteristic multiplier with modulus bigger than one, or it has a characteristic multiplier with modulus equal to one and its algebraic multiplicity is strictly bigger than its geometric multiplicity.

It is usually a very nontrivial problem to determine the characteristic multipliers. Sometimes the following information can of some use.

Using Liouville’s formula for the principal matrix solution $\Phi(t, t_0)$ I find that

$$\det \Phi(t, t_0) = \exp \int_{t_0}^t \text{tr} \ A(\tau) \ d\tau,$$

and therefore, due to periodicity of $P$,

$$\det e^{TB} = \exp \int_0^T \text{tr} \ A(\tau) \ d\tau = \rho_1 \ldots \rho_k,$$

and

$$\lambda_1 + \ldots + \lambda_k = \frac{1}{T} \int_0^T \text{tr} \ A(\tau) \ d\tau \pmod{2\pi i T}.$$

Example 3.37. Consider problem (3.17) with

$$A(t) = \begin{bmatrix} \frac{1}{2} - \cos t & b \\ a & \frac{3}{2} + \sin t \end{bmatrix}.$$
Since I have that
\[ \int_0^{2\pi} \text{tr } A(\tau) \, d\tau = 4\pi, \]
therefore
\[ \lambda_1 + \lambda_2 = 2 > 0 \]
and therefore there exists at least one one-parameter family of solutions to this system which becomes unbounded when \( t \to \infty \).

**Example 3.38.** An important and not obvious fact is that the eigenvalues of \( A(t), t \in \mathbb{R} \) cannot be used to infer the stability of the system. Consider
\[ A(t) = \begin{bmatrix} -1 + \frac{3}{2} \cos^2 t & 1 - \frac{3}{2} \sin t \cos t \\ -1 - \frac{3}{2} \sin t \cos t & -1 + \frac{3}{2} \sin^2 t \end{bmatrix}. \]

Therefore,
\[ \lambda_1 + \lambda_2 = -\frac{1}{2}. \]
Hence, no conclusion can be made about the stability. I can calculate the eigenvalues of \( A(t) \), which, surprisingly, do not depend on \( t \):
\[ \mu_{1,2} = (-1 \pm i \sqrt{7})/4, \]
which both have negative real part. However, as it can checked directly, the solution
\[ t \mapsto \begin{bmatrix} -\cos t \\ \sin t \end{bmatrix} e^{t/2} \]
solves the system, and hence the system is unstable.

**Example 3.39.** Actually, a converse to the previous example is also true. Consider
\[ A(t) = \begin{bmatrix} -\frac{11}{2} + \frac{15}{2} \sin 12t & \frac{15}{2} \cos 12t \\ \frac{15}{2} \sin 12t & -\frac{11}{2} - \frac{15}{2} \sin 12t \end{bmatrix}. \]
The eigenvalues can be calculated as 2 and \(-13\). However, the system with this matrix is asymptotically stable, as can be shown by finding the fundamental matrix solution\(^1\).

Unfortunately there exist no general methods to find matrices \( P(t) \) and \( B \), and whole books are devoted to the analysis of, e.g., *Hill’s equation*
\[ \ddot{x} + (a + b(t))x = 0, \]
where \( b(t) = b(t + \pi) \).

**Exercise 3.62.** Consider the system
\[ \dot{x} = A(t)x, \]
where \( t \mapsto A(t) \) is a smooth \( T \)-periodic matrix function, \( x(t) \in \mathbb{R}^k \).


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1. \( k = 1 \), \( A(t) = f(t) \). Determine \( P(t) \) and \( B \) in the Floquet theorem. Give necessary and sufficient conditions for the solutions to be bounded as \( t \to \pm \infty \) or to be periodic.

2. \( k = 2 \) and 
\[
A(t) = f(t) \begin{bmatrix} a & b \\ c & d \end{bmatrix}.
\]
Determine \( P(t) \) and \( B \) in the Floquet theorem. Give necessary and sufficient conditions for the solutions to be bounded as \( t \to \pm \infty \) or to be periodic.

3. Consider now
\[
A(t) = \begin{bmatrix} \cos t & \sin t \\ \sin t & -\cos t \end{bmatrix}.
\]
Note that not only \( \text{tr} A(t) = 0 \) but also all the terms in \( t \to A(t) \) have the average zero value through one period. Are the solutions bounded?

Exercise 3.63. Consider a non-homogeneous problem
\[
\dot{x} = A(t)x + f(t),
\]
where both \( A \) and \( f \) are \( T \)-periodic. Prove that if the homogeneous system has no \( T \)-periodic vanishing solution then the non-homogeneous system has one and only one \( T \)-periodic solution.

3.10 Appendix

3.10.1 Calculating the matrix exponent

I did not give full details how to compute the matrix exponent for an arbitrary matrix \( A \) in the main text. Here I give a brief description of one procedure that is often convenient for the matrices of not very high order. This procedure (interpolation method) actually can be applied to calculate other than exponent functions of matrices\(^2\).

Let \( A \in \mathbb{R}^{k \times k} \) and \( f(\lambda) = e^{\lambda t} \). I need to determine \( f(A) \). I need to find first the characteristic polynomial for \( A \), denote it \( P(\lambda) = \prod_{j=1}^{m}(\lambda - \lambda_j)^{a_j} \), where all the \( \lambda_j \) are distinct. Define
\[
g(\lambda) = \alpha_0 + \alpha_1 \lambda + \ldots + \alpha_{k-1} \lambda^{k-1},
\]
where \( \alpha_j \) are some constants to be determined. They are, in fact, are the unique solution to \( k \) equations:
\[
g^{(n)}(\lambda_j) = f^{(n)}(\lambda_j), \quad n = 1, \ldots, a_j, j = 1, \ldots, m.
\]
I claim that \( f(A) = g(A) \), the motivation for this is the Cayley–Hamilton theorem that says that all powers of \( A \) greater than \( k - 1 \) can be expressed as a linear combination of \( A^n \), \( n = 1, \ldots, k - 1 \). Thus all the terms of order greater than \( k - 1 \) in the definition of the matrix exponent can be written in terms of these lower powers as well.

Exercise 3.64. Fill in the details in the previous paragraph and prove that \( g \) gives the appropriate linear combination (interpolation) for \( e^{tA} \).

\(^2\)I am borrowing the description and example from Laub, A. J. (2005). Matrix analysis for scientists and engineers. SIAM.
Example 3.40. Let
\[
A = \begin{bmatrix}
-1 & 1 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}.
\]
I find \( P(\lambda) = - (\lambda + 1)^3 \), so \( m = 1 \) and \( a_1 = 3 \). I have
\[
\begin{align*}
g(-1) &= f(-1) \implies \alpha_0 + \alpha_1 + \alpha_2 = e^{-t}, \\
g'(-1) &= f'(-1) \implies \alpha_1 - 2\alpha_2 = te^{-t}, \\
g''(-1) &= f''(-1) \implies 2\alpha_2 = t^2 e^{-t}.
\end{align*}
\]
Solving this system for \( \alpha_j \) I get
\[
\begin{align*}
\alpha_2 &= \frac{t^2}{2} e^{-t}, \\
\alpha_1 &= t e^{-t} + t^2 e^{-t}, \\
\alpha_0 &= e^{-t} + t e^{-t} + \frac{t^2}{2} e^{-t},
\end{align*}
\]
and hence
\[
e^{tA} = g(A) = \alpha_0 I + \alpha_1 A + \alpha_2 A^2,
\]
which yields
\[
\begin{bmatrix}
e^{-t} & te^{-t} & 0 \\
0 & e^{-t} & 0 \\
0 & 0 & e^{-t}
\end{bmatrix}.
\]

Exercise 3.65. Use the interpolation method to compute
\[
e^{tA} = e^{t \begin{bmatrix}
-4 & 4 \\
-1 & 0
\end{bmatrix}}.
\]

3.10.2 More on the implicit function theorem