The set of semidualizing complexes is a metric space

Anders Frankild

Sean Sather-Wagstaff

13 March 2004

Throughout, $(R, \mathfrak{m}, k)$ is a Cohen-Macaulay local ring.

I. MOTIVATION

Much research in commutative algebra is devoted to duality.

(Grothendieck-Hartshorne) Investigate $\mathrm{Hom}_R(-, D)$ when R admits a canonical module D.

(Auslander-Bridger) Investigate $\mathrm{Hom}_R(-, R)$.

(Foxby-Golod) Investigate $\mathrm{Hom}_R(-, K)$ when K is semidualizing.

Definition. A finite (i.e., finitely generated) R-module K is *semidualizing* if

(a) The natural homothety homomorphism

$$\chi_K^R \colon R \to \mathrm{Hom}_R(K, K)$$

is an isomorphism, and

(b) $\mathrm{Ext}_R^i(K, K) = 0$ for each $i \neq 0$.

Example. D and R are semidualizing.

Example. (Avramov-Foxby) The dualizing module D_φ of a local homomorphism $\varphi\colon Q \to R$ of finite G-dimension is semidualizing.

Notation. $\mathfrak{S}(R)$ is the set of isomorphism classes of semidualizing R-modules.

Example. If R is Gorenstein, then $\mathfrak{S}(R) = \{R\}$.

Example. If $\mathfrak{m}^2 = 0$, then $\mathfrak{S}(R) = \{R, D\}$.

Example. (Foxby) $\mathfrak{S}(R)$ may have elements other than D and R. Let $\varphi\colon Q \to R$ be a finite flat local homomorphism and ω a canonical module for Q. The following R-modules are in $\mathfrak{S}(R)$:

$$D = \operatorname{Hom}_Q(R, \omega), \quad \operatorname{Hom}_Q(R, Q), \quad \omega \otimes_Q R, \quad R = Q \otimes_Q R.$$

Question. Is $\mathfrak{S}(R)$ finite?

General motivation. Increase the understanding of $\mathfrak{S}(R)$.

Theorem 1. $\mathfrak{S}(R)$ *admits a (nontrivial) metric.*

II. REFLEXIVITY

Let K, M be finite R-modules with K semidualizing.

Definition. M is *totally K-reflexive* if

(a) The natural biduality homomorphism

$$\delta_M^K \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,K),K)$$

is an isomorphism, and

(b) $\operatorname{Ext}_R^i(M,K) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(M,K),K)$ for each $i \neq 0$.

Example. R is totally K-reflexive.

Example. The nonzero totally D-reflexive modules are the MCMs.

Example. The nonzero totally R-reflexive modules are the modules of G-dimension 0.

Notation. For K, L in $\mathfrak{S}(R)$ write $K \leq L$ whenever L is totally K-reflexive.

Fact. This ordering is reflexive: $K \leq K$.
It is also antisymmetric: If $K \leq L$ and $L \leq K$, then $K \cong L$.

Question. Is the ordering transitive?

Example. Let $\varphi\colon Q \to R$ be a finite flat local homomorphism and ω a canonical module for Q. In $\mathfrak{S}(R)$ there are inequalities

$$D \leq \operatorname{Hom}_Q(R, Q) \leq R \qquad \text{and} \qquad D \leq \omega \otimes_Q R \leq R.$$

If Q is not Gorenstein, then

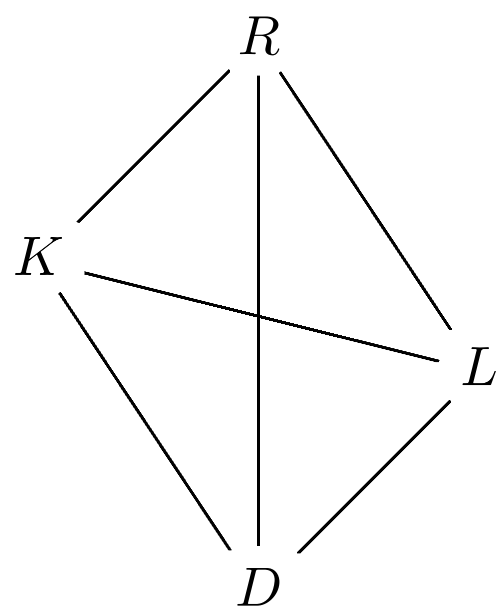
$$\operatorname{Hom}_Q(R, Q) \not\leq \omega \otimes_Q R$$

and we can construct examples where

$$\omega \otimes_Q R \not\leq \operatorname{Hom}_Q(R, Q).$$

Idea. Define the proximity of K and L when they are comparable in the ordering. Then define the distance between general K and L via sequences of pairwise comparable elements of $\mathfrak{S}(R)$.

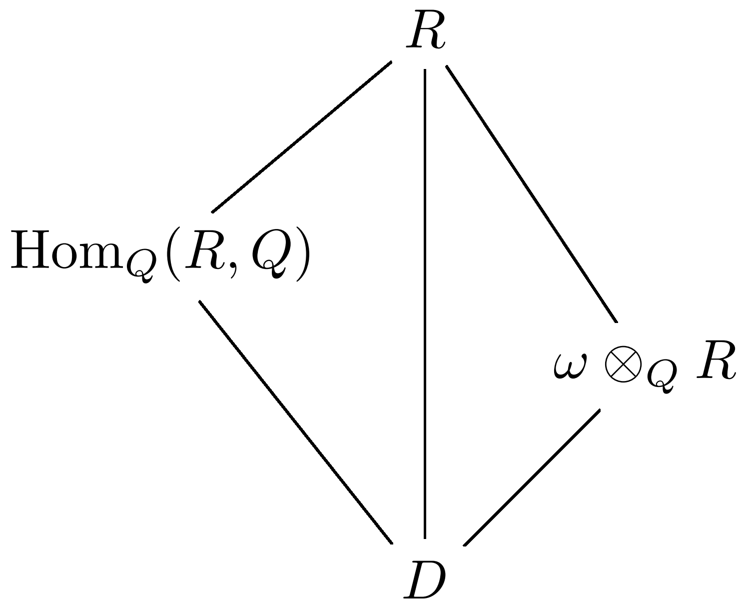
Construction. Let $\Gamma(R)$ be the graph with vertex set $\mathfrak{S}(R)$ and an edge $K - L$ whenever $K \leq L$ or $L \leq K$.



Note. The path $K - R - L$ shows that $\Gamma(R)$ is connected.

Note. The graph need not be complete.

Example. Let $\varphi\colon Q \to R$ be a finite flat local homomorphism and ω a canonical module for Q.



III. THE METRIC

Fact. Fix K, L in $\mathfrak{S}(R)$ with $K \leq L$.

(a) The R-module $\operatorname{Hom}_R(L, K)$ is semidualizing.

(b) $\operatorname{Hom}_R(K, L) \cong R$ if and only if $K \cong L$.

Definition. (Avramov) The *curvature* of a finite R-module M is

$$\operatorname{curv}_R(M) = \limsup_{n\to\infty} \sqrt[n]{\beta_n^R(M)}.$$

Fact. There are inequalities $0 \leq \operatorname{curv}_R(M) < \infty$, and TFAE:

(i) $\operatorname{curv}_R(M) < 1$, (ii) $\operatorname{curv}_R(M) = 0$, (iii) $\operatorname{pd}_R(M) < \infty$.

When M is semidualizing, these are equivalent to

(iv) $M \cong R$.

Putting the facts together, this yeilds:

Proposition 2. *For K, L in $\mathfrak{S}(R)$ with $K \leq L$, TFAE:*

(i) $\operatorname{curv}_R(\operatorname{Hom}_R(L, K)) < 1$,

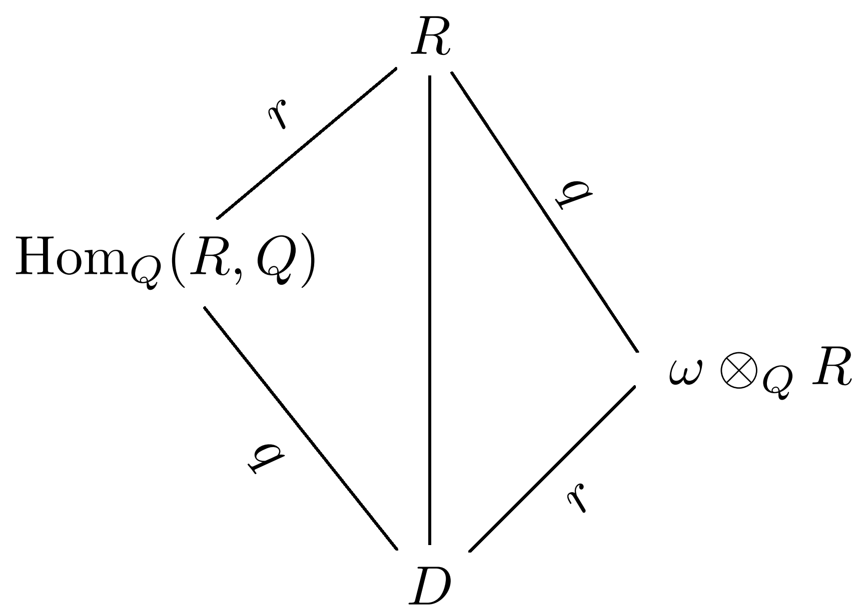
(ii) $\operatorname{curv}_R(\operatorname{Hom}_R(L, K)) = 0$,

(iii) $K \cong L$.

Notation. For K, L in $\mathfrak{S}(R)$ with $K \leq L$, set

$$\sigma_R(K, L) = \sigma_R(L, K) = \mathrm{curv}_R(\mathrm{Hom}_R(L, K)).$$

Example. Let k be a field, $Q = k[X_1, \ldots, X_q]/(X_1, \ldots, X_q)^2$ and $R = Q[Y_1, \ldots, Y_r]/(Y_1, \ldots, Y_r)^2$ where $q, r \geq 2$ are integers. The natural inclusion $\varphi\colon Q \to R$ is finite, flat, and local, and Q admits a dualizing module ω since it is Artinian.



Definition. Given a path $\gamma = (K - K_1 - \cdots - K_n - L)$ in $\mathsf{\Gamma}(R)$, the *length* of γ is

$$\operatorname{length}_R(\gamma) = \sigma_R(K, K_1) + \sigma_R(K_1, K_2) + \cdots + \sigma_R(K_n, L).$$

The *distance* from K to L is

$$\operatorname{dist}_R(K, L) = \inf\{\operatorname{length}_R(\gamma) \mid \gamma \text{ a path } K \text{ to } L\}.$$

Note. It is straightforward to show that dist is a metric on $\mathfrak{S}(R)$.

Theorem 3. *For K, L in $\mathfrak{S}(R)$ with $K \leq L$, there is an equality $\operatorname{dist}_R(K, L) = \sigma_R(K, L)$.*

IV. STANDARD OPERATIONS

Theorem 4. *Let $\varphi\colon Q \to R$ be a local homomorphism of finite flat dimension. The functor $- \otimes_Q R$ maps $\mathfrak{S}(Q) \to \mathfrak{S}(R)$, and*

$$\operatorname{dist}_R(K \otimes_Q R, L \otimes_Q R) \leq \operatorname{dist}_Q(K, L)$$

for all $K, L \in \mathfrak{S}(Q)$, with equality when $K \leq L$.

Theorem 5. *Let* $\varphi\colon Q \to R$ *be a surjective local homomorphism with kernel generated by a* Q*-sequence. When* Q *is complete, the map* $\mathfrak{S}(Q) \to \mathfrak{S}(R)$ *given by sending* K *to* $K \otimes_Q R$ *is an isometry.*

Theorem 6. *Localization* $(-)_{\mathfrak{p}}$ *at a prime ideal of* R *maps* $\mathfrak{S}(R) \to \mathfrak{S}(R_{\mathfrak{p}})$*, and for all* $K, L \in \mathfrak{S}(R)$

$$\operatorname{dist}_{R_{\mathfrak{p}}}(K_{\mathfrak{p}}, L_{\mathfrak{p}}) \leq \operatorname{dist}_R(K, L).$$

V. DAGGER DUALITY

Assume that R admits a canonical module D.

Theorem 7. *The function* $\Delta\colon \mathfrak{S}(R) \to \mathfrak{S}(R)$ *given by sending* K *to* $\operatorname{Hom}_R(K, D)$ *is an isometric involution.*

Question. When does Δ have a fixed point?

Refined Question. If Δ has a fixed point, must R be Gorenstein?

Proposition 8. *If* $K \in \mathfrak{S}(R)$ *satisfies* $K \cong \operatorname{Hom}_R(K, D)$ *and* $\mathfrak{m}^3 = 0$*, then* R *is Gorenstein.*