Ascent Properties for Derived Functors

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28 April 2013
AMS Central Section Meeting
Iowa State University
arxiv:1302.4151

Introduction

Assumption

The term "ring" is short for "commutative, noetherian ring with identity". Fix a ring homomorphism $R \to S$.

Big Picture

- To study a ring R, study its modules.
- To study how close R is to S, compare the R-modules and the S-modules.
  - Each S-module is an R-module by restriction of scalars.
  - Each R-module M gives rise to two S-modules: $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$.

Question (Ascent)

When does a given R-module M have a compatible S-module structure?

A Recent Ascent Result

Fact (B. Anderson, Frankild, SW, Wiegand; '08$\times$2 and in press)

Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$ be a flat local ring homomorphism, and let M be a finitely generated R-module. TFAE:

(i) *M has a compatible S-module structure.*

(ii) $\operatorname{Hom}_R(S, M) \xrightarrow{\cong} M$ *by* $f \mapsto f(1)$.

(iii) $M \xrightarrow{\cong} S \otimes_R M$ *by* $n \mapsto 1 \otimes n$.

(iv) $\operatorname{Ext}^i_R(S, M) = 0$ *for all* $i \geq 1$.

(v) $\operatorname{Ext}^i_R(S, M)$ *is f.g. over* S *for* $i = 1, \ldots, \dim_R(M)$.

(vi) $S \otimes_R M$ *is finitely generated over* R.

(vii) $R/\operatorname{Ann}_R(M) \xrightarrow{\cong} S/\operatorname{Ann}_R(M)S$.

(viii) *For all* $\mathfrak{p} \in \operatorname{Min}_R(M)$, *we have* $R/\mathfrak{p} \xrightarrow{\cong} S/\mathfrak{p}S$.

The case $S = \widehat{R}$

Fact (B. Anderson, Frankild, SW, Wiegand; '08×2 and in press)

Let M be a finitely generated R-module. TFAE:

(i) *M is complete.*

(ii) $\operatorname{Hom}_R(\widehat{R}, M) \xrightarrow{\cong} M$ *by* $f \mapsto f(1)$.

(iii) $\operatorname{Ext}^i_R(\widehat{R}, M) = 0$ *for all* $i \geq 1$.

(iv) $\operatorname{Ext}^i_R(\widehat{R}, M)$ *is f.g. over* $\widehat{R}$ *for* $i = 1, \ldots, \dim_R(M)$.

(v) $\widehat{M}$ *is finitely generated over* R.

(vi) $R/\operatorname{Ann}_R(M) \xrightarrow{\cong} \widehat{R}/\operatorname{Ann}_R(M)\widehat{R}$.

(vii) *For all* $\mathfrak{p} \in \operatorname{Min}_R(M)$*, we have* $R/\mathfrak{p} \xrightarrow{\cong} \widehat{R}/\mathfrak{p}\widehat{R}$.

Question

Let $R = k[X, Y]_{(X,Y)}$ and $S = \widehat{R} = k[\![X, Y]\!]$.
For which i do we have $\operatorname{Ext}^i_R(S, R) \neq 0$?

Questions about Ascent of Pairs of Modules

Question

Given R-modules M and N, what conditions guarantee that $\operatorname{Ext}^i_R(M,N)$ and $\operatorname{Tor}^R_i(M,N)$ have compatible S-module structures?

Answer

If M or N has a compatible S-module structure.

Example

Let k be a field, and set $R = k[X,Y]_{(X,Y)}$. Then $\widehat{R} \cong k[\![X,Y]\!]$.
R/XR and R/YR do not have compatible $\widehat{R}$-module structures.
However, $\operatorname{Ext}^i_R(R/XR, R/YR)$ and $\operatorname{Tor}^R_i(R/XR, R/YR)$ do have compatible $\widehat{R}$-module structures.

A More Recent Ascent Result

Theorem (SW '13)

Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, K)$ be a flat local ring homomorphism, and let M and N be finitely generated R-modules. TFAE:

(i) $M \otimes_R N$ *has a compatible S-module structure.*

(ii) $\operatorname{Tor}_i^R(M, N)$ *has a compatible S-module structure* $\forall i \geq 0$.

(iii) $\operatorname{Ext}_R^i(M, N)$ *has a compatible S-module structure* $\forall i \geq 0$.

(iv) $\operatorname{Ext}_R^i(M, N)$ *has a compatible S-module structure for* $i = 0, \ldots, \dim_R(N) - 1$.

(v) $\operatorname{Ext}_R^i(S \otimes_R M, N)$ *is finitely generated over R for all* $i \geq 1$.

(vi) $\operatorname{Ext}_R^i(S \otimes_R M, N) \xrightarrow{\cong} \operatorname{Ext}_R^i(M, N)$ *for all* $i \geq 0$.

(vii) $\forall \mathfrak{p}$ *that are minimal elements of* $\operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N)$, *we have* $R/\mathfrak{p} \xrightarrow{\cong} S/\mathfrak{p}S$.

The Result is Sharp

In the previous result, if $\operatorname{Ext}^i_R(M,N)$ has a compatible S-module structure for $i = 0, \ldots, \dim_R(N) - 1$, then $\operatorname{Ext}^i_R(M,N)$ has a compatible S-module structure $\forall i \geq 0$, as does $M \otimes_R N$.

Example

Let $R = k[X_1, \ldots, X_d]_{(X_1,\ldots,X_d)}$, and choose $j \in \{0, \ldots, d-1\}$.
Set $M = R/(X_1, \ldots, X_j)$ and $N = R/(X_{j+1}, \ldots, X_{d-1})$.

$$\operatorname{Ext}^i_R(M,N) \cong \begin{cases} 0 & \text{for all } i \neq j = \dim_R(N) - 1 \\ N & \text{for all } i = j = \dim_R(N) - 1. \end{cases}$$

$\operatorname{Ext}^i_R(M,N)$ is an $\widehat{R}$-module for $i = 0, \ldots, \dim(N) - 2$.
$\operatorname{Ext}^{\dim_R(N)-1}_R(M,N)$ does not have an $\widehat{R}$-module structure.
Same for $M \otimes_R N \cong R/(X_1, \ldots, X_{d-1})$.